Chapter 5

Embedding $\mathbb{R}^2$ into $\mathbb{RP}^2$; Duality in Projective Geometry

We have an embedding (a continuous injection) $\alpha : \mathbb{R}^2 \to \mathbb{RP}^2$ given by

$$\alpha(x, y) = \ell_{(x,y,1)},$$

where $\ell_{(x,y,1)}$ is the line joining the point $(x, y, 1)$ to the origin in $\mathbb{R}^3$. (What does continuity mean here?) Thus α maps the tangent plane $\{z = 1\}$ to the unit sphere at the north pole $(0, 0, 1)$ to $\mathbb{RP}^2$. In the model of $\mathbb{RP}^2$ given by $S^2/(v \sim -v)$, we have

$$\alpha(x, y) = \left[\frac{1}{\sqrt{x^2 + y^2 + 1}}(x, y, 1)\right] = \left\{\pm\frac{1}{\sqrt{x^2 + y^2 + 1}}(x, y, 1)\right\}.$$

The image of α is all of $\mathbb{RP}^2$ except for horizontal lines, i.e., lines in the xy-plane. Note that α takes lines in $\mathbb{R}^2$ to lines in $\mathbb{RP}^2$ (why?), so it is a "map of geometries."

Homogeneous Coordinates on $\mathbb{RP}^2$*:* If ℓ is the line on the origin and (x, y, z), we denote the homogeneous coordinates of ℓ by $[x, y, z]$. The square brackets indicate the ambiguity:

$$[x, y, z] = [\lambda x, \lambda y, \lambda z] \text{ for } \lambda \in \mathbb{R}^*,$$

since the lines determined by these two coordinates are the same. Here $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. (This may seem shocking, but polar coordinates have a similar ambiguity.)

For example, if $z \neq 0$, we have $[x, y, z] = [x/z, y/z, 1]$, so we can almost always set the last coordinate to be 1. This just means that every line on the origin and not lying in the xy-plane hits the tangent plane $\{z = 1\}$ in a unique point, as is clear from a picture.

If ℓ is in the xy-plane, then its homogeneous coordinates are of the form $[x, y, 0]$. Thus if a family of non-horizontal lines ℓ_t, $t \in [0, 1)$ "fall down" to a horizontal line ℓ_1, the homogeneous coordinates of $\ell_t = [x_t, y_t, 1]$ seem to jump discontinuously to the homogeneous coordinates of $\ell_1 = [x_1, y_1, 0]$. Why is this not a problem?

Anyway, in homogeneous coordinates,

$$\alpha(x, y) = [x, y, 1],$$

a nice formula.

Lines $\mathcal{L}$ in $\mathbb{RP}^2$ are planes on the origin in $\mathbb{R}^3$, so have equation $\mathcal{L} = \mathcal{L}_{(a,b,c)} = \{(x, y, z) : ax+by+cz = 0\}$ for some choice of $(a, b, c) \neq (0, 0, 0)$. Since $(x, y, z) \in \mathcal{L} \Leftrightarrow (\lambda x, \lambda y, \lambda z) \in \mathcal{L}$, we can write

$$\mathcal{L} = \{[x, y, z] : ax + by + cz = 0\}.$$

We should really call this line $\mathcal{L}_{[a,b,c]}$ – Why?

Special example: $\mathcal{L}_\infty = \{[x, y, z] : z = 0\}$ is the set of horizontal lines in $\mathbb{R}^3$, or "horizontal points" in $\mathbb{RP}^2$. So this "line at infinity" is precisely the points of $\mathbb{RP}^2$ not in the image of α :

$$\mathrm{Im}(\alpha) = \mathbb{RP}^2 \setminus \mathcal{L}_\infty.$$

Points on the line at infinity are called points at infinity. Note that α of a line in the $z = 1$ plane is all of a line in $\mathbb{RP}^2$ except for one point at infinity. For $\alpha(ax + by = c) = \{[x, (c - bx)/a, 1]\}$ (if $a \neq 0$), and all these points satisfy $ax + by - cz = 0$. There is one more point on the line $\mathcal{L}_{[a,b,-c]}$, namely the point at infinity $[-b/a, 1, 0]$ if $a \neq 0$.

Important note: There is nothing special about the north pole and its tangent plane $z = 1$. For each choice of (antipodal) point(s) on S^2, we get an embedding of $\mathbb{R}^2$ into $\mathbb{RP}^2$. If we choose the points on the line $[x_0, y_0, z_0]$ then the image of the embedding is all of $\mathbb{RP}^2$ except the line $\mathcal{L}_{[x_0,y_0,z_0]}$.

Calculations in Homogeneous Coordinates:

Example.

- In $\mathbb{R}^2$, $\lim_{b\to\infty}(7, b)$ does not exist, of course.
- In $\mathbb{RP}^2$,

$$\lim_{b\to\infty} \alpha(7, b) = \lim_{b\to\infty} [7, b, 1] = \lim_{b\to\infty} [7/b, b/b, 1/b] = [0, 1, 0]!$$

  So the limit exists, but the limit point $[0, 1, 0]$ is a point at infinity – from the previous example, this limit point can't be a point in $\alpha(\mathbb{R}^2)$.

- Similarly, $\lim_{b\to\infty} \alpha(10, b) = [0, 1, 0]$. Thus the line parallel lines $x = 7$ and $x = 10$ cannot meet in $\mathbb{R}^2$, but $\alpha(\mathcal{L}_{[7,0,0]}) \cap \alpha(\mathcal{L}_{[10,0,0]}) = \{[0, 1, 0]\}$. So while it is incorrect to say that parallel lines meet at infinity, it is correct to say that parallel lines in $\mathbb{R}^2$ embed into lines in $\mathbb{RP}^2$ that meet at a point at infinity.
- Check that the lines $y = ax + b_0, y = ax + b_1$ with $b_0 \neq b_1$ embed into lines in $\mathbb{RP}^2$ that meet at $[1/a, 1, 0]$ if $a \neq 0$. Do the case where $a = 0$.

Note: In the glued sphere model of $\mathbb{RP}^2$, a line $\mathcal{L}$ is given by a great circle with antipodal points glued. This is the same as taking a semicircle (half of the great circle) and gluing the endpoints – Why? This clearly gives a circle again, so *lines in $\mathbb{RP}^2$ have the shape of a circle.* Note that $\mathbb{RP}^1$, the set of lines on the origin in $\mathbb{R}^2$, is modeled by a circle with antipodal points glued, so *lines in $\mathbb{RP}^2$ are copies of/homeomorphic to $\mathbb{RP}^1$.* That's satisfying, as lines in $\mathbb{R}^2$ are copies of $\mathbb{R} = \mathbb{R}^1$.

Other Projective Planes

We can try to form projective planes for $\mathbb{C}^2$ or $\mathbb{Z}_3^2$ or $\mathbb{Z}_p^2$ for prime p. (Recall that $\mathbb{Z}_m$ for composite m is a bad plane.) We can work either synthetically or analytically.

For the synthetic approach, we note that to pass from $\mathbb{R}^2$ to $\mathbb{RP}^2$, we have added one new point "at infinity" for each maximum family of parallel lines. Recall that a line The lines in $\mathbb{RP}^2$ consist of (a) the old lines in $\mathbb{R}^2$ with one extra point at infinity, where the point at infinity added to the affine line ℓ is the new point added to the family of lines parallel to ℓ; and (b) one new line consisting of all the points at infinity. We should check that this works, in the synthetic sense that we get a geometry where two distinct points determine a unique line, and that there are now no parallel lines. [Check this!] So we can try to do the exact same for $\mathbb{C}^2$ and $\mathbb{Z}_p^2$. Check that it works for $\mathbb{C}^2$. For $\mathbb{Z}_3\mathbb{P}^2$, It's a little tricky, as the synthetic definition of a point in $\mathbb{Z}_3\mathbb{P}^2$ is a line on the origin in $\mathbb{Z}_3^3$, and the set of lines has to be labeled laboriously if we're not allowed to use coordinates for the 27 points in $\mathbb{Z}_3^3$.

It's much easier to do this using homogeneous coordinates. For a line on the origin and $(x, y, z) \in \mathbb{Z}_3^3$, we assign homogeneous coordinates $[x, y, z]$ with the understanding that

$$[x, y, z] = [\lambda x, \lambda y, \lambda z] \text{ for } \lambda \in \mathbb{Z}_3^* = \{1, 2\}.$$

We have the same α map as before. The new points at infinity are $[1, 0, 0], [1, 2, 0], [2, 0, 0], [2, 1, 0]$, and these points lie on the new line $z = 0$. So while the affine geometry $\mathbb{Z}_3^2$ has 9 points and 12 lines, with 3 points on a line, the projective geometry $\mathbb{Z}_3^2\mathbb{P}^2$ has 13 points and 13 lines, with 4 points on a line. You can draw a nice picture.

Exercise. Draw a picture of $\mathbb{Z}_2\mathbb{P}^2$. This should remind you of something we've done.

Similarly, $\mathbb{CP}^2$ is given by $\{[x, y, z] : (x, y, z) \neq (0, 0, 0); x, y, z \in \mathbb{C}\}$ with

$$[x, y, z] = [\lambda x, \lambda y, \lambda z] \text{ for } \lambda \in \mathbb{C}^*.$$

The α map embeds $\mathbb{C}^2$ into $\mathbb{CP}^2$, with image missing a (complex) line at infinity. While it's hard to visualize $\mathbb{CP}^2$, it is a very important space, as we'll see later.

Question. What does a line in $\mathbb{CP}^2$ look like? Is it a copy of $\mathbb{CP}^1$?

We'll return to this later.

All these examples indicate the following:

Theorem. *Any affine plane embeds into a projective plane.*

Proof. Just as above, add one new point to each line by the procedure in (a) and (b) above, and collect all the new points into one new line. Check that this works – the new set of points and lines form a projective geometry. □

What about reversing this procedure? Can we delete a line from a projective plane and get an affine plane? There is no particular way to determine which line is the "line at infinity," so which line should we delete? Looking at $\mathbb{RP}^2$, we mentioned that we can choose any glued up great circle to be the line at infinity for an embedding α. Similarly, looking at $\mathbb{Z}_2\mathbb{P}^2$, you can check that no matter what line you delete, the result gives $\mathbb{Z}_2^2$.

Exercise. What happens when you embed $\mathbb{R}^2$ into $\mathbb{RP}^2$ using α and then go back to $\mathbb{R}^2$ using the inverse of another α? Specifically, let $\beta : \{(1, y, z) :, (y, z) \in \mathbb{R}^2\} \to \mathbb{RP}^2$ be the analogue of the α map but for the tangent plane at the "east pole" $(1, 0, 0)$. What is the line at infinity for β? What is the domain and range for the map $\beta^{-1} \circ \alpha$? This is an interesting map from most of $\mathbb{R}^2$ to most of $\mathbb{R}^2$. Does it take lines to lines? (Give a synthetic proof: a line in the $z = 1$ plane determines a plane on the origin, which meets the $x = 1$ plane in a line. Then give an analytic proof: take $ax + by = c$ in the $z = 1$ plane, apply $\beta^{-1} \circ \alpha$ to this line and show that the resulting set of points satisfies an equation of the form $a'y + b'z = c', x = 1$.) Take two lines in the $z = 1$ plane that intersect on the y-axis. What is the image of these lines under $\beta^{-1} \circ \alpha$? Does $\beta^{-1} \circ \alpha$ take circles to circles? Conics to conics?

This motivates the following result:

Theorem. *Let $\mathcal{P}$ be a projective plane, and let $\mathcal{L}$ be a line in $\mathcal{P}$. Set $\mathcal{A} = \mathcal{P} \setminus \mathcal{L}$. Then $\mathcal{A}$ is an affine plane, where the lines in $\mathcal{A}$ are the lines in $\mathcal{P}$ minus any points on $\mathcal{L}$.*

Proof. Here's a sketch that $\mathcal{A}$ has the parallel postulate. Take a line ℓ in $\mathcal{A}$ and a point $P \notin \ell$. Let Q be the unique point on $\mathcal{L}$ such that $\ell \cup \{Q\}$ is a line in $\mathcal{P}$. Set $\mathcal{L}' = \overline{PQ}$, a line in $\mathcal{P}$. Then the line $\ell' = \mathcal{L}' \setminus \{Q\}$ goes through P and is parallel to ℓ. You fill in the details. □

Moving lines to infinity

Say we want to prove a concordance theorem in Euclidean geometry, *i.e.*, a theorem that just involves an arrangement of points and lines – no references to angles, areas, etc. Examples: (a) Two distinct lines cannot intersect in more than one point; (b) Desargues Theorem. Since the α map takes lines in $\mathbb{R}^2$ to lines in $\mathbb{RP}^2$ and intersecting lines to intersecting lines, if a concordance theorem is true in $\mathbb{R}^2$, then it is true in $\mathbb{RP}^2$. *Warning:* This needs to be interpreted correctly, since the parallel postulate (which is a concordance result) holds in $\mathbb{R}^2$, but not in $\mathbb{RP}^2$. So if a concordance theorem involves parallel lines $\ell_1 \| \ell_2$ in $\mathbb{R}^2$, either in the statement or the conclusion, then this becomes "ℓ_1 meets ℓ_2 on the line at infinity" in $\mathbb{RP}^2$.

In particular, under the $\beta^{-1} \circ \alpha$ map, intersecting lines may go to parallel lines and *vice versa.* This gives a very useful technique for proving concordance theorems, called moving a line to infinity.

Example. Consider Pappus' Theorem, from around 200 AD:

Theorem. [Pappus' Theorem] *Pick points P_1, P_2, P_3 on a line ℓ_1, and points Q_1, Q_2, A_3 on a line ℓ_2 in $\mathbb{R}^2$. Set $T = P_1Q_2 \cap P_2Q_1$, $S = P_1Q_3 \cap P_3Q_1$, $R = P_2Q_2 \cap P_3Q_2$. Then T, S, R are collinear.*

This is a concordance theorem. If you draw a picture, you'll see that this is a mess to prove directly. However, draw this configuration in the $z = 1$ plane so that T, S lie on the y-axis. Then $\alpha(y-\text{axis})$ is the line at infinity for the β map, so $\beta^{-1} \circ \alpha$ is not defined on T, S. If we let $P_i' = \beta^{-1} \circ \alpha(P_i)$ and similarly for Q_i', then we see that

$$P_1'Q_2' \| P_2'Q_1',\ P_1'Q_3' \| P_3'Q_1'.$$

Thus $R \in ST$ iff R' doesn't exist, *i.e.*, iff $P_2'Q_3' \| P_3'Q_2'$. In old fashioned language we would say that we have moved TS to the line at infinity, and we have to show that R is also on the line at infinity.

So now we have reduced the proof of Pappus' Theorem to the following special case – in the lemma, we drop the primes from the notation.

Lemma. *Pick points P_1, P_2, P_3 on a line ℓ_1, and points Q_1, Q_2, A_3 on a line ℓ_2 in $\mathbb{R}^2$. Assume $P_1Q_2 \| P_2Q_1, P_1Q_3 \| P_3Q_1$. Then $P_2Q_3 \| P_3Q_2$.*

If you draw a picture, you'll see so many similar triangles that a proof should be reasonable.

Proof. There are two cases depending on whether ℓ_1 meets ℓ_2 or not. Let's assume they meet at U; you can do the other case. Then

$$\Delta UP_1Q_2 \sim \Delta UP_2Q_1 \Longrightarrow \frac{|UP_1|}{|UQ_2|} = \frac{|UP_2|}{|UQ_1|}.$$

Similarly,

$$\frac{|UP_1|}{|UQ_3|} = \frac{|UP_3|}{|UQ_1|}.$$

But now a little algebra allows us to combine these two equations and conclude

$$\frac{|UP_2|}{|UQ_3|} = \frac{|UP_3|}{|UQ_2|},$$

which implies $P_2Q_3 \| P_3Q_2$. $\square$

Recall Desargues' Theorem from around 1625– draw a picture.

Theorem. [Desargues' Theorem] *Let ΔABC and $\Delta A'B'C'$ be in perspective from a point O: i.e., AA', BB', CC' meet at O. Set $R = AB \cap A'B'$, $S = AC \cap A'C'$, $T = BC \cap B'C'$. Then R, S, T are collinear.*

There are several ways to prove this. You can "move O to infinity," which reduces the theorem to the case where AA"$\|BB'\|CC'$. Alternatively, you can "move RS to infinity and prove that T is on the line at infinity." This means you assume $AB\|A'B', AC\|A'C'$ and show that $BC\|B'C'$. The second method is fairly straightforward, and you should work this out.

These are two concordance theorems. Why did Desargues' Theorem appear over a century after Pappus' Theorem? The answer lies in the motivation for Desargues, which was to give a mathematical justification of the *ad hoc* rules of perspective drawing developed in the 1400s. To see this, let's give the standard 3D proof of Desargues' Theorem.

Proof. (Draw a picture.) Let ΔABC and $\Delta A'B'C'$ be two triangles in $\mathbb{R}^3$ in perspective from O. Thus an artist with one eye at O (and the other closed) would see these triangles as identical. Let $\mathcal{P}, \mathcal{P}'$ be the planes containing $\Delta ABC, \Delta A'B'C'$, respectively. (We'll assume that O is not in either plane, and you can handle this other case.) Set $\ell = \mathcal{P} \cap \mathcal{P}'$. (Again, you handle the case where $\mathcal{P}\|\mathcal{P}'$.) Since O, A, A', B, B' lie on two lines, they lie in a plane $\mathcal{P}_1$. Thus AB and $A'B'$ cannot be skew: they are either parallel in $\mathcal{P}_1$ (another case to handle) or $R = AB \cap A'B'$ exists in both $\mathcal{P}_1$ and in $\mathcal{P} \cap \mathcal{P}' = \ell$. The same argument shows that $S, T \in \ell$.

It seems like we are done with the proof: If we project the 3D setup to a plane, like the sheet of paper you drew the picture on, then "concordance projects to concordance," (*i.e.*, intersecting lines project to intersecting lines), so the projected triangles satisfy Desargues. But we have to show something more: given two triangles $\Delta A_1B_1C_1, \Delta A_1'B_1'C_1'$ in perspective from O_1 in $\mathbb{R}^2$, there exist triangles $\Delta ABC, \Delta A'B'C'$ in in perspective from O in $\mathbb{R}^3$ such that $\Delta ABC, \Delta A'B'C', O$ project to $\Delta A_1B_1C_1, \Delta A_1'B_1'C_1', O'$, respectively.

Exercise. Prove this.

The Exercise finishes the proof. $\square$

It is very unusual that a geometry proof is easier in 3D than in 2D, which reflects that the origin of Desargues Theorem really is a 3D problem.

Duality in Projective Geometry

Look at the axioms of a projective plane. If you replace each occurrence of "point" with "line" and *vice versa,* and replace each occurrence of "meet" (*i.e.*, intersection) with "join" (*i.e.*, the line determined by two points is called their join), the axioms do not change. For example, "each pair of distinct lines intersect at a unique point " becomes "each pair of distinct points join to/determine a unique line."

In contrast, if you try this with the axioms of an affine plane, the parallel postulate turns into "Given a point P and a line ℓ not on P, there exists a unique point P' such that P' is on ℓ and there is no join/line determined by P and P'." This is completely wrong, as P and P' definitely determine a line.

This is important: if you take a concordance (!) theorem and its proof in a particular projective plane, you can do the replacements above in the statement and each line of the proof. You have then given a (probably) new statement in the projective plane and its proof. So this is a method to double your theorems in projective geometries for free! This is not possible in affine geometry.

How can you tell if a theorem stated in an affine plane is really a theorem in the associated projective plane? Check if the theorem holds when you move a "reasonable" line in the statement to infinity, as we did with Pappus and Desargues. (In Pappus, you can't move the line containing P_1, P_2, P_3 to infinity, as the statement then makes no sense.) Then we argued above that the affine version of the theorem must hold.

Warning: I'm not claiming that Pappus or Desargues holds in any projective plane. In fact, this is deeply connected to the "coordinate ring" of the plane – see Artin's book.

Let's see what the dual of Desargues' Theorem is. First we dualize the hypothesis. In ΔABC, let $X = AB, Y = BC, Z = AC$. Denote the lines dual to A, B, C by $\bar{A}, \bar{B}, \bar{C}$, and the points dual to X, Y, Z by $\bar{X}, \bar{Y}, \bar{Z}$. Then "$A$ join B equals X" dualizes to $\bar{A} \cap \bar{B} = \bar{X}$. Continuing, the dual of ΔABC is a triangle $\Delta \bar{X}\bar{Y}\bar{Z}$ with sides $\bar{A}, \bar{B}, \bar{C}$. Similarly, the dual of $\Delta A'B'C'$ is a triangle $\Delta \bar{X}'\bar{Y}'\bar{Z}'$ with sides $\bar{A}', \bar{B}', \bar{C}'$.

Continuing, we dualize the hypothesis "AA', BB', CC' meet at O" to "the three points $\bar{A} \cap \bar{A}'$, $\bar{B} \cap \bar{B}'$, $\bar{C} \cap \bar{C}'$ join at the line $\bar{O}$." This means that $\bar{A} \cap \bar{A}' = \bar{X}\bar{Z} \cap \bar{X}'\bar{Z}'$,$\bar{B} \cap \bar{B}' = \bar{X}\bar{Y} \cap \bar{X}'\bar{Y}'$, $\bar{A} \cap \bar{C}' = \bar{X}\bar{Y} \cap \bar{X}'\bar{Y}'$ are collinear.

Now you dualize the conclusion of Desargues. You should get that "$AB \cap A'B' = (A \cup B) \cap (A' \cup B')$ and the two other points are collinear"[1] dualizes to $\bar{X}\bar{X}', \bar{Y}\bar{Y}', \bar{Z}\bar{Z}'$ meet at a point.

If you draw a picture for the dual statement (= hypothesis + conclusion), you'll see that the dual to Desargues' Theorem is exactly the converse of Desargues' Theorem! This is great: we don't have to do any thinking to prove the

[1] Just as it's convenient to write $\ell \cap \ell'$ for intersections, it's convenient to write $A \cup B$ for joins. Then the dual of $A \cup B$ is $\bar{A} \cap \bar{B}$.

converse, we just tediously change every step of the proof of Desargues into its dual statement.

Exercise. Determine the dual of Pappus' Theorem

This discussion of duality in projective geometry has been synthetic – no coordinates. Let's now look at duality analytically – with coordinates.

A point Q in $\mathbb{RP}^2$ has homogeneous coordinates $Q = [a, b, c]$. To Q we associate its *polar line* $\ell_Q = \ell_{[a,b,c]} = \{[x, y, z] : ax + by + cz = 0\}$. Conversely, to a line $\ell = \ell_{[a,b,c]}$ we associate its *pole* $P = P_\ell = [a, b, c]$.

Exercise. Check that duality really works here: $\ell_Q \cap \ell_R = P_{Q \cup R}; P_{\ell_1} \cup P_{\ell_2} = \ell_{\ell_1 \cap \ell_2}$.

Note that duality "usually" works in $\mathbb{R}^2$ thought of as the plane $z = 1$ as before: the dual to the point $(a, b) \leftrightarrow [a, b, 1]$ is the line $ax + by + 1 = 0$, and the dual to the line $ax + by = c$ is the point $(-a/c, -b/c)$ – Why? Of course we have trouble if $c = 0$, and you should check that this is precisely the case where the pole is on the line at infinity.