Chapter 6

Conics

We have a pretty good understanding of lines/linear equations in $\mathbb{R}^2$ and $\mathbb{RP}^2$. Let's spend some time studying quadratic equations in these spaces. These curves are called conics for the very good reason that they arise as the intersection of a plane in $\mathbb{R}^3$ with a cone.

Recall that a cone on the origin has the equation $x^2 + y^2 = \alpha^2 z^2$ for some $\alpha > 0$. (Why is this a cone?) The angle formed between the positive z-axis and the side of the cone is $\tan^{-1}(\alpha^{-1})$ (Why?); this is called the cone angle, but sometimes we'll just say the cone angle is α.

Set $\alpha = 1$ for convenience. If we intersect the cone $\mathcal{C}$ with the vertical plane $\mathcal{P} : a'x + b'y = c'$, $c' \neq 0$, you can check that we get a hyperbola in (x, z) coordinates (as long as $a' \neq 0$). Note that this really means that the projection of $\mathcal{C} \cap \mathcal{P}$ into the (x, z)-coordinate plane is a hyperbola; a further check is needed to show that $\mathcal{C} \cap \mathcal{P}$ itself is a hyperbola. Note that if $c' = 0$, the intersection is a pair of lines.

Exercise. Show that (a) if we intersect a general cone $\mathcal{C}$ with a horizontal plane, we get a circle or a point; (b) if we intersect $\mathcal{C}$ with a plane that is parallel to a generating line of $\mathcal{C}$ (a line lying on $\mathcal{C}$), then we get a parabola; note that the angle this plane makes with the xy-plane equals the cone angle; (c) if we intersect $\mathcal{C}$ with a plane whose angle with the xy-plane is less than the cone angle, then we get an ellipse or a point; (d) if we intersect $\mathcal{C}$ with a plane whose angle with the xy-plane is more than the cone angle, then we get a hyperbola. In all cases, it's not hard to check that the projection of the intersection into a coordinate plane is an ellipse/parabola/hyperbola, but it's harder to check that the intersection itself is of this type.

When you do this Exercise, you will see that the intersections are of the form

$$ax^2 + bxy + cy^2 + ex + fy + g = 0, \tag{1}$$

with at least one of a, b, c nonzero, as long as $c' \neq 0$ in the plane $\mathcal{P} : a'x + b'y + c'z = 0$. (What happens if $c' = 0$?) This is the general expression of a quadratic equation in two variables, so we're justified in saying that

the set of conics is the same as the set of curves in $\mathbb{R}^2$ *satisfying a quadratic equation.*

This definition of conics allows a single point and a pair of intersecting lines to be a conic; these are called degenerate (= stupid) conics. If $b = 0$ (no "cross term"), you learned in high school how to complete the square to see what conditions on a, c, e, f, g determine whether (**??**) is an ellipse, parabola or hyperbola. If $b \neq 0$, you may or may not have seen how to rotate the xy-plane so that the equation in (**??**) in the rotated coordinate has no cross term.

Exercise. Let R_θ be the matrix which rotates the plane counterclockwise by the angle θ. Define new coordinates/axes x', y' on the plane by

$$R_\theta \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Given the equation (**??**), find θ in terms of a, b, c, d, e, f such that (**??**) in x', y'-coordinates has no cross term.

Exercise. PODASIP: Every ellipse can be obtained by intersecting the standard cone ($\alpha = 1$) with a plane. Every parabola can be obtained by intersecting the standard cone ($\alpha = 1$) with a plane. Every hyperbola can be obtained by intersecting the standard cone ($\alpha = 1$) with a plane.

Let's say we want to prove a concordance theorem about ellipses. Fix an ellipse $\mathcal{E}$. Find a cone $\mathcal{C}$ and a plane $\mathcal{P}$ such that $\mathcal{E} = \mathcal{C} \cap \mathcal{P}$. Note that $\mathcal{C} \cap \{z = 1\}$ is a circle. In fact, the concordance theorem holds for $\mathcal{E}$ iff it holds for the circle. Note that there is a bijection between points on the circle and points on the ellipse just by seeing that every generating line of the cone contains exactly one point of the circle and one point of the ellipse.

Let's try the same argument for a parabola. Now the bijection breaks down at exactly one point of the circle, which has no corresponding point on the parabola. However, if we extend the α map of Chapter 5 to a map (also called) $\alpha : \mathbb{R}^3 \setminus \{0\} \to \mathbb{RP}^2$ by $\alpha(x, y, z) = [x, y, z]$, then we see that α of the circle in the $z = 1$ plane is a circle on S^2 and hence topologically a circle on $\mathbb{RP}^2$, while α of the parabola is a the same circle on S^2 minus a point. It is pretty natural to add this point in, so we can say that the circle and the parabola are "projectively equivalent." (This is a non-standard use of this term.) (What happens when we do the same argument with a hyperbola?)

In all cases, we see that *to prove a concordance theorem about conics, it's enough to prove the concordance theorem for circles.* Once again, there is a technical issue with the "missing" point on a parabola or hyperbola which we'll gloss over.

From this point of view, Pappus' Theorem is a result about points on a degenerate conic, a pair of lines. It's a great leap, one that took over a millennium, to think that this result should hold on any conic. (Sorry for the double labeling of points.)

Theorem. **[Pascal's Theorem]** *($\approx$ 1600) Let $A = Q_2, B = P_3, C = Q_1, D = P_2, E = Q_3, F = P_1$ be six points on a nondegenerate conic. Set*

$$T = P_1Q_2 \cap P_2Q_1, S = P_3Q_1 \cap P_1Q_3, R = P_2Q_3 \cap P_3Q_2.$$

Then R, S, T are collinear.

As above, we may assume that the conic is a circle.

The proof is taken from Barager, where it is taken from other sources. We begin the proof with a lemma. (Draw a picture.)

Lemma. *Let Γ_1, Γ_2 be circles with intersect at E, F. Take $C, D \in \Gamma_1$ and set $A = CE \cap \Gamma_2, B = DF \cap \Gamma_2$. Then $AB \| CD$.*

Proof. Let O_1 be the center of Γ_1. Set $\angle\alpha = \angle DO_1E$ on the side facing C, and set $\angle\beta = \angle DO_1E$ on the side facing F. Then

$$\angle ECD + \angle EFD = \frac{1}{2}\angle\beta + \frac{1}{2}\angle\alpha = \pi.$$

This implies $\angle ECD = \pi - \angle EFD = \angle EFB$. But by a similar argument, $\angle EFB = \pi - \angle EAB$. Thus $\angle ECD = \pi - \angle EAB$, which implies the result. □

Proof of the Theorem. (Draw a picture to make the proof comprehensible.) Set $G = AB \cap CD, H = DE \cap FA$. Let Γ be the circle on E, F, H. (Recall that such a circle exists and is unique provided E, F, H are not collinear. Can they be?) Set $P = BE \cap \Gamma, Q = CF \cap \Gamma$. The Lemma implies

$$(BG = BA) \| PH, (CD = CG) \| QH, BC \| PQ.$$

Therefore $\Delta BGC \sim \Delta PHQ$ and the corresponding sides of these two triangles are parallel. Put another way, the corresponding sides of these triangles meet on the line at infinity. By the converse to Desargues' Theorem, these triangles are in perspective from a point O. (Draw a picture: O is probably "between" the two triangles.) In other words, HG, BP, QC meet at O. Since $QC = FC, BP = BE$, we get that $\Delta BGC, \Delta EHF$ are also in perspective from O. So by Desargues' Theorem,

$$HF \cap GC = AF \cap DC = T, HE \cap GB = DE \cap AB = R, EF \cap BC = S$$

are collinear. □

Exercise. Can you prove Pascal's Theorem by moving TS to infinity? In other words, assume that $P_1Q_2 \| P_2Q_1, S = P_3Q_1 \| P_1Q_3$, and prove that $P_2Q_3 \| P_3Q_2$. Of course, you can assume that the conic is a circle. Warning: I haven't worked this out.

Cryptic Remarks. We used Desargues' Theorem about triangles to prove a theorem about conics. However, a triangle is a degenerate cubic curve. Is Desargue's Theorem a degenerate case of a theorem about pairs of cubics, or a theorem about degree six curves? If Desargues fails, the underlying coordinate ring (assuming it exists) has no zero divisors, and Pappus exists iff the coordinate ring is commutative. The failure of commutativity is presumably measured by an ℓ_∞ structure. How does that translate into higher degree versions of Pappus? What is the algebraic interpretation of higher degree versions or failures of Deargues? Ref: Artin's *Geometric Algebra* and `http://en.wikipedia.org/wiki/Desargues'_theorem`.

Duality for Conics

Duality takes lines to points and *vice versa.* Take a conic $\mathcal{C}$ in e.g. the $z = 1$ plane. Each tangent line ℓ to $\mathcal{C}$ has a pole P_ℓ. We claim that the set of all such P_ℓ themselves lie on a conic $\mathcal{C}'$, the dual conic to $\mathcal{C}$. This is worked out in coordinates in the *Geometry* text.

Example. Let $\mathcal{C}$ be the circle $x^2 + y^2 = \alpha^2$. (This is a conic/circle in the cone with cone angle α.) Take a point $Q = (x_0, y_0, 1)$ on $\mathcal{C}$. Since $(-y_0, x_0, 0)$ is perpendicular to Q (thought of as a vector in $\mathbb{R}^3$ – Why?), the tangent plane to the cone at Q contains the origin, Q and any vector $Q' = Q+\mu(-y_0, x_0, 0)$. Take e.g. $\mu = 1$. After some nasty algebra (or if you know that a normal vector to this plane is given by the cross product $Q\times Q'$), we get that the tangent plane satisfies $x_0x + y_0y - \alpha^2 z = 0$. Thus the polar point for this projective line $\ell_{[x_0,y_0,-\alpha^2]}$ is $[x_0, y_0, -\alpha^2] = [x_0/\alpha^2, y_0/\alpha^2, 1]$, which lies on the conic $x^2 + y^2 = \alpha^{-2}$. This is the dual conic $\mathcal{C}'$ to $\mathcal{C}$. Note that the dual of $\mathcal{C}'$ is $\mathcal{C}$.

Thus duality for conics is a little tricky, as it takes a tangent line to $\mathcal{C}$ to a point of the dual conic $\mathcal{C}'$. It is not clear that the dual of $\mathcal{C}'$ is always $\mathcal{C}$, but this does work out. In particular, duality for conics implies that *a configuration theorem for points on a conic $\mathcal{C}'$ has a dual configuration theorem for tangent lines on the dual conic $\mathcal{C}$.* (Note that his procedure generalizes duality for points and lines, as the tangent line to a line is always just the line itself.)

Example. Let's dualize Pascal's Theorem. In the hypothesis of Pascal's Theorem, the six points $P_i, Q_i, I = 1, 2, 3$ on $\mathcal{C}'$ dualize to six tangent lines ℓ_{P_i}, ℓ_{Q_i} on the dual conic $\mathcal{C}$. The line $P_1Q_2 = P_1 \cup Q_2$ dualizes to $\ell_{P_1} \cap \ell_{Q_2} = P_{P_1Q_2}$, so the intersection point $P_1Q_2 \cap Q_1P_2 = T$ dualizes to $P_{P_1Q_2} \cup P_{P_2Q_1} = \ell_T$. The conclusion that R, S, T are collinear, or $T\cup S = R\cup S$, dualizes to $\ell_T \cap \ell_S = \ell_R \cap \ell_S$, *i.e.*, ℓ_T, ℓ_R, ℓ_S are concurrent. Thus we have proved

Theorem (Branchion's Theorem). *Let $\ell_{P_i}, \ell_{Q_i}, i = 1, 2, 3$ be six distinct tangent lines on a conic $\mathcal{C}$. Set $P_{P_1Q_2} = \ell_{P_1} \cap \ell_{Q_2}$, etc. Then the three lines*

$$P_{P_1Q_2} \cup P_{P_2Q_1} = \ell_T,\ P_{P_1Q_3} \cup P_{P_3Q_1} = \ell_S,\ P_{P_2Q_3} \cup P_{P_3Q_2} = \ell_R$$

are concurrent.

Constructing Conics

We know that two general (or "generic") points determine a unique line, where in this case "generic" just means "distinct." How many generic points determine a unique conic? It's tempting to say three, since three points determine a unique parabola, but this assumes the parabola is upwards/downwards pointing, which is an added condition. In fact, since a hyperbola and ellipse can intersect in four points, we certainly need at least five generic points to determine a conic.

We can solve this problem algebraically. As a warm-up, take the equation of a line $ax + by + c = 0$. This has three unknowns a, b, c. Take two distinct points $(x_1, y_1), (x_2, y_2)$ and plug them into this equation. Of course, we don't get a unique solution for a, b, c, as we would need three points for a unique solution. We are *not* saying that three points determine a line! If you check the algebra, after you plug in the two points, you get that (a, b, c) are unique up to multiplication by a constant: if $(a, b, c), (a', b', c')$ are solutions of

$$ax_1 + by_1 + c = 0,\ ax_2 + by_2 + c = 0,$$

then there exists $\lambda \in \mathbb{R}$ with $a' = \lambda a, b' = \lambda b, c' = \lambda c$ (except in the stupid case $a = b = c = 0$). Now the algebra matches up with our geometric insight: the set of points in the plane lying on the line $ax + by + c = 0$ is the same set as the line $\lambda ax + \lambda by + \lambda c = 0$, as long as $\lambda \neq 0$. So two points determine a unique line, but the equation of this line is unique only up to multiplication by a nonzero real number.

As a check. If you try to solve the system $ax_i + by_i + c = 0, i = 1, 2, 3$ for three *non collinear* points (x_i, y_i), you will see that the only solution is $a = b = c = 0$, and of course $0x + 0y + 0 = 0$ is an equation satisfied by all points in $\mathbb{R}^2$, not just a line of points.

Similarly, take a general conic $ax^2 + bxy + cy^2 + ex + fy + g = 0$ with $(a, b, c,) \neq (0, 0, 0)$ and take five points $(x_i, y_i), i = 1, \ldots, 5$. Since there are six unknown constants a, b, c, e, f, g, we expect to be able to solve for the unknowns up to a scalar multiple, which is fine. Of course, we need some genericity condition on the five points.

Exercise. PODASIP. We can solve for the six constants up to a scalar multiple iff no three of the points (x_i, y_i) are collinear.

In summary, five (generic) points determine a unique conic.

Warning: For cubic and higher degree curves, determining the number of points is trickier. This is called Cramer's paradox; see `http://www.mathpages.com/home/kmath207/kmath207.htm`.

Now we give an incredible straightedge (alone) construction of arbitrarily many points on a conic. Take five generic points A, B, C, E, F. We want to construct more points D on the unique conic determined by the five points.

We'll need the converse of Pascal's Theorem. Take six points A, B, C, D, E, F such that the appropriate diagonals ("$P_1Q_2 \cap P_2Q_1$", etc.) intersect at collinear

points R, S, T. (Draw a picture.) Take the unique conic $\mathcal{C}$ on A, B, C, D, E, assumed to be generic. Set $F' = CR \cap \mathcal{C}$. By Pascal's Theorem applied to A, B, C, D, E, F', $S = AE \cap RT \in BF'$. Thus F, F' both lie on BS and CR, so $F = F'$. Thus A, B, C, D, E, F lie on the conic $\mathcal{C}$.
(Ref: `http://www.maths.gla.ac.uk/wws/cabripages/conpascal.html`)

Back to the five points A, B, C, D, F lying on a unique conic $\mathcal{C}$. For any line ℓ on B, we expect to find a unique point D on $\ell \cap \mathcal{C}$. (Can you justify this algebraically?) Take $S = BF \cap AE$ as usual, and set $T = \ell \cap CE, R = CF \cap ST$. Finally, set $D = \ell \cap AR$. (Draw a picture.) Then by the converse of Pascal's Theorem, D must lie on $\mathcal{C}$.