Hilbert's Axioms for Euclidean Plane Geometry

Undefined Terms  Point, line, on, between, and congruence

GROUP I: Axioms of Connection

I-1. Through any two distinct points \( A, B \), there is always a line \( m \).
I-2. Through any two distinct points \( A, B \), there is not more than one line \( m \).
I-3. On every line there exist at least two distinct points. There exist at least three points which are not on the same line.

GROUP II: Axioms of Order

II-1. If point \( B \) is between points \( A \) and \( C \), then \( A, B, C \) are distinct points on the same line, and \( B \) is between \( C \) and \( A \).
II-2. For any two distinct points \( A \) and \( C \), there is at least one point \( B \) on the line \( AC \) such that \( C \) is between \( A \) and \( B \).
II-3. If \( A, B, C \) are three distinct points on the same line, then only one of the points is between the other two.

Definition  By the segment \( AB \) is meant the set of all points which are between \( A \) and \( B \). Points \( A \) and \( B \) are called the endpoints of the segment. The segment \( AB \) is the same as segment \( BA \).

II-4. (Pasch's Axiom.) Let \( A, B, C \) be three points not on the same line and let \( m \) be a line which does not pass through any of the points \( A, B, C \). Then if \( m \) passes through a point of the segment \( AB \), it will also pass through a point of segment \( AC \) or a point of segment \( BC \).

Note: This postulate may be replaced by the separation axiom.
II-4'. A line $m$ separates the points which are not on $m$ into two sets such that if two points $X$ and $Y$ are in the same set, the segment $XY$ does not intersect $m$, and if $X$ and $Y$ are in different sets, the segment $XY$ does intersect $m$. In the first case $X$ and $Y$ are said to be on the same side of $m$; in the second case, $X$ and $Y$ are said to be on opposite sides of $m$.

**Definition**  By the ray $\overrightarrow{AB}$ is meant the set of points consisting of those which are between $A$ and $B$, the point $B$ itself, and all points $C$ such that $B$ is between $A$ and $C$. The ray $\overrightarrow{AB}$ is said to emanate from point $A$. A point $A$, on a given line $m$, divides $m$ into two rays such that two points are on the same ray if and only if $A$ is not between them.

**Definition**  If $A$, $B$, and $C$ are three points not on the same line, then the system of three segments $AB$, $BC$, $CA$, and their endpoints is called the triangle $\triangle ABC$. The three segments are called the sides of the triangle, and the three points are called the vertices.

**GROUP III: AXIOMS OF CONGRUENCE**

**III-1.** If $A$ and $B$ are distinct points on line $m$, and if $A'$ is a point on line $m'$ (not necessarily distinct from $m$), then there is one and only one point $B'$ on each ray of $m'$ emanating from $A'$ such that the segment $A'B'$ is congruent to the segment $AB$, written $AB \cong A'B'$.

**III-2.** If two segments are each congruent to a third, then they are congruent to each other. (From this it can be shown that congruence of segments is an equivalence relation; i.e., $AB \cong AB$; if $AB \cong A'B'$, then $A'B' \cong AB$; and if $AB \cong CD$ and $CD \cong EF$, then $AB \cong EF$.)

**III-3.** If point $C$ is between $A$ and $B$, and $C'$ is between $A'$ and $B'$, and if $AC \cong A'C'$ and $CB \cong C'B'$, then $AB \cong A'B'$.

**Definition**  By an angle is meant a point (called the vertex of the angle) and two rays (called the sides of the angle) emanating from the point.

If the vertex of the angle is point $A$ and if $B$ and $C$ are any two points other than $A$ on the two sides of the angle, we speak of the angle $\angle BAC$ or $\angle CAB$ or simply of angle $\angle A$.

**III-4.** If $\angle BAC$ is an angle whose sides do not lie on the same line and if $\overrightarrow{A'B'}$ is a ray emanating from $A'$, then there is one and only one ray $\overrightarrow{A'C'}$ on a given side of line $A'B'$, such that $\angle B'A'C' \cong \angle BAC$. In short, a given angle can be laid off on a given side of a given ray in one and only one way. Every angle is congruent to itself.

**Definition**  If $\triangle ABC$ is a triangle then the three angles $\angle BAC$, $\angle CBA$, and $\angle ACB$ are called the angles of the triangle. Angle $\angle BAC$ is said to be included by the sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$ of the triangle.
III-5. [SAS] If two sides and the included angle of one triangle are congruent, respectively, to two sides and the included angle of another triangle, then each of the remaining angles of the first triangle is congruent to the corresponding angle of the second triangle.

GROUP IV: AXIOM OF PARALLELS

IV-1. (Playfair's axiom.) Through a given point $A$ not on a given line $m$ there passes at most one line which does not intersect $m$.

GROUP V: AXIOMS OF CONTINUITY

V-1. (Archimedian axiom.) If $AB$ and $CD$ are arbitrary segments, then there exists a number $n$ such that if the segment $CD$ is laid off $n$ times on the ray $AB$ starting from $A$, then a point $E$ is reached, where $n \cdot CD = AE$, and where $B$ is between $A$ and $E$.

V-2. (Axiom of linear completeness.) The system of points on a line with its order and congruence relations cannot be extended in such a way that the relations existing among its elements as well as the basic properties of linear order and congruence resulting from axiom groups I, II and III and axiom V-1 remain valid.

Note: These axioms may be replaced by Dedekind's axiom of continuity. For every partition of the points on a line into two nonempty sets such that no point of either lies between two points of the other, there is a point of one set which lies between every other point of that set and every point of the other set.