Handout on Common Groups

Here is a list of groups we will use this summer. Usually $n = 2$ or 3 for us.

- $\mathbf{Gl}(n, \mathbb{R})$ is the group of $n \times n$ invertible matrices with real coefficients. This is equivalent to:

  1. The set of $n \times n$ matrices with real coefficients and $\det \neq 0$.
  2. The set of $n \times n$ matrices with real coefficients such that no column vector lies in the set of vectors given by sums of the other $n-1$ column vectors.
  3. The set of bijective linear transformations $A : \mathbb{R}^n \to \mathbb{R}^n$.

  $\mathbf{Gl}(n, \mathbb{R})^+$ is the subgroup of $\mathbf{Gl}(n, \mathbb{R})$ consisting of orientation preseriving invertible matrices.

- $\mathbf{Gl}(n, \mathbb{C})$, $\mathbf{Gl}(n, \mathbb{Q})$, $\mathbf{Gl}(n, \mathbb{Z}_p)$ are the groups of $n \times n$ invertible matrices with coefficients in $\mathbb{C}$, $\mathbb{Q}$, and $\mathbb{Z}_p$, respectively. They have the same three equivalent characterizations as above.

- $\mathbf{Gl}(n, \mathbb{Z})$ is the group of $n \times n$ invertible matrices with integer coefficients such that the inverse also has integer coefficients. (This condition on the inverse is automatic for the previous four groups, since the coefficients lie in a field.) This is equivalent to

  1. The set of $n \times n$ matrices with $\mathbb{Z}$ coefficients and determinant ± 1.
  2. The set of bijective linear transformations $A : \mathbb{Z}^n \to \mathbb{Z}^n$.

- $\mathbf{O}(n)$ is the set of $n \times n$ matrices with real coefficients and with all columns of length one and mutually perpendicular (as vectors in $\mathbb{R}^n$). By characterization 2 of $\mathbf{Gl}(n, \mathbb{R})$, we see that $\mathbf{O}(n) \subset \mathbf{Gl}(n, \mathbb{R})$. Elements of $\mathbf{O}(n)$ have determinant ± 1.

- $\mathbf{SO}(n)$ are those elements of $\mathbf{O}(n)$ which have determinant one. Thus by definition $\mathbf{SO}(n) = \mathbf{O}(n) \cap \mathbf{Sl}(n, \mathbb{R})$. Equivalently, $\mathbf{SO}(n)$ are the elements of $\mathbf{O}(n)$ which preserve orientation.

- $\mathbf{Sl}(n, \mathbb{R})$ is the set of $n \times n$ matrices with real coefficients and determinant one. Equivalently, this is the set of $n \times n$ matrices with real coefficients and which preserve volumes and orientations of (reasonable) figures. By characterization 2 of $\mathbf{Gl}(n, \mathbb{R})$, we see that $\mathbf{Sl}(n, \mathbb{R}) \subset \mathbf{Gl}(n, \mathbb{R})$.

- $\mathrm{Isom}(\mathbb{R}^n)$ is the set of isometries (= distance preserving maps) of $\mathbb{R}^n$. Every $A \in \mathrm{Isom}(\mathbb{R}^2)$ has a unique decomposition $A = TRF$, where T is a translation, R is a rotation around the origin, and $F = \mathrm{Id}$ (resp. the reflection across the x-axis) if A is orientation preserving (resp. orientation reversing).

- $\mathrm{Isom}_0(\mathbb{R}^n)$ is the set of isometries of $\mathbb{R}^n$ which fix the origin. It is a *theorem* that $\mathrm{Isom}_0(\mathbb{R}^n) = \mathbf{O}(n)$.

- $\mathbf{PGl}(3, \mathbb{R})$ is the set of elements of $\mathbf{Gl}(3, \mathbb{R})$ where elements A and λA are considered to be "the same" for all $\lambda \in \mathbb{R} \setminus \{0\}$. More precisely, $\mathbf{PGl}(3, \mathbb{R})$ is the quotient group of $\mathbf{Gl}(3, \mathbb{R})$ by the normal subgroup $\{\lambda \mathrm{Id} : \lambda \in \mathbb{R} \setminus \{0\}\}$. Equivalently, $\mathbf{PGl}(3, \mathbb{R})$ is the set of elements of $\mathbf{Sl}(3, \mathbb{R})$ where elements A and $-A$ are considered to be "the same." It is a *theorem* that the isometry group of the projective plane plane is $\mathbf{PGl}(3, \mathbb{R})$.

- $\mathbf{PSl}(2,\mathbb{R})$ is the set of elements of $\mathbf{Sl}(2,\mathbb{R})$ where elements A and $-A$ are considered to be "the same." More precisely, $\mathbf{PSl}(2,\mathbb{R})$ is the quotient group of $\mathbf{Sl}(2,\mathbb{R})$ by the normal subgroup $\{\pm\mathrm{Id}\}$. It is a *theorem* that the orientation preserving isometry group of the hyperbolic plane is $\mathbf{PSl}(2,\mathbb{R})$, or equivalently $\mathbf{PGl}(2,\mathbb{R})^+$, or equivalently the set of *linear fractional transformations* $\{A : \mathcal{H} \to \mathcal{H} : A(z) = \frac{az+b}{cz+d}; a,b,c,d, \in \mathbb{R}, ad-bc > 0\}$. All isometries f of the hyperbolic plane are given by $f = \mathcal{R} \circ A$, where $A \in \mathbf{PSL}(2,\mathbb{R})$ and $\mathcal{R} = Id$ if f is orientation preserving, and $\mathcal{R}$ is the reflection on the y-axis if f is orientation reversing,

- The *unit quaternions* are $\mathcal{Q} = \{(x,y,z,w) = x+yI+zJ+wK \in \mathbb{R}^4 : |x|^2+|y|^2+|z|^2+|w|^2 = 1\}$, with multiplication determined by

$$I^2 = J^2 = K^2 = 1, IJ = K, JK = I, KI = J, JI = -K, KJ = -I, IK = -J$$

  and the distributive law. $\mathcal{Q}$ has the subgroup $\{(x+yI : |x|^2+|y|^2 = 1\}$, the unit complex numbers.

- The *isometry group* or *symmetry group* $\mathrm{Isom}(F)$ of a figure $F \subset \mathbb{R}^n$ is $\{A \in \mathrm{Isom}(\mathbb{R}^n) : A(F) = F\}$. *Warning:* We usually insist that A must be in $\mathrm{Isom}_0(\mathbb{R}^n) = \mathbf{O}(n)$, but not for e.g. $F = \mathbb{R}^n$. The *direct symmetry group* $\mathrm{Isom}_d(F)$ of F is those symmetries of determinant one (equiv. orientation preserving symmetries). Thus $\mathrm{Isom}_d(F) = \mathrm{Isom}(F) \cap \mathbf{SO}(n)$.

- The *permutation group* Σ_n is the set of bijections of the set $\{1,2,\ldots,n\}$. The alternating group A_n is the subgroup of Σ_n consisting of all elements which can be written as a product of an even number of transpositions.

- The symmetry groups of the Platonic solids are:
  - The tetrahedron has direct symmetry group A_4 and full symmetry group Σ_4.
  - The cube and octahedron have direct symmetry group Σ_4 and full symmetry group $\Sigma_4 \times \mathbb{Z}_2$
  - The icosahedron and dodecahedron have direct symmetry group A_4 and full symmetry group $A_4 \times \mathbb{Z}_2$.

- A group $G = \langle g_1, \ldots, g_n\rangle / \langle R_1, \ldots R_k\rangle$ given by generators $\{g_i\}$ and relations R_j (a fixed string in the g_i and the symbols g_i^{-1}) consists of all strings of $g_i^{\pm 1}$ (and the empty word) with all appearances of $g_i^{\pm 1} g_i^{\mp 1}$ and R_k forbidden. Multiplication is by concatenation of strings, with any newly created forbidden string deleted. The *free group* on n generators is the group G above with no relations. $F_1 \simeq \mathbb{Z}$.