Chapter 1

Lecture 1: What is geometry?

Warm-up problem: want to form a series of 3-person teams from 7 children such that:

i) every two children are on a unique team

ii) any two (distinct) teams share exactly one child

In what sense is this a geometry problem?
One solution:

So teams are, for example, $\{1, 4, 2\}$, $\{4, 7, 3\}$, $\{4, 5, 6\}$ are teams, but $\{3, 4, 6\}$ is not.

Dictionary:

child $\leftrightarrow$ "pt" $= \{1, 2, \ldots, 7\}$

team $\leftrightarrow$ "lines" $= \{\{1, 4, 2\}, \{4, 7, 3\}, \{4, 5, 6\}\}$

There are seven points and seven lines.
Rule (i) is like Euclidean geometry, meaning two points determine a line.
Rule (ii) is not, since not all Euclidean lines intersect in one point (parallel lines).
How unique is the solution? Any relabeling of $\{1, 2, \ldots 7\}$gives another solution, so we get 7! solutions.
There is an important subset of relabelings that produce no new teams. Examples are relabeling given by flipping on one of the three axes, or rotating by 0, 120, 240 degrees.
Let's define "relabeling" in a precise way:

Definition. $f : X \to Y$ is **one-to-one** (**injective**) if $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$.

$f : X \to Y$ is **onto** (**surjective**) if $\forall y \in Y, \exists x \in X$ such that $f(x) = y$.

$f : X \to Y$ is **bijective** if f is both injective and surjective.

So a relabeling is a bijective function.

Question. Is every solution given by a relabeling of the solution we gave?

In terms of our dictionary, a *symmetry* of our "geometry" is a bijection taking lines to lines. We found six symmetries above, as the relabelings which produce no new teams.

Question. Did we find all the symmetries? For every pair of lines ℓ_1, ℓ_2 in our geometry, is there a symmetry taking ℓ_1 to ℓ_2? How many relabelings produce at least one new team? Can a relabeling produce exactly one new team? Two new teams? etc.

1.1 Synthetic vs. Analytic Geometry

Roughly speaking, synthetic geometry means reasoning from axioms, and analytic reasoning means using coordinates.

1.1.1 Synthetic Geometry; Euclid's Axioms

An *axiomatic system* consists of

1. A set of undefined terms;
2. A list of axioms establishing relationships between the undefined terms.

We use *accepted logical rules* to derive consequences from the axioms, which are called theorems, lemmas, corollaries, propositions. We can make new defintions in terms of the undefined terms or of previous defintions.

Axioms must be *consistent* (not leading to contradictions), so e.g. we shouldn't have axioms saying that (1) there are blorgs, (2) All blorgs are blue, (3) No blorgs are blue. Axioms shouldn't be redundant.

Euclid's Axioms:

1. Every pair of (distinct) points determine a (unique) line segment.
2. Every line segment is infinitely extendable.
3. All right angles are the same.
4. All circles of a given center and radius exist.
5. "Through a given point P not on a line ℓ, there exists a unique parallel line to ℓ passing through P."

Question. What are the undefined terms? Are lines made of points? Where is e.g. SAS? Big question: is the Parallel Postulate redundant? Strange question: Are Euclid's axioms consistent?

An example of a "theorem:" Two distinct lines cannot intersect in more than one point. An example of further defintions: "An acute angle is an angle that fits inside a right angle." (Uh-oh.) "An obtuse angle is an angle that is neither a right angle nor an acute angle."

Contrast the vagueness of Euclid's Axioms with the definition of a group.

First, some notation:

Definition. (i) if X, Y are sets, then $X \times Y = \{(x, y) | x \in X, y \in Y\}$ is the *product* of X and Y. (So the product is just the set of ordered pairs, with the first element from X and the second from Y.)

(ii) A *binary operation* on a set X is a function $f : X \times X \to X$. (So f takes an ordered pair of elements from X and gives you a new element.) Rather than write $f(x_1, x_2)$, whenever possible we'll write e.g. $x_1 \otimes x_2$.

Definition. A *group* is a set G with a binary operation $\otimes$ such that

- Closure: For all $gh \in G, g \otimes h \in G$.
- Identity: $\exists e \in G$ such that $g \otimes e = e \otimes g = g, \forall g \in G$.
- Inverses: $\forall g \in g, \exists h \in G$ such that $g \otimes h = h \otimes g = e$.
- Associativity: $\forall g, h, k \in G, (g \otimes h) \otimes k = g \otimes (h \otimes k)$.

(Closure is redundant.) Notice that we don't assume Commutativity: $\forall g, h \in G, g \otimes h = h \otimes g$. We denote a group by $(G, \otimes)$ to emphasize that we need to know the set and the binary operation.

Question. What are the undefined terms? Do you agree that $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +)$ are groups? If we prove a theorem about groups, is this theorem true in every example of a group? If we prove a theorem about $(\mathbb{Z}, +)$, is it true for all groups? Is there more than one example of "points, lines, angles, circles" satisfying Euclid's Axioms? Why didn't Euclid ask this question?

Example of a theorem in group theory: Every group has a unique identity element.

1.1.2 Analytic Geometry; Cartesian Coordinates

Here the axiomatic system is the real numbers $\mathbb{R}$ and it's arithmetic $(+, -, \times, \div)$. We take the plane to be $\mathbb{R}^2 \stackrel{\text{def}}{=} \mathbb{R} \times \mathbb{R}$ (so are we assuming all of the axioms of set theory?), and we now define all the previously undefined terms in Euclid's axioms:

Definition. (i) Given $a, b, c \in \mathbb{R}$, the *line* determined by a, b, c is $\{(x, y) \in \mathbb{R}^2 : ax + by = c\}$. (ii) A *circle* of radius $r > 0$ and center $(x_0, y_0) \in \mathbb{R}^2$ is $\{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 = r^2\}$.

Question. How do we define angles? How do we define "parallel"?

This seems much more precise than the synthetic approach. Problems: (1) What are the axioms of set theory? (2) What are the axioms for $\mathbb{R}$?

Without answering these difficult questions, let's appreciate how difficult $\mathbb{R}$ is by playing the UN game.

1.1.3 The UN Game

The UN (i.e. its General Assembly) is made up of one person from each nation. To make life easy, assume that (1) no one is a citizen of more than one nation (no dual citizenships), (2) everyone is a citizen of one nation (no stateless people). Certainly if there are a finite number of people on the planet, then there are a finite number of ways to make up a UN.

Now we form "UN"s from various sets of numbers.

Example.

1. Declare two elements x, y of $\mathbb{Z}$ to be citizens of the same country iff $11|x - y$. Construct a UN. Write down what a general UN has to look like.

2. Do the same for $\mathbb{Z}$ is x, y are citizens of the same country iff x and y have the same sign. (0 is assigned its own sign of 0.)

3. Do the same for $\mathbb{Z}$ if x, y are citizens of the same country iff $x + y > 5$.

4. Do the same for $\mathbb{Z}$ if x, y are citizens of the same country iff $x \geq y$.

OK, we better impose some constraints on the UN game on a set S. Denote "x is in the same country as y" by $x \sim y$. Our rules must satisfy (1) (relexivity) $x \sim x$, (2) (symmetry) $x \sim y$ iff $y \sim x$, (3) (transitivity) $x \sim y$ and $y \sim z$ implies $x \sim z$.

Back to the game:

5. For $\mathbb{Z}$, the rule is $x \sim y$ iff $x = y$.

6. For $\mathbb{Z}$, the rule is $x \sim y$ for all x, y.

These games are stupid. Here are some better ones.

7. On $\mathbb{Q}$, the rule is $x \sim y$ iff $x - y \in \mathbb{Z}$.

8. On $\mathbb{C}$, the rule is $x \sim y$ iff $x - y \in \mathbb{R}$.

9. On $\mathbb{R}$, the rule is $x \sim y$ iff $x - y \in \mathbb{Q}$.

Example 9 is so hard that reasonable mathematicians can argue about whether a UN exists! The basic issue is: if there is no way of listing the countries, or even describing what a list of countries would look like, how can we pick a citizen from each country?

If you believe that no matter what UN game you play, (1) there is a definite collection of all the countries, and (2) we can select one citizen from each country, then you are a believer in the Axiom of Choice.

The Axiom of Choice: Given any collection of sets F_α, indexed by α in some set $\mathcal{A}$, there exists a function $f : \mathcal{A} \to \cup_{\alpha \in \mathcal{A}} F_\alpha$ such that $f(\alpha) \in F_\alpha$.

If $\mathcal{A}$ is the set $\{1.2., ..., N\}$, where N is the number of countries in the world (e.g. 1 corresponds to Uzbekistan, 2 to Paraguay, etc.), and F_α is the set of citizens of the country indexed by α, then $f(1)$ must be a citizen of Uzbekistan, $f(2)$ must be a citizen of Paraguay, etc. Therefore f selects a UN.

For this course, we'll assume the Axiom of Choice. (It is hard, but not impossible, to find mathematicians who reject AC.)

Summary: (1) The real numbers are incredibly complicated. "Assuming we understand $\mathbb{R}$" is a risky statement. (2) Both the synthetic and analytic approaches to geometry have deep difficulties.