Chapter 1

Lecture 3: $\mathbb{R}^2$ and $\mathbb{RP}^2$

Remark. More examples of groups:

(i) The group of symmetries of the seven point geometry, i.e. the bijections on the set of points which takes lines to lines. This is clearly a group. What is its order?

(ii) The symmetries of a square, defined as in (i). This consists of four rotations (including rotation by 0 degrees, the identity) and four flips. This *dihedral group* D_4 is nonabelian. What is the symmetry group D_3 of the triangle, D_5 of the pentagon?

(iii) The symmetries of the letter Z are those isometries of $\mathbb{R}^2$ that take Z to itself. Show that this symmetry group is isomorphic to $\mathbb{Z}_2$.

1.0.1 Back to the TRF decomposition

Remark. We can extract some information from the proof of the TRF theorem.

(0) First of all, we can finish the proof that $\mathcal{I}$ is a group. We only need to show that isometries are surjective. But every isometry has a TRF decomposition. Translations, rotations and flips are clearly surjective. So their composition is also surjective.

(i) Let $f = R_\theta \mathscr{F}$ have no translational part, so $f \in O(2)$. We know $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since the two columns are $g\vec{i}, g\vec{j}$ which must have length one, we get $a^2 + c^2 = b^2 + d^2 = 1$. Since the two columns must be perpendicular, we get $ab + cd = (a, c) \cdot (b, d) = 0$. Thus

$$\begin{aligned} 0 &= (ab + cd)^2 = a^2b^2 + c^2d^2 + 2abcd \\ 1 &= (a^2 + c^2)(b^2 + d^2) = a^2b^2 + c^2d^2 + b^2c^+a^2d^2 \end{aligned}$$

Subtracting, we get

$$1 = b^2c^2 + a^2d^2 - 2abcd = (ad - bc)^2.$$

Therefore elements of $O(2)$ have $\det f \stackrel{\text{def}}{=} ad - bc = \pm 1$.

(ii) If $f = R_\theta$, then we know that f is a standard rotation matrix, which by direct computation has $\det f = 1$. If $f = R_\theta F_x$, we again know its matrix, and can check that $\det f = -1$. Therefore the sign of the determinant is ± 1 iff f is orientation preserving/reversing. (Actually, since we know the rotation matrices, we could easily skip the calculation in (i).)

(iii) In the homework, we work out that $|\det f|$ is the area scaling factor for any linear transformation f of $\mathbb{R}^2$. Thus the determinant of a linear transformation precisely encodes the scaling factor and the orientation.

We don't really understand a group until we understand its multiplication rules. We have to take two elements of $\mathcal{I}$ with their TRF decomposition and get the TRF decomposition of their product. The difficulty is that the group is nonabelian.

Say $f_1 = T_x R_{\theta_1} F_1$, $f_2 = T_y R_{\theta_2} F_2$. We know $f_3 = f_1 \circ f_2 = T_x R_{\theta_1} F_1 T_y R_{\theta_2} F_2$, but we want to write $f_3 = f_1 \circ f_2 = T_z R_{\theta_3} F_3$ and find z, θ_3, F_3.

1. Finding F_3 is easy. F_3 is orientation preserving $\Leftrightarrow$ F_1 and F_2 are both orientation-preserving or both orientation-reversing $\Leftrightarrow$ $F_3 = F_1 \circ F_2$.

2. Finding z isn't bad.

$$z = f_3(\vec{O}) = (f_1 \circ f_2)(\vec{O}) = f_1(f_2(\vec{O})) = f_1(y).$$

3. Finding R_{θ_3} involves some work. We know we need $R_{\theta_3}\vec{\imath} = (T_{-z} \circ f_3)(\vec{\imath})$, but $(T_{-z} \circ f_3)(\vec{\imath}) = R_{\theta_3} F_3$, so

$$R_{\theta_3}(\vec{\imath}) = (T_{-z} \circ f_3)(\vec{\imath}) = T_{-f_1(y)} f_3 \vec{\imath} = T_{-f_1(y)} T_x R_{\theta_1} F_1 T_y R_{\theta_2} F_2(\vec{\imath})$$

Now $T_{-f_1(y)} T_x = T_{-R_{\theta_1} F_1(y)}$, since

$$\begin{aligned} T_{-f_1(y)} T_x &= T_{-f_1(y)+x} = T_{-(T_x R_{\theta_1} F_1(y))+x} = T_{-(x+R_{\theta_1} F_1(y))+x} \\ &= T_{-(R_{\theta_1} F_1(y))} \end{aligned}$$

Let $R_{\theta_2} F_2(\vec{\imath}) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Then

$$\begin{aligned} R_{\theta_3}\vec{\imath} &= T_{-f_1(y)} T_x R_{\theta_1} F_1 T_y R_{\theta_2} F_2(\vec{\imath}) \\ &= T_{-R_{\theta_1} F_1(y)} R_{\theta_1} F_1 T_y \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= T_{-R_{\theta_1} F_1(y)} R_{\theta_1} F_1 \begin{pmatrix} \alpha + y_1 \\ \beta + y_2 \end{pmatrix} \\ &= T_{-R_{\theta_1} F_1(y)} R_{\theta_1} \begin{pmatrix} \alpha + y_1 \\ \pm(\beta + y_2) \end{pmatrix} \\ &= R_{\theta_1} \begin{pmatrix} \alpha + y_1 \\ \pm(\beta + y_2) \end{pmatrix} - R_{\theta_1} \begin{pmatrix} y_1 \\ \pm y_2 \end{pmatrix} \end{aligned}$$

Since R_θ is a linear transformation,

$$
\begin{aligned}
R_{\theta_3}\vec{i} &= R_{\theta_1}\left(\begin{pmatrix}\alpha + y_1\\ \pm(\beta + y_2)\end{pmatrix} - \begin{pmatrix} y_1 \\ \pm y_2\end{pmatrix}\right) = R_{\theta_1}\begin{pmatrix}\alpha\\ \pm\beta\end{pmatrix}\\
&= R_{\theta_1}F_1R_{\theta_2}F_2(\vec{\imath}) = R_{\theta_1}F_1R_{\theta_2}(\vec{\imath})\\
&= \begin{cases} R_{\theta_1+\theta_2}(\vec{\imath}) & F_1 = \mathrm{Id}\\ R_{\theta_1}R_{-\theta_2}\mathscr{F}_x(\vec{\imath}) & F_1 = F_x\end{cases}
\end{aligned}
$$

Conclusion: For $f_3 = f_1 \circ f_2$ and notation as above,

$$F_3 = F_1 \circ F_2$$

$$T_z = T_{f_1(y)}$$

$$R_{\theta_3} = \begin{cases} R_{\theta_1+\theta_2} & F_1 = \mathrm{Id}\\ R_{\theta_1-\theta_2} & F_1 = F_x\end{cases}$$

Example. Let f be a glide reflection with a flip on the y-axis, then translate by $(0,4)$. Find TRF decomposition of f and f^{-1}

Solution: It is clear that f has $\mathscr{F} = F_x$. $f(0) = (0,4)$, to the translation part is $T_{(0,4)}$. $T_{-(0,4)}f\vec{i} = -\vec{i}$, so the rotational part is R_π. Therefore $f = T_{(0,4)}R{-}\pi F_x$.

For f^{-1}: justify that "$(ab)^{-1} = b^{-1}a^{-1}$." (You put on your socks before your shoes, but only with great difficulty can you remove your socks before your shoes.)

$$f^{-1} = F_x^{-1}R_\pi^{-1}T_{(0,4)}^{-1} = F_xR_\pi T_{(0,-4)}$$

Set $R_\pi = f_1, T_{(0,-4)} = f_2$. Then

$$R_\pi T_{(0,-4)} = T_{f_1(0,-4)}R_{\pi+0}\circ \mathrm{Id} = T_{(0,4)}R_\pi.$$

(Note the noncommutativity.) So far, $f^{-1} = F_xT_{(0,4)}R_\pi$. Now setting $f_1 = F_x, f_2 = T_{(0,4)}R_\pi$, we get

$$f^{-1} = T_{f_1(0,4)}R_{0-\pi}F_x = T_{(0,-4)}R_\pi F_x$$

A good trick to make life easier: (i) the rotation by angle θ around a point v is given by $T_vR_\theta T_{-v}$. (Why?) (ii) if v is a vector from the origin to a line ℓ, then reflection across ℓ is given by $T_vR_\theta F_xR_{-\theta}T_{-v}$, where θ is the angle that ℓ makes with the x-axis.

$\mathscr{I} = \mathrm{Isoms}(\mathbb{R}^2)$ is a group that has interesting subgroups G, i.e., $G \subset \mathscr{I}$ with closure: $g_1, g_2 \in G \Rightarrow g_1 \circ g_2 \in G$ and inverses: $g \in G \Rightarrow g^{-1} \in G$. (The associative property is automatic, since $\mathcal{I}$ is associative, and closure implies $gg^{-1} = I \in \mathcal{I}$.)

Example. $(\mathbb{Z},+)$ is a group, $2\mathbb{Z} = \{2n : n \in \mathbb{Z}\}$ (the evens) is a subgroup. Also $\mathbb{Z}$, $\{0\}$, $3\mathbb{Z}$, $4\mathbb{Z}$, etc. Is that all?

Consider {all isoms fixing $\vec{O}$} $= O(2)$ (O stands for "orthogonal.") We have

$$SO(2) \subset O(2) \subset \mathscr{I}.$$

For $f \in SO(2)$, easily $f = T_{\vec{v}} R_\theta F$ has $\vec{v} = \vec{O}$ and $F = \text{Id}$. Thus $f = R_\theta$, so $SO(2) = \{R_\theta : \theta \in [0, 2\pi)\}$. Similarly, $O(2) = \{R_\theta \mathscr{F}\}$. Note that the columns of a matrix in $O(2)$ are indeed orthogonal.

Note:

1. $F_x \in O(2) - SO(2)$
2. $SO(2)$ is abelian, as $R_\theta R_\psi = R_{\theta+\psi} = R_{\psi+\theta} = R_\psi R_\theta$.
3. Is $O(2)$ abelian? No, since $R_\theta \circ \mathscr{F}_x = \mathscr{F}_x \circ R_{-\theta}$
4. $SO(2)$ looks like S^1, the unit circle via the map $R_\theta \mapsto (\cos\theta, \sin\theta)$

1.1 Remarks on Topology and Algebra

What do we mean when we say $SO(2) = \{R_\theta : \theta \in [0, 2\pi]\}$ looks like S^1?

First, d is a **metric/distance function** on a set X if $d : X \times X \to \mathbb{R}$ has

(i) $d(x,y) \geq 0$, $d(x,y) = 0 \Leftrightarrow x = y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,y) + d(y,z) \geq d(x,z)$ (Triangle inequality)

Example. (a) The usual distance on $\mathbb{R}^2$

(b) On any X, set

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

(c) The taxicab metric on $\mathbb{R}^2$: For $X = (x_1, x_2)$ and $Y = (y_1, y_2)$, set$d(X,Y) = |y_1 - x_1| + |y_2 - x_2|$.

If X, Y have metrics, $f : X \to Y$ is *continuous* if $d_X(x_i, x) \to 0$ as $i \to \infty$ implies $d_Y(f(x_i), f(x)) \to 0$ as $i \to \infty$. Here $d_X(x_i, x) \to 0$ in $\mathbb{R}$ has the usual $\delta - \epsilon$ definition, but for us it's good enough to say "the distance between x_i and x gets smaller and smaller as i gets larger and larger." We can abbreviate $d_X(x_i, x) \to 0$ by $x_i \to x$.

We will say that a function $f : X \to Y$ between metric spaces is *continuous* if for all $x \in X$,

$$x_i \to x \Rightarrow f(x_i) \to f(x).$$

X, Y are two sets with distance functions, then X 'looks like" Y will mean that X is **homeomorphic** to Y: i.e. there exists a bijection $f : X \longrightarrow Y$ with f continuous and f^{-1} continuous.

Example. Now we can show that $SO(2)$ is homeomorphic to S^1, i.e. has the same shape as S^1.

$$f : SO(2) \to S^1,\ R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mapsto (\cos\theta, \sin\theta).$$

Here we set $d_{SO(2)}(R_{\theta_1}, R_{\theta_2}) = |\theta_1 - \theta_2| \pmod{2\pi}$. OK, you do the details that f is a homeomorphism.

Example. $O(2) =$ isometries fixing $\vec{O} = \{R_\theta \mathscr{F} : \mathscr{F} = F_x \text{ or } \mathrm{Id}\}$.

$$\exists \text{ bij. } f : O(2) \to SO(2) \times \mathbb{Z}_2 \approx S^1 \times \mathbb{Z}_2$$

$$R_\theta F \mapsto \begin{cases} (R_\theta, 0) & F = \mathrm{Id} \\ (R_\theta, 1) & F = F_x \end{cases}$$

On $\mathbb{Z}_2$, set $d(0,0) = d(1,1) = 0$, $d(0,1) = d(1,0) = 1$. Put a product metric on $SO(2) \times \mathbb{Z}_2$ as on $\mathbb{R} \times \mathbb{R}$: $d(R_1, x_1), (R_2, x_2)) = \sqrt{d(R_1, R_2)^2 + d(x_1, x_2)^2}$. Check that $SO(2) \times \mathbb{Z}_2$ looks like $SO(2) \times \{0\} \cup SO(2) \times \{1\} \approx S^1 \sqcup S^1$.

So much for topological/shape structure of $SO(2)$ and $O(2)$.

Algebraic Structure

Say G, H are groups.

$$G \times H = \{(g,h) : g \in G, h \in H\}$$

$G \times H$ is a group (product group) with

$$(g_1, h_1) \cdot (g_2, h_2) \stackrel{\text{def}}{=} (g_1 \cdot_G g_2, h_1 \cdot_H h_2) \in G \times H$$

with identity element $= (e_G, e_H)$.

It is associative, and

$$(g_1, h_1) \cdot (g_1^{-1}, h_1^{-1}) = (g_1 g_1^{-1}, h_1 h_1^{-1}) = (e_G, e_H)$$

so inverses exist.

The point: Check carefully that if groups G and H are isomorphic, then G is abelian iff H is abelian. Check that G_1, G_2 abelian implies $G_1 \times G_2$ is abelian. So $SO(2) \times \mathbb{Z}_2$ is abelian, while $O(2)$ is definitely not. So these two groups are bijective as sets, homeomorphic as metric spaces, but not isomorphic as groups.

There are lots of examples of isomorphic groups having the same properties:

Example. Let G be the isometries of the square. Thus,

$$G = \{\mathrm{Id}, R, R^2, R^3, \mathscr{F}_/, \mathscr{F}_\backslash, \mathscr{F}_-, \mathscr{F}_|\}$$

(The slashes indicate lines in the square to be flipped over.) G is not abelian $\Rightarrow G \not\approx \mathbb{Z}_8$. (Check this carefully.) Similarly, $G \not\approx Z_2 \times Z_4$, even though both groups have order 8.

1. Some subgroups:
$$H_1 = \{\mathrm{Id}, R, R^2, R^3\} \simeq \mathbb{Z}_4$$
$f : H_1 \to \mathbb{Z}_4$ given by $\mathrm{Id} \mapsto 0$, $R \mapsto 1$, $R^2 \mapsto 2$, $R^3 \mapsto 3$. f is an isomorphism.

2. $f' : H_1 \to \mathbb{Z}_4$ given by $\mathrm{Id} \mapsto 0$, $R \mapsto 3$, $R^2 \mapsto 2$, $R^3 \mapsto 1$. f' is an isomorphism. So Note: isomorphisms between two groups need not be unique.

3. Is
$$H_2 = \{\mathrm{Id}, R^2, \mathscr{F}_-, \mathscr{F}_|\} \overset{?}{\simeq} \mathbb{Z}_4$$
It appears to be abelian, so it looks good. Try to build an isomorphism $f : H_2 \to \mathbb{Z}_4$ given by $\mathrm{Id} \mapsto 0$. Then $R^2 \mapsto 2$ because R^2 can't map to 0 and $(R^2)^2 = \mathrm{Id}$ must map to 0. But by the same reasoning, $\mathscr{F}_| \mapsto 2$, and $\mathscr{F}_- \mapsto 2$. So H_2 is not isomorphic to $\mathbb{Z}_4$.

This shows that if $G \simeq H$, then every group property of G holds in H and *vice versa.*

An Example of a Big Group

Let X be any set, and $\mathscr{F} = \{f : X \to X\}$ with composition as operation. It is not a group because inverses don't (necessarily) exist. But
$$\mathscr{F}' = \{f : X \to X | f \text{ is bijective}\}$$
is a group. Say $|X| = n$, that is, X has order (size) n. Then
$$|\mathscr{F}'| = n!$$
Suppose $X = \mathbb{R}$. Then $f \in \mathscr{F}'$ must cross every horizontal line and every vertical line exactly once. (Why?)

1.2 Other Geometries

1. $\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$; lines are $\{\vec{v} + \lambda\vec{w} | \lambda \in \mathbb{R}\}$ where $\vec{v}, \vec{w} \in \mathbb{R}^n$.

2. $\mathbb{C}^n = \{(z_1, \dots z_n) | z_i \in \mathbb{C}\}$, lines as above with $\lambda \in \mathbb{C}$.

3. $\mathbb{Z}_p^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{Z}_p\}$, p prime, lines as above with $\lambda \in \mathbb{Z}_p$.

Look at Z_3^2. A line is $\{\vec{v} + \lambda\vec{w} | \lambda \in \mathbb{Z}_3\}$. There are 9 points in this geometry.

It has 12 lines, because you can form a line from any two points, but any choice of two of the three points on a line determine the same line: $12 = \frac{\binom{9}{2}}{\binom{3}{2}}$. The parallel postulate holds ($\mathbb{Z}_3^2$ is an affine plane).

4. $\mathbb{RP}^2 =$ projective plane, the "geometry of what you see if your eyeball is at the origin, you close the other eye, and you can see in back of your head." For examples, two triangles with corresponding vertices on lines on your eyeball appear as one triangle. So,

$$\mathbb{RP}^2 = \{\text{what you see}\} = \{\text{lines through } \vec{O}\}$$

Lines in $\mathbb{RP}^2$ are what you see as lines, i.e. planes on $\vec{O}$. How does $\mathbb{RP}^2$ fare with Euclid's 5 axioms?

   (a) 2 distinct lines on $\vec{O}$ determine a unique plane on $\vec{O}$, so ✓
   (b) ✓
   (c) skip – we don't have a notion of circle.
   (d) skip – we don't have a notion of angle.
   (e) FAIL. There exist no parallel lines (!!), since any two distinct planes on $\vec{O}$ meet in a line on $\vec{O}$.

1.3 $\mathbb{RP}^2$: It's Shape and Symmetry Group

The Shape of $\mathbb{RP}^2$

$\mathbb{RP}^2 = \{\text{lines on } \vec{O} \text{ in } \mathbb{R}^3\}$—the "points" in $\mathbb{RP}^2$ are lines in $\mathbb{R}^3$, and lines in $\mathbb{RP}^2$ are planes in $\mathbb{R}^3$ on $\vec{O}$.

Q: What does $\mathbb{RP}^2$ look like? Every line on the origin in $\mathbb{R}^3$ hits the upper hemisphere of the unit sphere exactly once, except that horizontal lines hit the equator twice. So there is a bijection from $\mathbb{RP}^2$ to the strict upper hemisphere and your choice of half the equator. Write this as $f : \mathbb{RP}^2 \to S^2_+$. But f cannot be made continuous. This is maybe hard to prove, but e.g. if we set $f(x-\text{axis}) = (1, 0, 0)$, then $f(y = mx, z = 0)$ is forced to be $(m/|m|, 0, 0)$ for $m > 0$. Swinging the line past the y-axis, we see that f must assign the "back half of the equator) to lines in the xy-plane. But after we swing the x-axis through π radians, we see that f jumps discontinuously from a point on the back equator close to $(-1, 0, 0)$ back to $(1, 0, 0)$. Draw a picture.

The solution is to "glue" every point $\vec{x}$ on the equator to the point $-\vec{x}$ on the equator. This relieves us of having to make a choice of antipodal points. That is, set

$$Y = \{\text{pts in strict upper hemisphere}\} \sqcup \{\{\vec{x}, -\vec{x}\} : \vec{x} \in \text{equator}\}$$

From now on, drop the vector notation and denote $\{x, -x\}$ by $[x] = [-x]$.

We can put a distance function on Y by setting $d(x, y)$ to be the length of the great circle arc from x to y for x, y not on the boundary, and $d([x], y)$ to be the shorter length of the arcs from y to $\pm x$. There is a similar definition of $d([x], [y])$. Then

$$f : \mathbb{RP}^2 \to Y, \quad f(\text{line}) = \begin{cases} \text{line} \cap \text{upper hemis}, & \text{if line not horizontal} \\ [x], & \text{if line} \cap \text{sphere} = \pm\vec{x} \end{cases}$$

is a homeomorphism rather trivially if we define the distance between two lines to be the shortest great circle distance between their intersections with the unit sphere.

Another model for (i.e. set homeomorphic to) $\mathbb{RP}^2$: every line hits unit sphere S^2 in 2 antipodal points $\pm\vec{x}$. Set $Z = \{[\vec{x}] | \vec{x} \in S^2\}$. $g : \mathbb{RP}^2 \to Z$ is a bijection where line $\mapsto$ line $\cap S^2$.

A third model for $\mathbb{RP}^2$: looking straight down on Y, it looks like a disk with antipodal boundary points glued. In this model, any line in $\mathbb{RP}^2$ has the shape of a circle S^1. Thus *projective lines are homeomorphic to circles.*

All these models are examples of a UN game. For the second model, the set of points/people is S^2, and the citizenship relation is $\vec{x} \sim \pm\vec{x}$. So the country of $\vec{x}$ is $[\vec{x}] = \{\pm\vec{x}\}$.

Example. As a dumbed down example, we define $\mathbb{RP}^1 = \{\{\vec{x}, -\vec{x}\} | \vec{x} \in S^1\}$, i.e. by gluing opposite points on S^1. Show that the shape of $\mathbb{RP}^1$ is S^1. So we have a 2-to-1 map from S^1 to $\mathbb{RP}^1 \approx S^1$.

1.4 The Symmetries of $\mathbb{RP}^2$

We imposed a distance function on $\mathbb{RP}^2$, so we could talk about angles and circles. But it turns out to be more useful to forget about this distance function. (But we still want to talk about the shape of $\mathbb{RP}^2$. This means there is a notion of homeomorphism that depends only on the topology of $\mathbb{RP}^2$, not any distance function on it – but that's beyond these notes.)

If we forget about distances, then a symmetry of $\mathbb{RP}^2$ is as for the 7 point geometry: it should be a bijection taking points to points and lines to lines, and that's it. So

$$\mathcal{F} = \{f : \mathbb{R}^3 \to \mathbb{R}^3 | f \text{ bijective}, f(\text{line on } \vec{O}) = \text{line on } \vec{O}, \quad (1)$$

$$f(\text{plane on } \vec{O}) = \text{plane on } \vec{O}\}$$

gives us symmetries of $\mathbb{RP}^2$.

Technical points: (i). Since symmetries of $\mathbb{RP}^3$ are just maps defined on lines on the origin in $\mathbb{R}^3$, we could presumably have a symmetry of $\mathbb{RP}^2$ which does not come from a map of $\mathbb{R}^3$. Based on the original motivation of perspective drawing, we rule this possibility out by fiat.

(ii) We are not claiming that the map from $\mathcal{F}$ to $\mathrm{Sym}(\mathbb{RP}^2)$ is a bijection, but by (i) we are forcing it to be a surjection. We'll discuss this more carefully later.

Long Warmup. Replace $\mathbb{R}^3$ by $\mathbb{R}^2$ in the (1), so we are looking for the symmetries of $\mathbb{RP}^1$. (Of course the condition on planes is automatic, so let's drop it.) This isn't a good definition. For example, in polar coordinates $f(r, \theta) = (r^{701}, \theta)$ would be a symmetry of $\mathbb{RP}^1$, but it is an awful map of the plane, as lines not on the origin go to complicated curves. So let's consider

$$\mathcal{S} = \{f : \mathbb{R}^2 \to \mathbb{R}^2 | f \text{ bijective}, f(\text{any line}) = \text{line}, f(\vec{O}) = \vec{O}\} \quad (2)$$

Note that we added that f fixes the origin, so that it takes projective points to projective points.

Take $f \in \mathcal{S}$. Draw the parallelogram in the plane determined by $\vec{v}, \vec{w}$, and call the two sides of the parallelogram not on the origin ℓ_1, ℓ_2, i.e. (dropping the vector notation) $v + w = \ell_i \cap \ell_2$. This implies $f(v+w) \in f(\ell_1)$ and $f(v+w) \in f(\ell_2)$. Since f is bijective, $f(\ell_1) \neq f(\ell_2)$, and so $f(v+w) = f(\ell_1) \cap f(\ell_2)$. Again since f is bijective, it takes parallel lines to parallel lines, so the v, w-parallelogram gets mapped over to a $f(v), f(w)$-parallelogram (draw a picture). Thus $f(v) + f(w) = f(\ell_1) \cap f(\ell_2)$. So we've shown

$$f(v) + f(w) = f(v+w).$$

Recalling the definition of a linear transformation, you see where we're going: the next step is to show that $f(\lambda v) = \lambda f(v)$ for $\lambda \in \mathbb{R}$. Surprisingly, we cannot show this – see Homework Set 5 where we play a new UN game – without an additional assumption on $\mathcal{S}$. So we redefine $\mathcal{S}$ to be

$$\mathcal{S} = \{f : \mathbb{R}^2 \to \mathbb{R}^2 | f \text{ bijective}, f \text{ continuous}, f(\text{any line}) = \text{line}, f(\vec{O}) = \vec{O}\} \quad (3)$$

Let $w_i \to v$ be a sequence of vectors. Because f is continuous,

$$f(2v) = \lim_{i\to\infty} f(v + w_i) = \lim_{i\to\infty} f(v) + f(w_i) = f(v) + f(v) = 2f(v).$$

Now we follow a well known argument. Iterating (i.e. an induction argument) gives $f(nv) = nf(v)$ for $n \in \mathbb{Z}^+$. Then

$$f(-nv) + f(nv) = f(-nv + nv) = f(0) = 0 \quad (4)$$

implies $f(nv) = nf(v)$ for $n \in \mathbb{Z}$. (Actually, in (4) we should take $w_i \to -nv$, which is left to the reader.) Now take $a, b \in \mathbb{Z}, b \neq 0$. Then

$$af(v) = f(av) = f(b \cdot \frac{a}{b} v) = bf(\frac{a}{b} v) \Longrightarrow f(\frac{a}{b} v) = \frac{a}{b} f(v) \quad \forall \frac{a}{b} \in \mathbb{Q}.$$

Finally, for $\lambda \in \mathbb{R}$ take $\frac{a_i}{b_i} \to \lambda$. Then

$$\lambda f(v) = \lim_{i\to\infty} \frac{a_i}{b_i} f(v) = \lim_{i\to\infty} f(\frac{a_i}{b_i} v) = f(\lambda v)$$

by the continuity of f.

Thus $f \in \mathcal{S}$ must be a linear transformation of $\mathbb{R}^2$ and so is given by a 2×2 matrix

$$f \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad f \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}, \quad f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

Since f is bijective, the types of matrices we can get are restricted:

Lemma. *A linear transformation f as above is bijective iff $\begin{pmatrix} a \\ c \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$ is not on the line $\left\{ \lambda \begin{pmatrix} a \\ c \end{pmatrix} | \lambda \in \mathbb{R} \right\}$.*

Proof. ($\Rightarrow$) If $\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then $f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = f\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so f is not injective, and hence not bijective. If $\begin{pmatrix} a \\ c \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ but $f\begin{pmatrix} b \\ d \end{pmatrix} = \lambda_0 \begin{pmatrix} a \\ c \end{pmatrix}$, then check that $f\begin{pmatrix} \lambda_0 \\ 0 \end{pmatrix} = f\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So again f is not injective.

($\Leftarrow$) A picture proof: you supply the details. For surjectivity of f, draw the (bona fide) parallelogram determined by $v = \begin{pmatrix} a \\ c \end{pmatrix}, w = \begin{pmatrix} b \\ d \end{pmatrix}$. Call the line on v and the origin ℓ_1, and the line on w and the origin ℓ_2. For any $P = (r, s)$ in the plane, drop a line parallel to ℓ_2 from P to ℓ_1, so the intersection point is some $\lambda_1 v$. Drop a line parallel to ℓ_1 from P to ℓ_2 to determine a vector $\lambda_2 w$. Then clearly $(r, s) = \lambda_1 v + \lambda_2 w$. (Note how the proof fails if $\begin{pmatrix} b \\ d \end{pmatrix} = \lambda \begin{pmatrix} a \\ c \end{pmatrix}$.)

For injectivity, if $f\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = f\begin{pmatrix} \gamma \\ \delta \end{pmatrix}$, then easily

$$\vec{0} = f\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix} - \begin{pmatrix} \gamma \\ \delta \end{pmatrix}\right) = f\begin{pmatrix} \alpha - \gamma \\ \beta - \delta \end{pmatrix} = (\alpha - \gamma)v + (\beta - \gamma)w.$$

But then v is a multiple of w, a contradiction. □

Bonus: (i) Note that the proof actually shows that f is surjective iff the right hand side of the Lemma holds, and f is injective iff the right hand side of the Lemma holds. Thus f is injective iff it is surjective iff it is bijective.

To connect this with high school algebra, note that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is surjective $\Leftrightarrow$ the system

$$\begin{aligned} ax + by &= z, \\ cx + dy &= w \end{aligned}$$

can be solved for any (z, w) (why?), and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is injective $\Leftrightarrow$ the system

$$\begin{aligned} ax + by &= z, \\ cx + dy &= w \end{aligned}$$

has at most one solution (x, y) for a given (z, w) (why?).

(ii) Enclose the v, w-parallelogram by a rectangle with two sides on the axes. By chopping the part of the rectangle outside the parallelogram into easy rectangles and right triangles, you can directly show that the area of the parallelogram is $|ad - bc| = |\det(f)|$. Of course this area is nonzero iff the parallelogram is a bona fide parallelogram, i.e. not a line segment of a point.

So we conclude:

f is bijective $\Leftrightarrow$ $ax + by = z, cx + dy = w$ always has a unique solution $\Leftrightarrow$ $\det(f) \neq 0$.

You probably have seen the explicit solution for (x, y) in terms of (z, w): it involves division by $\det(f)$, so this works iff $\det(f) \neq 0$. We've given a geometric interpretation of all this algebra.

(iii) Since f is bijective, it has a left and right inverse. Not surprisingly, its inverse is again a linear transformation. For f is surjective, so we can solve

$$f\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}, f\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}.$$

This implies

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

and you can check the same holds for the matrices multiplied in the reverse order. Note that the matrix $I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ is the identity matrix, i.e. $Iv = v$ for all vectors v. Thus $f^{-1} = \begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix}$.

We now know that f is invertible iff $\det(f) \neq 0$. The set of such matrices is an interesting and important group:

Definition.

$$GL(2, \mathbb{R}) = \{2 \times 2 \text{ matrices } A | \det A \neq 0\}$$

Proposition. *$GL(2, \mathbb{R})$ is a group for matrix multiplication.*

Proof.

- $\mathrm{Id} \in GL(2, \mathbb{R})$ as $\det(\mathrm{Id}) = 1$.
- Associativity: this can be done by a painful calculation that depends of course on the associativity of multiplcation in $\mathbb{R}$. One learns almost nothing from this proof.

  Alternatively, we can argue as follows. Composition of functions is associative – an easy homework problem. We also show in the homework that the linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ associated to the 2×2 matrix A satisfies

  $$T_{AB} = T_A \circ T_B. \tag{5}$$

  This is an important exercise, because this is precisely the motivation for defining matrix multiplication the way we do! We proved that the map $A \mapsto T_A$ is a bijection, so

  $$T_{(AB)C} = T_{AB} \circ T_C = (T_A \circ T_B) \circ T_C = T_A \circ (T_B \circ T_C) = T_A \circ T_{BC} = T_{A(BC)}$$

  implies $(AB)C = A(BC)$. I think you learn a lot from this proof.
- Closure: we have to show that $A, B \in GL(2, \mathbb{R})$ implies $(AB)^{-1}$ exists. In fact, $(AB)^{-1} = B^{-1}A^{-1}$, as

  $$ABB^{-1}A^{-1} = A \ \mathrm{Id} \ A^{-1} = \mathrm{Id}$$
  $$B^{-1}A^{-1}AB = B^{-1} \ \mathrm{Id} \ B = \mathrm{Id}$$

  Notice how associativity can be assumed and is used.

- Inverses: $A \in GL(2,\mathbb{R}) \Rightarrow A^{-1} \in GL(2,\mathbb{R})$ as $(A^{-1})^{-1} = A$ (which follows from staring at $AA^{-1} = A^{-1}A = I$).

□

We summarize our work with the following:

Theorem.

$$\mathcal{F} = \{f : \mathbb{R}^2 \to \mathbb{R}^2 | f(\textit{line}) = \textit{line}, f(\vec{0}) = \vec{0}, f \ \textit{bij.}, f \ \textit{cont.}\} \simeq GL(2,\mathbb{R}).$$

The proof that the bijection from $\mathcal{F}$ to $GL(2,\mathbb{R})$ is an isomorphism is buried in the proof of associativity of $GL(2,\mathbb{R})$, particularly (4)

This is the end of our long warmup.

Back to $\mathbb{RP}^2$

We now know we should study the symmetries of $\mathbb{RP}^2$ by first determining

$$\mathcal{S} = \{f : \mathbb{R}^3 \to \mathbb{R}^3 | f(\text{line}) = \text{line}, f(\text{plane}) = \text{plane}, f(\vec{0}) = \vec{0}, f \text{ bij.}, f \text{ cont.}\}.$$

Set $GL(3,\mathbb{R}) = \{3 \times 3 \text{ matrices } A | A^{-1} \text{ exists}\}$. It is a group.

Theorem.

$$\{f : \mathbb{R}^3 \to \mathbb{R}^3 | f(\textit{line}) = \textit{line}, f(\textit{plane}) = \textit{plane}, f(\vec{0}) = \vec{0}, f \ \textit{bij.}, f \ \textit{cont.}\} = GL(3,\mathbb{R})$$

Proof. The proof is the same as for $GL(2,\mathbb{R})$, but with parallelepipeds instead of parallelograms. Along the way, we have to get a criterion for a 3×3 matrix A to be bijective. The correct statement is:

$$A \text{ is surjective} \Leftrightarrow f\vec{i} \text{ is not in the } f\vec{j}, f\vec{k} \text{ plane.}$$

(Actually, there are some extra cases: when $f\vec{j}, f\vec{k}$ only determine a line or a point, A is not surjective.) This is proved as before. □

From this proof, we see that $A \in GL(3,\mathbb{R})$ iff the volume of the parallelepiped formed by $f(\vec{\imath}), f(\vec{\jmath}), f(\vec{k})$ is nonzero. This should have an expression in terms of the entries of A, and of course it does, namely $|\det A|$. The proof is in the homework. There should be a better proof, involving enclosing the parallelepiped inside a rectangular box and subtracting off volumes outside the parallelepiped. Can you find such a proof?

Anyway, we get the geometric interpretation that 3 equations in 3 unknowns has a solution for all (z, w, q) in the notation of the 2 equations case above iff it has a unique solution iff the volume of the parallelepiped is nonzero iff $\det(A) \neq 0$.

So far, we have a surjective map $\alpha : GL(3,\mathbb{R}) \to \text{Sym}(\mathbb{RP}^2)$.

Exercise: $\text{Sym}(\mathbb{RP}^2)$ is a group with respect to composition (since symmetries are by definition bijections). Show that $\alpha(AB) = \alpha(A) \circ \alpha(B)$.

Thus α is an isomorphism if it is injective. But it isn't injective!

Example.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the Id symmetry of $\mathbb{RP}^2$. Now consider

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

It's $2 \cdot \mathrm{Id}$ as a matrix, but it is also Id in $\mathbb{RP}^2$. In fact, you can replace the 2 by any nonzero real number and get Id.

Fortunately, this is as bad as it gets.

Lemma. $A, B \in GL(3,\mathbb{R})$ *give the same symmetry of* $\mathbb{RP}^2$ *iff*

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} B \quad (\text{written } A = \lambda B)$$

for some $\lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

Proof. ($\Leftarrow$) This is easy.

($\Rightarrow$) AB^{-1} fixes every line and plane, so

$$AB^{-1}\vec{\imath} = \lambda_1\vec{i}, AB^{-1}\vec{\jmath} = \lambda_2\vec{j}, AB^{-1}\vec{k} = \lambda_3\vec{k},$$

for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^*$. Since $AB^{-1}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{pmatrix}$ and AB^{-1} fixes the line of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, we must have $\lambda_1 = \lambda_2$. Similarly $\lambda_1 = \lambda_3$. $\square$

We now know "by how much" α fails to be injective. We can restore injectivity by gluing those elements of $GL(3,\mathbb{R})$ which α sends to the same symmetry. So we play the following UN game on elements A, B of $GL(3,\mathbb{R})$:

$$A \sim B \Leftrightarrow A = \lambda B, \lambda \in \mathbb{R}^*.$$

So the countries consist precisely of elements of $GL(3,\mathbb{R})$ which give the same symmetry of $\mathbb{RP}^2$.

Definition.

$$\{\text{countries in } GL(3,\mathbb{R})\} = PGL(3,\mathbb{R})$$

Thus if we denote the country of $A \in GL(3,\mathbb{R})$ by $[A] = \{\lambda A : \lambda \in \mathbb{R}^*\}$, then $PGL(3,\mathbb{R}) = \{[A] : A \in GL(3,\mathbb{R})\}$.

We now have a *bijection*, which we still call α,

$$\alpha : PGL(3,\mathbb{R}) \to \mathrm{Sym}(\mathbb{RP}^2),\ [A] \mapsto \text{whatever } A \text{ does to lines/planes on } \vec{O}. \tag{6}$$

1.4.1 Quotient Groups

To finish the discussion, we want to see that (6) is an isomorphism of groups. To do this, we have to see if $PGL(3,\mathbb{R})$ is a group!

Example. Look at the following UN game on $\mathbb{Z}$: $a \sim b \Leftrightarrow 5|a-b$. The countries are:

$$\begin{aligned}
\{5k|k\in\mathbb{Z}\} &= [0]=[\pm 5]=\dots\\
\{5k+1|k\in\mathbb{Z}\} &= [1]=[6]=\dots\\
\{5k+2|k\in\mathbb{Z}\} &= [2]=[7]=\dots\\
\{5k+3|k\in\mathbb{Z}\} &= [3]=[8]=\dots\\
\{5k+4|k\in\mathbb{Z}\} &= [4]=[9]=\dots
\end{aligned}$$

$\mathbb{Z}$ is a group and the set of countries $\{[0],[1],[2],[3],[4]\}$ is also a group by the rule $[a]+[b] \overset{\text{def}}{=} [a+b]$. This is well-defined, i.e., if $[a']=[a]$ and $[b']=[b]$, then $[a'+b']=[a+b]$. For

$$a' = a+5k_1, b' = b+5k_2 \Rightarrow a'+b' = a+b+5(k_1+k_2) \Rightarrow [a'+b'] = [a+b].$$

Now it is easy to check that the set of countries $\mathbb{Z}/5\mathbb{Z}$ is a group with respect to this addition. The identity is $[0]$, you check associativity (and this should be checked!), and $-[n]=[-n]$. In fact, $\mathbb{Z}/5\mathbb{Z} \simeq \mathbb{Z}_5$, via $[a] \mapsto a \mod 5$, as you can easily check.

If we try to carry over this UN game to a general group G, we should pick a subset $S \subset G$ and impose $a \sim b$ whenever $ab^{-1} \in S$. (Right? – when the group is $(\mathbb{Z},+)$, ab^{-1} corresponds to $a-b$.) Check that this relation is symmetric and transitive iff S is a subgroup of G. (One direction is easy.) Note that $5\mathbb{Z}$ is a subgroup of $\mathbb{Z}$.

We still haven't shown that G/S is a group whenever S is a subgroup of G, and in fact this is not true in general.

Anyway, let's try this with $GL(3,\mathbb{R})$. We have $A \sim B$ iff $A=\lambda B$ iff $AB^{-1} = \lambda I$ for some $\lambda \in \mathbb{R}^*$. Note that $\{\lambda I : \lambda \in \mathbb{R}^*\}$ is a subgroup of $GL(3,\mathbb{R})$, so we have a chance.

As with the UN game on $\mathbb{Z}$, we try the following product:

$$[A]\cdot[B] \overset{\text{def}}{=} [AB]$$

Is this well defined? If $A' = \lambda_1 A$, $B' = \lambda_2 B$, then

$$A'B' = \lambda_1 A \lambda_2 B = \lambda_1\lambda_2 AB = \lambda_3 AB. \tag{7}$$

Here we have used

$$A\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_1\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_1\end{pmatrix} A$$

So multiplication is well defined.
Warning: Let G be a nonabelian group and S a subgroup. Set $a \sim b$ iff $ab^{-1} \in S$. Set $[A] \cdot [B] = [AB]$. The proof above that this multiplication is well defined breaks down: we have $A' = S_1 A$, $B' = S_2 B$ in the obvious notation, but because in general $AS_2 \neq S_3 A$ for some $S_3 \in S$, we can't push (7) through.

Lemma. *$PGL(3, \mathbb{R})$ is a group.*

Proof. The multiplication is well defined and closed. The identity element is $[I]$. Associativity should be checked. $[A]^{-1} = [A^{-1}]$ as $[A][A^{-1}] = [AA^{-1}] = [I]$. □

Finally we get the result we want.

Theorem (Big Theorem)**.**

$$PGL(3, \mathbb{R}) \simeq \mathrm{Sym}(\mathbb{RP}^2)$$

Proof. We now have a bijection $\alpha : PGL(3, \mathbb{R}) \to \mathrm{Sym}(\mathbb{RP}^2)$, $\alpha[A](\ell) = T_A \ell$, for any line ℓ on the origin. Recall that T_A is the linear transformation associated to A – if you don't like it, just write $A\ell$. Then using (5), we get

$$\begin{aligned} \alpha([A][B])(\ell) &= \alpha[AB](\ell) = T_{AB}(\ell) = (T_A \circ T_B)(\ell) \\ &= T_A(T_B \ell) = [A]([B](\ell)) = (\alpha[A] \circ \alpha[B])(\ell), \end{aligned}$$

so $\alpha([A][B]) = \alpha[A] \circ \alpha[B]$. □

Think of $GL(3, \mathbb{R})$ as a blob on the blackboard, and the countries $[A]$ as stripes on the blob. Rather than work with $PGL(3, \mathbb{R})$, it would be nice to have a nice *slice* of $GL(3, \mathbb{R})$, namely a subset that crosses each country exactly once. Such a slice exists, and is actually a subgroup of $GL(3, \mathbb{R})$.

Definition.

$$SL(3, \mathbb{R}) = \{A \in GL(3, \mathbb{R}) | \det A = 1\}$$

Fact: $\det(AB) = \det A \cdot \det B$. If we know from the homework that $\det(A)$ is the oriented volume of the Ai, Aj, Ak parallelepiped, then this is immediate. Recall that A^{-1} exists iff $\det(A) \neq 0$.

Lemma. *$SL(3, \mathbb{R})$ is a subgroup of $GL(3, \mathbb{R})$.*

Proof.

- $A, B \in SL(3, \mathbb{R}) \Rightarrow \det(AB) = \det A \cdot \det B = 1 \cdot 1 = 1 \Rightarrow AB \in SL(3, \mathbb{R})$.
- $\det I = 1$, so the identity element $I \in SL(3, \mathbb{R})$.
- $A \in SL(3, \mathbb{R}) \Rightarrow 1 = \det I = \det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(A^{-1}) \Rightarrow A^{-1} \in SL(3, \mathbb{R})$.

□

Note that $\det(\lambda A) = \lambda^3 \det(A)$. Every country $[A] \in PGL(3, \mathbb{R})$ has a unique representative in $SL(3, \mathbb{R})$, namely $(\det(A))^{-1/3} A$. (Check this is independent of $A' \in [A]$.) Thus $SL(3, \mathbb{R})$ is the slice we want.

Corollary. $PGL(3,\mathbb{R}) \simeq SL(3,\mathbb{R})$ *by* $[A] \mapsto (\det(A))^{-1/3}A$.

You can check that the bijection just set up is an isomorphism.

Remark. All this dumbs down to $\mathbb{RP}^1$, and we can show that $PGL(2,\mathbb{R}) \simeq \mathrm{Sym}(\mathbb{RP}^1)$. However, $SL(2,\mathbb{R})$ is not a slice of $PGL(2,\mathbb{R})$, since for 2×2 matrices $\det(\lambda A) = \lambda^2 A$. Thus the country of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has no representative of determinant one, and $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ has two representatives, $\pm I$. It's strange that the difference between odd and even (dimensions) shows up only here.

To be consistent across dimensions, it's better to use $PGL(n,R)$.

A Final Remark on the Topology/Shape of $\mathbb{RP}^2$

The following is "intuitive," i.e. vague.

First, note that $\mathbb{RP}^2$ is a surface, that is, every point $x \in \mathbb{RP}^2$ has a neighborhood that is homeomorphic to a small open disk in $\mathbb{R}^2$. (Draw some pictures.)

List of surfaces: plane, sphere, open Möbius band, torus (with any number of holes), infinite cylinder, plane with the origin removed, etc.

A surface is **compact** if every infinite sequence $x_1, x_2, \dots \in S$ has a convergent subsequence (e.g., $x_{17}, x_{204}, \dots$) to some $x \in S$. That is, S "doesn't go to infinity" and "has no singularities."

So the plane, the infinite cylinder, the punctured plane etc. are not compact. The sphere and torus are. So is $\mathbb{RP}^2$ by the standard divide and conquer argument – chop $\mathbb{RP}^2$ into two pieces, pick a piece with an infinite number of elements of the sequence, continue, argue that the limit set is a point.

An intuitive argument that a sphere is not homeomorphic to a torus: you can draw a loop on a torus "around the donut and through the hole" that cannot be shrunk to a point without leaving the surface of the torus. But any loop you draw on the surface of a sphere can be shrunk to a loop.

$\mathbb{RP}^2$ has a loop γ with γ unshrinkable, but γ^2, i.e. going twice around γ, is shrinkable. You can see γ by going halfway around the boundary of the disk model of $\mathbb{RP}^2$ – note that this is a loop on $\mathbb{RP}^2$. It's intuitively clear that you can't shrink γ, and you can see an explicit shrinking of γ^2. Since none of the other ones have this property, $\mathbb{RP}^2$ is a new surface for us.

To be fair, a Klein Bottle is another surface with such a loop γ, but $\mathbb{RP}^2$ is characterized among surfaces as the only one with only one (deformation class) of unshrinkable loops, and this one has γ^2 shrinkable.