Chapter 1

Hyperbolic Geometry

We introduce the third of the classical geometries, hyperbolic geometry.

1.1 Hyperbolic Geometry

$$\mathbb{H} = \mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 | y > 0\} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$$

Lines are

(i) vertical lines in $\mathbb{H}$

(ii) semicircles centered on x-axis.

How do Euclid's first 5 axioms fare?

(i) 2 distinct points determine a unique line segment? Yes – if $P, Q \in \mathbb{H}$ are not a vertical line, take the intersection of their perpendicular bisector with the x-axis as the center of the circle.

(ii) lines are infinitely extendable? It seems no, but we want yes, otherwise, it's not a very good geometry. So yes, but we need to redefine distance in $\mathbb{H}$.

(iii) later – see the last Remark.

(iv) later

(v) Parallel postulate? No, but it fails differently from $\mathbb{RP}^2$: there are infinitely many lines on $P \notin \ell$ parallel to a given line ℓ.

Symmetries of $\mathbb{H}$

How should we define the symmetries of $\mathbb{H}$? Since we don't have a notion of distance, let's mimic our definition of symmetries of the 7 point geometry:

$$f : \mathbb{H} \to \mathbb{H}$$
$$f \text{ bij., continuous, and } f(\mathbb{H} - \text{line}) = \mathbb{H} - \text{ line}$$

Notice that we snuck in continuity, by which we mean $d_{\mathbb{R}^2}(z_i, z) \to 0 \Rightarrow d_{\mathbb{R}^2}(f(z_i), f(z)) \to 0$. This assumes whatever distance function we eventually put on $\mathbb{H}$ is compatible in some sense with the Euclidean distance, so this is somewhat of a cheat. We should all read the proof of the symmetry group of $\mathbb{H}$ to see if we are really using continuity.

Examples of symmetries:

Translations: only in horizontal directions. as nonhorizontal translations are not bijections of $\mathbb{H}$.

Rotations: No – same problem as above.

Reflections: only across vertical lines.

Dilations: i.e. scalings $z \mapsto \lambda z$ for a fixed $\lambda > 0$. Note that this is the first example of a hyperbolic symmetry which is not a Euclidean isometry.

Great Trick: Take $a, b, c, d \in \mathbb{R}, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has associated **linear fractional transformation** (LFT), denoted L_A. Warning: we often write $A \cdot z$ for $L_A(z)$.

$$L_A : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \frac{az + b}{cz + d}.$$

Where do LFTs come from? If we let the entries of A be complex numbers, LFTs are the simplest functions from $\mathbb{C}$ to $\mathbb{C}$ after polynomials, so it's natural that they were studied. A deeper motivation is in the next chapter. We'll see that LFTs have great properties.

Warning: if $z = x + iy$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \neq \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Q: When does $L_A : \mathbb{H} \to \mathbb{H}$, i.e. given $\mathrm{Im}(z) > 0$, when is $\mathrm{Im}(L_A(z)) > 0$?

$$L_A(z) = \frac{az + b}{cz + d} = \frac{a(x + iy) + b}{c(x + iy) + d} = \frac{(ax + b) + iay}{(cx + d) + icy}$$

Rationalize the denominator to get

$$(\text{real junk}) + \frac{(ad - bc)y}{(cx + d)^2 + (cy)^2} i$$

for $z = x + iy$. Since $y > 0$ (and you can check that the numerator is nonzero), we need $ad - bc = \det A > 0$.

Conclusion: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0 \Leftrightarrow L_A : \mathbb{H} \to \mathbb{H}$

Claim: All such L_A's are in $\mathrm{Sym}(\mathbb{H}^2)$. The proof in steps, building up from examples.

Example. (0) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $L_A z = \frac{z+0}{0 \cdot z+1} = z$; $L_A = \mathrm{Id}$.

(1) $T = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. $L_T z = \frac{1 \cdot z + b}{0 \cdot z + 1} = z + b$, so L_T is horizontal translation by $b \in \mathbb{R}$.

(2) $A = \begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}$. $L_A z = \frac{-5z}{-5} = z$, thus, $L_A = \mathrm{Id}$. So $L_A = L_{\lambda A}$ for $\lambda \in \mathbb{R}^*$.

(3) $A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $a > 0$, $L_A \cdot z = \frac{az+0}{0 \cdot z+1} = az$, $L_A \in \mathrm{Sym}(\mathbb{H}^2)$. So this gives us a scaling. (This will be an isometry of $\mathbb{H}^2$, but we haven't defined distance yet.)

(3.5) Combine (1) and (3):

$$L_{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}} \circ L_{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}}(z) = L_{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}}(az) = az + b$$

Note that $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ and

$$L_{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}}(z) = \frac{ax+b}{0z+1} = ax + b$$

that is, $L_{BA} = L_B \circ L_A$. We'll come back to this important formula.

(4) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. $L_A(z) = \frac{0 \cdot z+1}{-z+0} = -\frac{1}{z}$.

Claim: $L_A \in \mathrm{Sym}(\mathbb{H}^2)$.

Proof. For $z = x + iy = (r, \theta)$ in polar coordinates, we have $1/z = -\frac{x}{x^2+y^2} + \frac{y}{x^2+y^2} i = (1/r, -\theta)$ [check this], so $-1/z = (1/r, \pi - \theta)$. For $a, b, c \in \mathbb{R}$,

$$a(x^2 + y^2) + bx + c = 0$$

is a circle with center on the x-axis of radius $\sqrt{-c/a}$ (if $a \neq 0$ and $c < 0$) or a vertical line (if $a = 0$). Since $r^2 = x^2 + y^2, x = r\cos\theta$, this equation is

$$ar^2 + brcos\theta + c = 0$$

in polar coordinates. Under $z \to -1/z$, this equation becomes

$$a/r^2 + \frac{b}{r}\cos(\pi - \theta) + c = 0, i.e. a - br\cos(\theta) + cr^2 = 0.$$

So circles of radius $\sqrt{-c/a}$ go to circles of radius $(\sqrt{-c/a})^{-1}$, except some circles go to vertical lines (which ones?), and lines go to circles (except the y-axis goes to itself). □

(5) The case of a general LFT: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det A > 0$. We claim that $L_A \in \operatorname{Sym}\mathbb{H}$. First, if $c = 0$, then

$$L_{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}} = L_{\begin{pmatrix} \frac{a}{d} & \frac{b}{d} \\ 0 & 1 \end{pmatrix}}$$

This is scaling by a/d plus a horizontal translation by b/d, which is a symmetry of $\mathbb{H}$. Note that we cannot have $c = d = 0$.

If $c \neq 0$, we compute $\frac{az+b}{cz+d}$ via long division: and get

$$\begin{aligned} \frac{az+b}{cz+d} &= \frac{a}{c} + \frac{b - \frac{da}{c}}{cz+d} \\ &= \frac{a}{c} + \frac{bc - da}{c} \cdot \frac{-1}{cz+d} \end{aligned}$$

i.e. $L_A(z) = L_{\begin{pmatrix} 1 & \frac{a}{c} \\ 0 & 1 \end{pmatrix}} L_{\begin{pmatrix} \frac{ad-bc}{c} & 0 \\ 0 & 1 \end{pmatrix}} L_{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} L_{\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}} L_{\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}}(z)$

a composition of 5 symmetries. This proves the claim. It also gives a not very useful decomposition theorem for LFTs analogous to the TRF decomposition.

But we're missing the symmetries which are reflections on vertical lines. Set R = reflection on y-axis: $R \in \operatorname{Sym}(\mathbb{H})$. Note that reflection on the line $x = b$ is $T_b \circ R \circ T_{-b} \in \text{LFT}$.

Goal: $\operatorname{Sym}(\mathbb{H}^2) = \{\text{LFT}, R \circ \text{LFT}\}$.

Terminology:

$$R \circ L_{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} : (r, \theta)_p \mapsto (\frac{1}{r}, \theta)_p$$

is called **inversion on the unit circle**. It is a symmetry of $\mathbb{H}$.

To work towards the goal, let $GL(2, \mathbb{R})^+$ be the 2×2 invertible matrices with real coefficients and positive determinant.

Claim: $GL(2, \mathbb{R})^+$ is a subgroup of $GL(2, \mathbb{R})$.

Proof. This follows from $\det(AB) = \det(A)\det(B)$, which can be proven by unrevealing direct calculation or by homework problems. For closure,

$$A, B \in GL(2,\mathbb{R})^+ \Rightarrow \det(AB) = \det(A)\det(B) > 0 \Rightarrow AB \in GL(2,\mathbb{R})^+.$$

For inverses, if $A \in GL(2,\mathbb{R})^+$, then

$$1 = \det(AA^{-1}) = \det(A)\det(A^{-1}) \Rightarrow \det(A^{-1}) = (\det(A))^{-1} > 0 \Rightarrow A^{-1} \in GL(2,\mathbb{R})^+.$$

As a good check, $\mathrm{Id} \in GL(2,\mathbb{R})^+$, and associativity follows from that of $GL(2,\mathbb{R})$. □

Now we check that LFT is a group with respect to composition.

Proof. Closure: For $L_A, L_B \in LFT$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} c & d \\ e & f \end{pmatrix}$,

$$\begin{aligned} (L_A \circ L_B)(z) &= L_A(L_B(z)) \\ &= L_A\left(\frac{ez+f}{gz+h}\right) \\ &= \frac{a\left(\frac{ez+f}{gz+h}\right)+b}{c\left(\frac{ez+f}{gz+h}\right)+d} \\ &= \frac{ae+by)z+(af+bh)}{(ce+dg)z+(cf+dh)} \\ &= L_{AB}(z) \end{aligned}$$

Since $\det(AB) = \det(A) \cdot \det(B) > 0$, the important formula

$$L_A \circ L_B = L_{AB} \tag{1}$$

implies closure.

Identity: L_{Id} is the identity map on $\mathbb{H}$.

Inverses:

$$L_{A^{-1}} \circ L_A = L_{A^{-1}\cdot A} = L_{\mathrm{Id}} = \mathrm{Id},$$

and similarly $L_A \circ L_{A^{-1}} = \mathrm{Id}$, so $(L_A)^{-1} = L_{A^{-1}}$.

Associativity: This follows immediately from the fact that $L_A : \mathbb{H} \to \mathbb{H}$ is a function! □

Consider the map

$$\alpha : GL(2,\mathbb{R})^+ \to LFT, \quad \alpha(A) = L_A.$$

Is α an isomorphism? It is certainly surjective, since every LFT is of the form L_A for some $A \in GL(2,\mathbb{R})^+$. The map α also has the "preservation of multiplication table" property, as

$$\alpha(AB) = L_{AB} = L_A \circ L_B = \alpha(A) \circ \alpha(B)$$

by (1). But α is *not* injective, since

$$L_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = L_{\begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}}$$

for all $\lambda \in \mathbb{R}$.

So α is not an isomorphism. Nevertheless, it is close enough to motivate the following definition.

Definition. Let G, H be groups, and let $f : G \to H$ have $f(g_1 \cdot_G g_2) = f(g_1) \cdot_H f(g_2)$. Then f is a **homomorphism** from G to H.

In this language, $\alpha : GL(2,\mathbb{R})^+ \to \text{LFT}$ is surjective homomorphism. To construct a group isomorphic to LFT, we glue together matrices that map to the same LFT. Specifically, we glue the matrices A and λA, $\lambda \in R^*$ to form

$$PGL(2,\mathbb{R})^+ = \{[A] : A \in GL(2,\mathbb{R})^+\}, \text{ with } [A] = \{\lambda A : \lambda \in \mathbb{R}^*\}.$$

Notice the analogy to forming $\mathbb{RP}^2$ by gluing vectors $\vec{v}$ to $\lambda\vec{v}$, but here we are gluing up symmetry group elements.

Claim: $PGL(2,\mathbb{R})^+$ is a group where we define $[A][B] \stackrel{\text{def}}{=} [AB]$.

The proof is just as for $PGL(3,\mathbb{R})$. We have to show the multiplication is well-defined, and then check the group axioms. (If you're familiar with abstract algebra, we have formed the quotient of $GL(2,\mathbb{R})^+$ by the kernel of α, which must be a normal subgroup.)

We expect to get an isomorphism from the glued up group $PGL(2,\mathbb{R})^+$ to LFT. Specifically, $\alpha : GL(2,\mathbb{R})^+ \to \text{LFT}$ induces a map $\tilde{\alpha} : PGL(2,\mathbb{R})^+ \to LFT$ and we claim that this map is an isomorphism. Here we define $\tilde{\alpha}[A] = \alpha(A)$. This map is well defined, since $\alpha(\lambda A) = \alpha(A)$. $\tilde{\alpha}$ is clearly surjective. It is a homomorphism, since

$$\tilde{\alpha}([A][B]) = \tilde{\alpha}[AB] = \alpha(AB) = \alpha(A)\alpha(B) = \tilde{\alpha}[A]\tilde{\alpha}[B].$$

So we have to check that $\tilde{\alpha}$ is injective. If $\tilde{\alpha}[A] = \tilde{\alpha}[B]$, then $\alpha(A) = \alpha(B)$, which implies $\alpha(AB^{-1}) = \text{Id}$ as an LFT. If $AB^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, this means $\frac{az+b}{cz+d} = z$ for all $z \in \mathbb{H}$. In other words, the quadratic equation $cz^2 + (d-a)z + b = 0$ has infinitely many roots. This is only possible if $c = b = 0, a = d$. Thus $AB^{-1} = \lambda\,\text{Id}$ for some nonzero λ, so $A = \lambda B$, i.e. $[A] = [B]$.

Thus we have proved:

Theorem. *$\tilde{\alpha} : PGL(2,\mathbb{R})^+ \to LFT$ is an isomorphism.*

In analogy with $PGL(3,\mathbb{R})$, we would like to find a nice slice for $PGL(2,\mathbb{R})^+$ in $GL(2,\mathbb{R})^+$, and the natural candidate is $SL(2,\mathbb{R})$, the 2×2 matrices of determinant one. This doesn't work, since if $A \in SL(2,\mathbb{R})$, so is $-A$, and $[A] = [A]$. This means that every country $[B]$ has two representatives from

$SL(2,\mathbb{R})$, and there is no natural way to select one representative from each country.

We know how to get around this problem by gluing: set

$$PSL(2,\mathbb{R}) = \{[A] = \{\pm A\} : A \in GL(2,\mathbb{R})^+\}.$$

Note that the country of A in $PSL(2,\mathbb{R})$ is just $\{A, -A\}$.

We claim that $PSL(2,\mathbb{R})$ is a group with respect to $[A][B] \stackrel{\text{def}}{=} [AB]$. The proof should be standard by now. We also claim that the map

$$\beta : PSL(2,\mathbb{R}) \to PGL(2,\mathbb{R})^+, \beta[A]^{SL} = [A]^{GL}$$

is an isomorphism, where $[A]^{SL} = \{\pm A\}$, $[A]^{GL} = \{\lambda A : \lambda \in \mathbb{R}^*\}$. Thus β takes a small country of A into a bigger country of A. The proof is easy. β is surjective because every GL country $[B]^{GL}$ has two representatives $B'_\pm = \pm(\sqrt{B})^{-1}B$ in $SL(2,\mathbb{R})$, so $\beta[B'_+]^{SL} = [B]^{GL}$. We also have

$$\beta([A][B]) = \beta[AB] = [AB]^{GL} = [A]^{GL}[B]^{GL} = \beta[A]\beta[B].$$

For injectivity, if $\beta[A] = \beta[B]$, then as before $\beta[AB^{-1}] = [\mathrm{Id}]$, so $AB^{-1} = \lambda\,\mathrm{Id}$. Since $\det(AB^{-1}) = 1$, $\lambda = \pm 1$, so $A = \pm B$ and $[A]^{SL} = [B]^{SL}$. Thus

Theorem. $PSL(2,\mathbb{R}) \simeq PGL(2,\mathbb{R})^+ \simeq LFT$.

OK, now we "understand" LFT. To prove every symmetry of $\mathbb{H}$ is of the form $R \circ LFT$ or LFT, we have to impose some more geometry. We could put a distance on $\mathbb{H}$ – and this is done in the exercises. However, we can get away with less than a distance geometry by studying the *conformal* or angle geometry of $\mathbb{H}$. (Later when we discuss the Riemannian geometry of $\mathbb{H}$, we'll see in what sense an angle geometry is less information than a distance geometry.)

We define the angle between two $\mathbb{H}$-lines to be the Euclidean angle between their tangent vectors at the point of intersection.

Lemma. *Linear fractional transformations are conformal, i.e. they preserve the angles between* $\mathbb{H}$*-lines.*

Proof. This is easy to show for translations and dilations, so we're done if we show the conformality of the map $z \mapsto -1/z$. Since this is reflection across the y-axis (which is conformal) composed with the *inversion* map $z \mapsto 1/\bar{z}$, or $(r,\theta) \mapsto (1/r,\theta)$, it's suffices to show that inversion I is conformal.

For this, we use the proof from Barager. First, since every angle can be written as the sum or difference of two angles, with one common side of the two angles on a ray on $O = \vec{0}$, we can assume that one side of our angle α is on such a ray ℓ'. Let ℓ be the line forming the other side of α. (Draw a picture.) We assume that ℓ is entirely outside the unit circle – you do the other cases. Find $P \in \ell$ with $OP \perp \ell$. Set $P' = I(P)$. Then $I(\ell)$ is a circle $\mathcal{C}$ on O (why?). Since $d(O,P)$ is minimal for all points on ℓ, $d(O,P')$ is maximal for all points on $\mathcal{C}$, i.e. OP' is a diameter of $\mathcal{C}$.

Let $A = \ell \cap \ell'$ be the base of α, and set $A' = I(A) \in \mathcal{C}$. Since $I(\ell') = \ell'$, it is easy to check that β, the angle corresponding to α under inversion, is the angle between ℓ' and the tangent line to $\mathcal{C}$ at A'. Now $\Delta OAP \sim \Delta OP'A'$, since these are right triangles sharing the angle $\angle AOP = \angle A'OP'$. From the picture, $\angle A'P'O = \angle OAP = \alpha$.

Let O' be the center of $\mathcal{C}$. Let $x = \angle OA'O'$. Then $\beta + x = \pi/2$. Since $\alpha = \angle OP'A'$ is an inscribed angle in $\mathcal{C}$, the angles in the isosceles triangle $\Delta OO'A'$ satisfy $2\alpha + 2x = \pi$, or $\alpha + x = \pi/2$. Thus $\alpha = \beta$. □

So now let's redefine the symmetries of $\mathbb{H}$ as follows:

Definition. $\text{Sym}(\mathbb{H}) = \{f : \mathbb{H} \to \mathbb{H} : f \text{ bijective}, f(\mathbb{H}-\text{line}) = \mathbb{H}-\text{line}, f \text{ conformal}\}$.

We should check if we use that f has some sort of continuity in the proof below.

Theorem. $f \in \text{Sym}(\mathbb{H}) \Rightarrow f = L_A$ or $f = R \circ L_A$, *for* R *the reflection on the* y*-axis and* $L_A \in LFT$.

Proof. **Case I:** f is orientation preserving. We want to show that $f = L_A$ for some A.

Say $f(P) = P', f(Q) = Q'$.

Claim 1: $\exists!$ LFT taking P to P', Q to Q'.

For uniqueness, say $L_A, L_B : P \mapsto P', Q \mapsto Q'$. Then $L_{B^{-1}} \circ L_A = P \mapsto P, Q \mapsto Q$.

Set $C = B^{-1}$A, so $L_{B^{-1} \circ A} = L_C$. Write $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and set $P = x + iy, Q = r + is$. Check that $L_C P = p, L_Q = Q$, which is four equations in four unknowns, has the family of solutions $a = d, b = c = 0$, so $L_C = L_{\text{Id}} = \text{Id}$.

The existence proof is similar, except we write $P' = x' + iy', Q' = r' + is'$ and solve for a, b, c, d.

Claim 2: If $f, g \in \text{Sym}(\mathbb{H})$ has $f(P) = g(P), f(Q) = g(Q)$ for distinct points $P, Q \in \mathbb{H}$, then $f = g$.

By considering $g^{-1}f$ as above, it suffices to show that an orientation preserving symmetry h of $\mathbb{H}$ that fixes two points is the identity. [Compare this with the homework problem: an orientation preserving isometry of $\mathbb{R}^2$ which fixes two points is the identity.]

Take $Y \in \mathbb{H}$ not on the $\mathbb{H}$-line $\overline{PQ}$. Take the hyperbolic triangle ΔPYQ. Then since h is conformal, it is easy to see that $h(Y)$ has only two possibilities, one of which is Y. (Draw a picture.) The other possibility Y' would easily make h orientation reversing. So $h(Y) = Y$. Now if $W \in \overline{PQ}$, then considering the hyperbolic triangle ΔPWY and arguing as above easily leads to $h(W) = W$.

Now to finish Case I: take $f \in \text{Sym}(\mathbb{H})$, and say $f(P) = P', f(Q) = Q'$. There exists an LFT L_A with $L_A(P) = P, L_A(Q) = Q$. But L_A is also in $\text{Sym}(\mathbb{H})$, so by Claim 2, $f = L_A$.

Case II: If f is orientation reversing, then $R \circ f$ is orientation preserving. Thus by Case I, $R \circ f = L_A$, so $f = R \circ L_A$. $\square$

Remark. Let r be reflection across the y-axis. For $P, Q \in \mathbb{H}$, find the LFT L_A taking P to i and Q to $2i$. Then $r_{PQ} \stackrel{\text{def}}{=} L_A \circ r \circ L_{A^{-1}}$ fixes P and Q (why?), and satisfies $r_{PQ}^2 = L_A \circ r \circ L_{A^{-1}} \circ L_A \circ r \circ L_{A^{-1}} = \text{Id}$. By conformality, r_{PQ} fixes every point on $\overline{PQ}$, so it is reasonable to call r_{PQ} the reflection across $\overline{PQ}$. Thus the y-axis is not some sort of special line in $\mathbb{H}$.

The Disk Model of Hyperbolic Geometry

Fix $z_0 \in \mathbb{C}$. For $z \in \mathbb{C}$, the map $z \mapsto z_0 z$ from $\mathbb{C}$ to $\mathbb{C}$ is conformal. For if $z_0 = (r_0, \theta_0), z = (r, \theta)$ in polar coordinates, then $z_0 z = (r_0 r, \theta_0 + \theta)$. (Check this.) So multiplication by z_0 scales z by a factor of r_0 and then rotates the result by θ_0 – and both of these actions are conformal.

Consider the complex linear fractional transformation

$$\alpha : \mathbb{C} \setminus \{-i\} \to \mathbb{C}, \quad \alpha(z) = \frac{z - i}{z + i}.$$

This is conformal, and you can check that it takes lines and circles to lines and circles. Since $\alpha(0) = -1, \alpha(1) = -i$, and

$$\alpha(\infty) \stackrel{\text{def}}{=} \lim_{t \to \infty} \alpha(t) = \lim_{t \to]infty} \frac{1 - \frac{i}{t}}{1 + \frac{i}{t}} = 1,$$

we conclude that α takes the x-axis to the unit circle S^1. Since $\alpha(i) = 0$ is inside S^1, we conclude that α takes $\mathbb{H}$ to the unit disk D. Since α is bijective with inverse $\alpha^{-1}(z) = \frac{-iz-i}{z-1}$ (why?), $\alpha(\mathbb{H}) = D$.

Here's an application that shows why we emphasized the conformal nature of LFTs. It is worthwhile to try to prove the following lemma directly in Euclidean geometry.

Lemma. *For each pair of distinct points $P, Q \in D$, there exists a unique circular arc (or diameter) on P and Q which meets S^1 at right angles.*

Proof. Find the unique hyperbolic line ℓ joining $P' = \alpha^{-1}(P)$ and $Q' = \alpha^{-1}(Q)$. Assume that ℓ is not a vertical line. Then ℓ meets the x-axis at right angles, so the conformal map $\alpha(\ell)$ joins P and Q and meets S^1 at right angles. The uniqueness of this circular arc or diameter follows from the uniqueness of ℓ.

if ℓ is vertical, and say $Im(Q') > Im(P')$, then the intersection of S^1 with the end of $\alpha(\ell)$ closest to P is at right angles. Inversion I takes ℓ to a circle or line whose end on the x-axis closest to $I(Q')$ meets the x-axis at a right angle. Since inversion is conformal, the intersection of S^1 with the end of $\alpha(\ell) = \alpha \circ I^{-1} \circ I(\ell)$ closest to Q is also at a right angle. $\square$

We can put a non-Euclidean geometry on D by setting lines to by $\alpha(\mathbb{H} -$ lines). By the Lemma, lines in D are either diameters or circular arcs hitting S^1

at right angles. Again, given a line ℓ and $P \notin \ell$, there are infinitely many lines on P and parallel to ℓ.

If we set

$$\mathrm{Sym}(D) = \{f : D \to D : f \text{ bijective}, f \text{ conformal}, f(D - \text{line}) = D - \text{line}\},$$

then we expect $\mathrm{Sym}(D) \simeq \mathrm{Sym}(\mathbb{H})$.

This is easy to show. Set $B : \mathrm{Sym}(\mathbb{H}) \to \mathrm{Sym}(D)$ by $B(g) = \alpha \circ g\alpha^{-1}$, i.e. $B(g)(P) = \alpha(g(\alpha^{-1}(P)))$. Since g and α are bijective and conformal, so is $B(g)$. By definition, $B(g)$ takes D-lines to D-lines. Thus $B(g) \in \mathrm{Sym}(D)$. B is a homomorphism, since $B(gh) = \alpha gh\alpha^{-1} = \alpha g\alpha^{-1}\alpha h\alpha^{-1} = B(g)B(h)$. Finally B is bijective because its inverse is $B^{-1}(f) = \alpha^{-1} \circ f \circ \alpha$. You can check that $BB^{-1} = B^{-1}B = \mathrm{Id}$.

Thus the geometry of $\mathbb{H}$ and the geometry of D are completely equivalent.

Remark. In the Homework, we introduce a distance on $\mathbb{H}$ such that each element of $\mathrm{Sym}(\mathbb{H})$ is an isometry. In this hyperbolic distance, it is easy to check that Euclid's first four axioms hold, but the parallel postulate fails. If the parallel postulate could be derived from the first four axioms, then it would hold in all examples/models of the first four axioms. Thus the existence of the hyperbolic distance on the hyperbolic plane shows that the parallel postulate is independent of the first four axioms.

Why did it take over 2000 years to determine that the parallel postulate is independent? Maybe Euclid and his followers didn't have a clear idea of axiomatic systems and their models – they were trying to axiomatize a particular geometry, and it may not have occurred to them that there might be "other geometries."