Geometry and Symmetry, The Open Door Set
Summer 2009

Here are some topics we didn't have time to discuss. Feel free to contact me at `sr@math.bu.edu` during the school year to discuss these. Have a good year.

Fundamental Domains

P1. Let a group G act on a set X. A set $\mathcal{F} \subset X$ is called a *fundamental domain* for this action if (i) $x \in \mathcal{F}, g \cdot x \in \mathcal{F} \Rightarrow g = e$, (ii) $\cup_{g \in G} g \cdot \mathcal{F} = X$. Find a fundamental domain for the action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{R}^2$ by vector translation. Find a fundamental domain for the action of $\mathbb{Z}_2 = \{\pm 1\}$ on S^2 given by $\pm 1 \cdot \vec{v} = \pm \vec{v}$. Find a fundamental domain for the action of $\mathbb{Q}$ on $\mathbb{R}$.

P2. Show that if $\mathcal{F}$ is a fundamental domain, then $X = \cup_{x \in \mathcal{F}} \mathcal{O}_x$. For a fixed point free group action, show that a maximal set with (i) is a fundamental domain. (Here a set F with (i) is *maximal* if $F \subset G$ and G has (i) implies $F = G$.) Show that a minimal set with (ii) is a fundamental domain. Show that a non-fixed point free group action cannot have a fundamental domain. *Hint:* if $g \cdot x = x$ for some $g \neq e$, and $x \in \mathcal{O}_y$ for $y \in \mathcal{F}$, show that there exists $g' \in G$ with $g' \cdot y = y$.

P3. Let X be a *metric space* (a set with a distance function) and let G act fixed point freely and by isometries. PODASIP: for $x \in X$, the set $\{y \in X : d(x, y) < d(x, g \cdot y) \text{ for all } g \neq e\}$ is a fundamental domain.

By now you should have seen that when a fundamental domain exists, it tends to include only part of its boundary. This is a pain to keep track of, so we'll modify our definition of the fundamental domain by including all the boundary points $\partial\mathcal{F}$ of our previous fundamental domain $\mathcal{F}$. (A point x is in the boundary of $\mathcal{F}$ in a metric space X if every small ball $B_x(\epsilon) = \{y \in X : d(x, y) < \epsilon\}$ centered at x has nontrivial intersection with both $\mathcal{F} \setminus \{x\}$ and $X \setminus (\mathcal{F} \cup \{x\})$.)

P4. Show that for any group action G on a metric space X, there exists a (newly defined) fundamental domain $\mathcal{F}$ with the properties (i) $x \in \mathcal{F}, g \cdot x \in \mathcal{F} \Rightarrow x \in \partial\mathcal{F}$, (ii) $\cup_{g \in G} g \cdot \mathcal{F} = X$. *Hint:* use either the maximal or minimal characterization of fundamental domains from P2. Your proof will use some version of the Axiom of Choice.

The Upper Half Plane and the Unit Disk

In the next five problems, we will show that $\mathcal{F} = \{z \in \mathbb{C} : |z| \geq 1, |\mathrm{Re}(z)| \leq \frac{1}{2}\}$ from Set 11, P3 is a fundamental domain for the action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{H}$. As a bonus, we will show that G of Set 11, P4 satisfies $G = \mathrm{SL}(2, \mathbb{Z})$.

P5. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Show that $\mathrm{Im}(A \cdot z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$. Pick $N > 0$. Show that the number of pairs (c, d) such that $|cz + d| < N$ is finite. Conclude there is an element $A_0 \in G$ such that $A_0 \cdot z$ has maximum $\mathrm{Im}(A_0 \cdot z)$.

P6. Show that some there exists an integer k such that $T^k A_0 \cdot z$ is in $\mathcal{F}$. *Hint:* Find k such that $T^k A_0 \cdot z$ has $|Re(T^k A_0 \cdot z)| \leq 1/2$. If $T^k A_0 \cdot z$ is not in $\mathcal{F}$, show that $|T^k A_0 \cdot z| < 1$. Then show that $Im(ST^k A_0 \cdot z) > Im(A_0 \cdot z)$, a contradiction.

P7. Show that for all $z \in \mathbb{H}$, there is an element $A \in G$ such that $A \cdot z \in \mathcal{F}$. (*Hint:* Multiply A_0 by an appropriate power of T.) Conclude that $\cup_{g \in G}\ g \cdot \mathcal{F} = \mathbb{H}$.

P8. Let z, z' be distinct points of $\mathcal{F}$ with $z' = A \cdot z$ for some $A \in \mathrm{SL}(2,\mathbb{Z})$. Show that either (i) $\mathrm{Re}(z) = \pm\frac{1}{2}$ and $z' = z \pm 1$ or (ii) $|z| = 1$ and $z' = -\frac{1}{z}$. (*Hint:* Suppose $\mathrm{Im}(z') \geq \mathrm{Im}(z)$. What restrictions does this place on c and d? To answer this, use $|\mathrm{Re}(z)| \leq \frac{1}{2}, |\mathrm{Re}(A \cdot z)| \leq \frac{1}{2}$.) Conclude that for z in the interior of $\mathcal{F}$, the only element of $\mathrm{SL}(2,\mathbb{Z})$ that sends z to $\mathcal{F}$ is I.

P9. Show that $\mathrm{SL}(2,\mathbb{Z}) = G$. (*Hint:* Pick $A \in \mathrm{SL}(2,\mathbb{Z})$ and z_0 in the interior of $\mathcal{F}$. Let $z = A \cdot z_0$ and find an element $A' \in G$ such that $A' \cdot z \in \mathcal{F}$. What does this tell you about A?) Conclude that $\mathcal{F}$ is a fundamental domain for $\mathrm{SL}(2,\mathbb{Z})$ acting on $\mathbb{H}$.

*P10. Let T_1 be the rotation by $2\pi/3$ around the vertex $\frac{1}{2} + \frac{\sqrt{3}}{2}$ of $\mathcal{F}$. Show that $\mathrm{SL}(2,\mathbb{Z})$ is isomorphic to the dictionary group generated by S, T_1 with the swear words S^2, T_1^3. (Since S generates a $\mathbb{Z}_2$ and T_1 generates a $\mathbb{Z}_3$, this is written $\mathrm{SL}(2,\mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$.)

P11. Recall the complex linear fractional transformation $B = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ of Set 11, P1, which takes $\mathbb{H}$ to $\mathcal{D}$, the unit disk. Take the "tiling" of $\mathbb{H}$ given by $\mathcal{F}$ and its images under elements of $\mathrm{SL}(2,\mathbb{Z})$. Using B, draw a picture of the corresponding tiling of $\mathcal{D}$.

P12. Take the standard tiling of the Euclidean plane by the unit square, a fundamental domain for $\mathbb{Z} \times \mathbb{Z} \subset \mathrm{Isom}(\mathbb{R}^2)$. Modify the unit square by sticking a little "head" out of the top boundary line and removing a congruent piece from the inside of the bottom of the square. Check that you get a new fundamental domain for $\mathbb{Z} \times \mathbb{Z}$. Using your imagination, you have now tiled the plane with robot shapes. Can you make your fundamental domain more interesting, so that you, like M. C. Escher, can tile the plane with an animal shape?

P13. Repeat P17 for the tiling of $\mathcal{D}$ given in P16. Compare with some Escher prints. Note that Escher's fundamental domains do not hit the boundary of the disk – he is using a group different from $\mathrm{SL}(2,\mathbb{Z})$.

The Wind on the Earth's Surface.

Think of the wind blowing around on the earth's surface, thought of as a bumpy or smooth sphere, at some fixed time. We'll show the famous theorem that there exists a point on the earth's surface where the wind must be still. Another interpretation: cover a sphere with hair, with each hair of some nonxero length. It is impossible to comb it flat – i.e. so that each hair lies in the tangent space to the sphere at its root. To see the connection between the two interpretations, think of the wind at each point of the sphere as determining a velocity vector tangent to the sphere.

P14. A tangent vector to S^2 at a point x is the tangent vector (usually denoted $\dot{\gamma}(0)$) at time zero to a curve $\gamma : (-1,1) \to S^2$ with $\gamma(0) = x$ (draw a picture). Note that $-\gamma$ is another curve on S^2 whose tangent vector is $-\dot{\gamma}(0)$ at $-x$. Now x and $-x$ are identified in $\mathbb{RP}^2$, and note that $\gamma(t)$ is identified with $-\gamma(t)$. So it makes sense to *define* a tangent vector to $[x] \in \mathbb{RP}^2$ to be $\{\pm\vec{v}\}$, where $\vec{v}$ is a tangent vector at x and $-\vec{v}$ is the opposite tangent vector at $-x$.

Take $A \in \mathrm{SO}(3)$. Take the axis of rotation ℓ_A, and choose a unit vector u_A along ℓ_A. Choose unit vectors $\vec{v}_1, \vec{v}_2 \in \ell_A^\perp$ such that $u_A, \vec{v}_1, \vec{v}_2$ obey the right hand rule. Then we can define a unit length vector $\vec{v}_A$ in the tangent space $T_{u_A}S^2$ to S^2 at u_A by $\vec{v}_A = A(\vec{v}_2)$. (Note that $\ell_A^\perp$ is parallel to $T_{u_A}S^2$, so we can identify $\vec{v}_A$ with a tangent vector at u_A.) If we repeat this procedure after choosing $-u_A$ and keeping $\vec{v}_1$ the same, then the new $\vec{v}_2$ is minus the old $\vec{v}_2$. Thus the new $\vec{v}_A$ is minus the old $\vec{v}_A$. Since there is no natural choice of u_A vs. $-u_A$, we conclude that for each $A \in \mathrm{SO}(3)$ and choice of $v_1 \in \ell_A^\perp$, we get a unit length tangent vector $[A(\pm\vec{v}_2)] \in T_{[u_A]}\mathbb{RP}^2$.

Now let A move around in $\mathrm{SO}(3)$. At least for small variations of A, we can choose v_1 to depend continuously on A. So small neighborhoods of A look like *all the unit length tangent vectors in a small neighborhood of* $[u_A]$ *in* $\mathbb{RP}^2$. Patch these neighborhoods together carefully to show that all of $\mathrm{SO}(3)$ looks like (i.e. has a continuous bijection with a continuous inverse to) the space of all unit length tangent vectors to $\mathbb{RP}^2$. *Hint:* We apparently cannot choose $\vec{v}_1$ continuously for all $A \in \mathrm{SO}(3)$. Show that if your choice of $\vec{v}_1$ jumps by a discontinuous amount at some A_0, the corresponding $A_0(\vec{v}_2)$ jumps by the same amount. So the patching process is the same on both spaces.

P15. Now we'll analyze this jumping more precisely. Suppose that S^2 had a "nonzero vector field," a continuously varying nonzero vector X_x in each tangent space T_xS^2. (This is precisely the case if the wind is nowhere still.) Show that this implies that $\mathrm{SO}(3)$ looks like $S^2 \times S^1$. *Hint:* By scaling, we can adjust X_x to have length one; the new vector field is still continuous. For each $A \in \mathrm{SO}(3)$, pick $\vec{v}_2$ so that $A(1,0,0), X_{A(1,0,0)}, \vec{v}_2$ are mutually perpendicular and obey the right hand rule. The last two vectors orient the plane $A(1,0,0)^{\perp}$, so we can unambiguously calculate the angle θ_A between $X_{A(1,0,0)}$ and $A\vec{j} \in A(1,0,0)^{\perp}$. Thus the map $A \mapsto (A(1,0,0), \theta_A)$ from $\mathrm{SO}(3)$ to $S^2 \times S^1$ is well defined.

From class, we know that $\mathrm{SO}(3)$ looks like $\mathbb{RP}^3$. Show that $\mathrm{SO}(3)$ has a closed loop which is not shrinkable, but going twice around the loop is shrinkable. (*Hint:* Consider a path in S^3 from the north pole to the south pole along a longitude line. Take the image of this path in $\mathbb{RP}^3$.) Show that $S^2 \times S^1$ has no such loop. (*Hint:* If $\gamma : [0,1] \to S^2 \times S^1$ were such a loop (with $\gamma(0) = \gamma(1)$), write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, so that $\gamma_2(t)$ is a loop on S^1. Show that $\gamma_2(t)$ is not shrinkable, but going twice around γ_2 is. That's hard to believe.) Conclude that S^2 has no continuous nonzero vector field.

P16. Bonus: conclude that (unlike S^1 and S^3) S^2 has no continuous group structure (i.e. a group structure with the multiplication map $\cdot : S^2 \times S^2 \to S^2$ and the map $g \mapsto g^{-1}$ continuous). *Hint:* Assume such a multiplication exists. Let X_e be a nonzero vector in T_eS^2. Let $\gamma : (-1,1) \to S^2$ be a curve on S^2 with $\gamma(0) = e$ and tangent vector $\dot{\gamma}(0) = X_e$. For $g \in S^2$, show that the curve $\gamma_g(t) = g \cdot \gamma(t)$ passes through g at time zero and has $\dot{\gamma}_g(0) \neq \vec{0}$. Subhint: To see that this last vector is nonzero, show that the map taking tangent vectors at e to tangent vectors at g given by letting X_e vary is a bijection – the inverse is given by applying g^{-1} to curves through g. So a group structure on S^2 would give a nonzero vector field on S^2.