Geometry and Symmetry Project Suggestions
Summer 2009

Here is a list of possible projects for the course. All of these projects have lots of variations, and there are many more project topics possible.

1 The Symmetries of the Regular n-gon

How many symmetries are there of an equilateral triangle? What is the group structure on the symmetry group? What are all the subgroups and what are all the normal subgroups? What are the corresponding quotient groups? Which symmetries are orientation preserving? What is the stabilizer subgroup of a vertex? What happens if you replace the triangle with a square, a regular pentagon,..., a regular n-gon? Are some symmetry groups subgroups of other symmetry groups?

What is the relation between the regular n-gon and the nth roots of unity? Can you *represent* the symmetry group of the n-gon by elements of $\mathbb{C}$?

2 The Symmetries of the Platonic Solids

Define and prove that there are only five Platonic solids (tetrahedron, cube, octahedron, etc.) How big is the symmetry group of each Platonic solid? Can you identify the group structure of each symmetry group? Can you identify the group structure of each stabilizer subgroup? Which symmetries are orientation preserving, and which are orientation reversing? Can you find symmetry groups of some regular n-gons inside the symmetry groups of the Platonic solids?

3 Coloring Stuff

How many different ways can you color a bracelet of 7 beads such that 3 beads are blue and 4 beads are green? What does different mean? How many different ways can you color the vertices of a cube if 4 vertices must be red and 4 must be yellow? Again, what does different mean? What is the relation between these coloring problems and the symmetry group of these figures? To answer these and related questions, you'll have to learn about Burnside's Theorem and Polya's theory of enumeration.

4 Symmetry Groups of Finite Affine and Projective Planes

Take the finite affine plane $A_p = \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. Can you write the symmetry group as a set of 2×2 matrices? What are the entries of the matrices? If you want to pinpoint a particular symmetry, how many points do you have to let it act on? How many symmetries are there of A_p? What is the stabilizer subgroup of a point? The symmetry group takes lines to lines. What is the stabilizer subgroup of a line?

Put A_p inside its associated projective plane P_p. Are the symmetries of A_p also symmetries of P_p? Are there new symmetries? What are the stabilizer subgroups of points and of lines?

5 Billiard Trajectories

Take a billiard table of dimensions $m \times n$, where $m, n \in Z^+$. Put a billiard ball at a spot on the table and hit it with the cue stick. Which trajectories close up so that the ball travels the same path over and over? How does this depend on your choice of initial position for the ball?

How does this depend on your choice of m, n? Can the ball return to its starting point but with a non-zero angle to the original path? If a path never closes up, does it hit every point on the table? Can it miss some large portion of the table? What if $m, n \in \mathbb{Q}^+$? What if $m = 1, n = \pi$?

6 The Geometry of Perspective Drawing

Given a row of telephone poles of height h and spaced d units apart, how can you accurately draw them "receding into the distance" on a canvas? Can you give simple rules to teach mathphobe artists how to do this?

Place an equilateral triangle on the floor. How would you draw it accurately on your canvas? What about a square, a pentagon, a regular n-gon?

What is the meaning of the "vanishing line" in perspective drawing in terms of projective geometry?

A sphere like the earth's surface is drawn as a circle on your canvas. The drawn latitude and longitude lines look like ellipses. Are they really ellipses? How would you accurately draw these latitude/longitude lines on your canvas?

7 Conics

Ellipses, hyperbolas and parabolas are conics, i.e. curves cut out by intersecting a cone with a plane. Are these all the conics? The standard conics have quadratic equations. Is every solution set of a quadratic equation a conic? A conic extends to a curve in projective space by possibly adding some "points at infinity." Given two conics, find a projective transformation taking one to the other. What theorems about conics reduce to theorems about circles?

8 Escher's Drawings and Hyperbolic Geometry

Some people like Escher's drawings; I don't. Regardless, many of Escher's drawings use discrete subgroups of the symmetry group of hyperbolic geometry. Pick a drawing and describe the subgroup in as detailed a way as possible. Find interesting subgroups of this symmetry group. Can you describe the stablizer subgroup of a point?

9 Analytic Finite Geometry

A point in the plane $\mathbb{R}^2$ has coordinates (x, y) with $x, y \in \mathbb{R}$ and lines are given by linear equations – that's the analytic approach to Euclidean geometry. Similarly, a point is the finite affine plane $\mathbb{Z}_p \times \mathbb{Z}_p$ has coordinates (m_0, n_0) with $m_0, n_0 \in \mathbb{Z}_p$, and lines satisfy linear equations. (We can also put homogeneous coordinates on the corresponding projective plane.) Note that $\mathbb{R}$ and $\mathbb{Z}_p$ have a lot of algebraic structure: they are *fields*. In fact, for any field k, we can form the affine plane $k \times k$ with coordinates.

Here's the converse problem: given an affine or projective plane, can we find a field k such that the points are given by ordered pairs of elements in k and the lines are given by linear equations with coefficients in k? When is the field commutative?

10 Spherical Geometry and the Geometry of Surfaces

The geometry of the sphere is supposed to model the geometry of the earth's surface, with line segments given by arcs of great circles. Is there a shortest path between any two points? What about if we delete one point from the sphere's surface? What is the sum of the angles in a triangle made up of shortest paths? Is there a formula for the area of a triangle similar to the Law of Cosines?

But come on, the earth is not a perfect sphere — in fact, it's not even a sphere! (Think of those natural arches out west.) On a bumpy surface like the earth, is there a shortest path between two points? What can we say about areas of triangles? (This part requires calculus.)

11 Hyperbolic Geometry and Number Theory

Take the set F of all points $z = x+iy$ in the upper half of the plane (i.e. $y > 0$) which lie outside (or on) the unit circle and have $|x| \leq 1/2$. The group of hyperbolic transformations $PSL(2, \mathbb{R})$ tiles the upper half plane with images of F. Which points on the real axis are boundary points of tiles? Relate this to the Farey sequence.

12 Linear Transformations and Number Theory

Prove that the n$^{\text{th}}$ Fibonacci number satisfies

$$\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(-\frac{-1+\sqrt{5}}{2}\right)^n\right)$$

and compare it to the proof using 2×2 matrices. Can you generalize this to other identities? Look at the linear algebra proof that there are infinitely many solutions to Pell's equation $x^2 - dy^2 = 1$. Can you generalize this?