Notes for Week I
Spring 2011

The real numbers $\mathbb{R}$

What is 5 to a child and to an adult? What is $\frac{3}{7}$? What is $\sqrt{2}$? What is π?

Do we have to define 5? Do we have to define the natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$? Do we have to define zero? Do we have to define $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$? Do we have to define $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$? Do we have to define $\mathbb{R}$? Is the number line the same as $\mathbb{R}$?

Fractions

Why can't you divide by zero?

Why are the following identities valid:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}; \quad \frac{a}{c} = \frac{ad}{cd}; \quad \frac{a}{c} \times \frac{b}{d} = \frac{ac}{bd}; \quad \frac{a}{c} \div \frac{b}{d} = \frac{ad}{bc}.$$

For $a, b, c, d > 0$, why is $\frac{a}{c} < \frac{b}{d}$ iff $ad < bc$?

Place value and arithmetic algorithms

Why is

$$17543.27 = 1 \cdot 10^4 + 7 \cdot 10^3 + 5 \cdot 10^2 + 4 \cdot 10 + 3 + 2 \cdot 10^{-1} + 7 \cdot 10^{-2}?$$

Could we replace 10 by 8? By 12? By 3? What would be the advantages and disadvantages of using base 2 vs. base 12 vs. base 13 vs. base 60?

Why does the standard algorithm for multiplication of two or three digit numbers work? A student turns in the following:

$$\begin{array}{r} 35 \\ \times\ 26 \\ \hline 30 \\ 180 \\ 100 \\ +\ 600 \\ \hline 910 \end{array}$$

What was this student doing? Is this ok?

Dividing 2749 by 13 gives 211 with a remainder of 6. Why does the standard algorithm for division work?

Place Value and Units

Listen to http://www.youtube.com/watch?v=zN9LZ3ojnxY&feature=related

Modular arithmetic

"Clock arithmetic" is the same as arithmetic mod 12. Notation: $a \equiv b \pmod{12}$ or $a = b$ in $\mathbb{Z}_{12}$ means 12 divides $a - b$. Why is this the same as "10 pm today is the same as 10 am two years ago, as far as my clock is concerned." In $\mathbb{Z}_7$, what is 25, -3, $\sqrt{2}$, $\sqrt{-3}$?

In $\mathbb{Z}_5$, is $x + y = 4$ the same equation as $2x + 2y = 8$? What about in $\mathbb{Z}_4$?

Do the multiplication and division algorithms of ordinary integers hold in $\mathbb{Z}_n$ for any $n \in \mathbb{N}$?

The role of proof at this level

What constitutes a rigorous proof in mathematics? Can we construct a machine to check proofs for accuracy? What level of rigor is appropriate for this week's topics?

Estimation

For each question, ask yourself if using your calculator will help or not before you get to work.

(a) Given two real numbers x, y, when is $x + y > xy$? Determine some easy cases, then try to work towards the more difficult cases.

(b) Is the infinite sum $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$ (what does this mean?) finite or infinite?

(c) What number less than 100 has the most factors? What number less that 1000? What number less than 10,000?

The role of the physical world

For each of the topics above, how much is their development motivated by attempts to quantify the physical world?