ST 112
Notes for Week II
Spring 2011

Algebraic algorithms

How do we prove e.g. $x + y = y + x$? How convincing is a picture proof? Do we prove it? We have the same three questions for $x + 0 = x$, $1 \cdot x = x$, $(x + y) + z = z + (y + z)$, etc. Why isn't $0 \cdot x = 0$ listed here?

The distributive law is $a(b + c) = ab + ac$. Draw a picture proof. From the distributive law, we can derive FOIL: $(a + d)(b + c) = ab + ac + db + dc$. Is a picture proof helpful here? Which rule is more basic?

(!) The multiplication and division algorithms are applications of the distributive law/FOIL.

Variables

Remember that we're just considering real numbers. Consider the equations $x + 2 = 5$ and $x + 0 = x$. Is the variable x the same or different in these equations? Are x, y the same in $x + y = y + x$ and in $xy = 1$ and in $x^2 + y^2 = 0$ and in $x^2 + y^2 = -1$?

Sets

We have an intuitive notion of sets, like the set of all students in this class, the set of all numbers greater than 4, the set of all real numbers whose square is -4. What is a precise notion of a set?

Set notation: $\{\text{students in this class}\}, \{x \in \mathbb{R} : x > 4\} = \{x | x > 4\}, \{x : x^2 = -4\}$. If X, Y are sets, then the product set is $X \times Y = \{(x, y) : x \in X, y \in Y\}$. Abbreviating $\mathbb{R} \times \mathbb{R}$ by $\mathbb{R}^2$, we see that $\mathbb{R}^2$ is the coordinate/analytic description of the plane. Similarly, $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ describes points in space. What does $\mathbb{R}^4, \mathbb{R}^5, ...$ describe?

Unions of sets $X \cup Y$; intersections of sets $X \cap Y$.

Sets of sets: Let x be a person, and let $\mathcal{C}_x = \{y : y \text{ is a citizen of the same country as } x\}$ be the country of x. Then $\mathcal{C} = \{\mathcal{C}_x : x \text{ is a person}\}$ is the set of all countries, and is a set of sets.

Functions

Intuition: Think of a set X as all the possible inputs to an experiment and another set Y as the set of all the possible outputs. Think of a function f as a particular experiment, which assigns to every possible input from X an output from Y. Shorthand notation for functions: $f : X \to Y$. If a particular $x_0 \in X$ is the input for the experiment labeled f and y_0 is the output for this experiment, we write $f(x_0) = y_0$. X is called the *domain* of f, and Y is called the *range* of f.

Two reasonable conditions on experiments: (i) Any input from X must be assigned to an output in Y; (b) Any input from X can be assigned to only one output in Y. So a function $f : X \to Y$ is a rule/experiment which assigns to each element of X a unique value in Y.

Examples: $X = \mathbb{R}, Y = \mathbb{R}^+, f(x) = x^2 + 1$. What if we keep X and f the same, but change Y to $\mathbb{R}$? What if $X = Y = \mathbb{R}, f(x) = \sqrt{x}$?

(*) The UN game: Let X be the set of all people. The choice of a UN is a function $f : \mathcal{C} \to \cup_{x\in X}\mathcal{C}_x$ with $f(\mathcal{C}_x) \in \mathcal{C}_x$. Does such a function exist for our difficult UN game in Set 1, P.15?

Composition of functions If $f : X \to Y$ and $g : Y \to Z$, we set $g \circ f : X \to Z$ by $(g \circ f)(x) = g(f(x))$. This just means "Apply experiment f, and then apply experiment g to the result." For $X = Y = Z = \mathbb{R}$, $f(x) = x^2$, $g(x) = 2x + 1$, what is $f \circ g$? What is $g \circ f$?

(!) Composition of functions is not commutative in general.

For $f(x) = x^2$ and $g(x) = \sqrt{x}$, what is $f \circ g$? What is $g \circ f$?

(!) Composition of functions may not be defined.

The role of proof at this level

What level of rigor is appropriate for this week's topics?

Units

Why are there no units in the discussion of algebraic algorithms and functions?

The role of the physical world

For each of the topics above, how much is their development motivated by attempts to quantify the physical world?