ST112
Notes for Week III
Spring 2011

Linear and nonlinear functions

A function $f : \mathbb{R} \to \mathbb{R}$ is *linear* if it is of the form $f(x) = ax + b$ for some $a, b \in \mathbb{R}, a \neq 0$. These are precisely the functions of constant (nonzero) slope. If the x-axis represents time and the y-axis represents distance traveled of a particle on a line, what type of "histories" are represented by linear functions?

Nonlinear functions are "everything else": quadratic, cubic, polynomial functions, trig functions, log and exponential functions, the absolute value function, functions like $y = \frac{1}{x}$, and "random" functions.

Inverse functions

$f : \mathbb{R} \to \mathbb{R}$ has inverse function $g : \mathbb{R} \to \mathbb{R}$ if $(f(x_0) = y_0 \Leftrightarrow g(y_0) = x_0)$.

This is only possible if f is a bijection. Examples include linear functions (the inverse is also a linear function – Why?), certain cubics, but no quadratics. When can we look at a graph of a function and tell if it has an inverse? How does the graph of a function compare to the graph of its inverse?

Everyone knows that $f(x) = x^2$ has inverse function $g(x) = \sqrt{x}$. But we have to be careful about the domain and range of f and g. In general, the existence of an inverse function depends on carefully specifying the domain of f.

Exponential and log functions

A function $f : \mathbb{R} \to \mathbb{R}$ grows/decays exponentially if the rate of increase of f at x_0 is proportional to the value $f(x_0)$, for all $x_0 \in \mathbb{R}$. We write this as $f'(x_0) = k \cdot f(x_0)$ with exponential growth if $k > 0$ and exponential decay if $k < 0$. Examples include $f(x) = C \cdot b^x$ for any $b > 0$ and $C \in \mathbb{R}$. Are there other examples? And what is $\pi^{\sqrt{2}}$?

An exponential function is a bijection $f : \mathbb{R} \to \mathbb{R}^+$ (for $C > 0$), so it has an inverse function $g : \mathbb{R}^+ \to \mathbb{R}$. By definition, the inverse function to $f(x) = b^x$ is denoted $g(x) = \log_b x$. So

$$b^x = y \Leftrightarrow \log_b y = x.$$

What are the domain and range of $\log_b$?

Growth rates of functions

What does it mean to say that f is growing faster than g (as $x \to \infty$)? How about $f(x) > g(x)$ for $x \gg 0$? How about $\frac{f(x)}{g(x)} > 2$ for all $x \gg 0$? How about $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$? Can we formulate what it means for f to decay faster than g as $x \to \infty$?

Which grows faster, $f(x) = 10^{-100}2^x$ or $g(x) = 10^{100}x^{9000}$?

Why do polynomials only have a finite number of roots?

Systems of equations

We can solve the systems

$$\begin{array}{rcl} x+y &=& 2 \\ 3x-4y &=& -1 \end{array}, \quad \begin{array}{rcl} x+y &=& 2 \\ 3x+3y &=& 2 \end{array}, \quad \begin{array}{rcl} x+y &=& 2 \\ 3x+3y &=& 6 \end{array}$$

algebraically. What is happening geometrically? What is happening geometrically when we solve 3 equations in 3 unknowns? 2 equations in 3 unknowns? 3 equations in 2 unknowns? m equations in n unknowns?

What happens if we try to solve systems of nonlinear equations like

$$\begin{array}{rcl} 2x-3y^2+3^z &=& 7 \\ xyz^2+\log_{10}(y+2z) &=& 4 \end{array}$$

Estimating solutions

I'm sure that the equation $x^3-9x^2+7x+3=5$ has a solution x_0 with $-100 < x_0 < 100$. How do I know that? How can I make my estimate more accurate? What is the tradeoff between a more accurate estimate and the time spent searching?

The role of the physical world

What physical processes are modeled by linear functions? Quadratic functions? Exponential functions? Log functions?