Example of Growth Rates

According to Set 3, P11, two functions $f, g : \mathbb{R}^+ \to \mathbb{R}$ *grow at the same rate* if there are positive constants C_1, C_2 with

$$C_1 \cdot |f(x)| \leq |g(x)| \leq C_2 \cdot |f(x)| \tag{1}$$

for all $x \gg 0$. (This means "for all x large enough." More precisely, this means that there is an $x_0 > 0$ such that for all $x > x_0$, equation (1) holds.)

In terms of the graphs of f and g, this means that for $x \gg 0$, the graph of $|g(x)|$ lies between two scaled versions of the graph of $|f(x)|$, namely between the graph of $C_1|f(x)|$ and $C_2|f(x)|$.

Here are three worked examples that should give you some insight into what this means.

An example of functions that grow at the same rate

$f(x) = x$, $g(x) = 2x$. Since $x > 0$, $|f(x)| = f(x), |g(x)| = g(x)$, so we can drop the absolute value signs. Set $C_1 = 1, C_2 = 2$. Then we certainly have

$$C_1 \cdot |f(x)| \leq |g(x)| \leq C_2 \cdot |f(x)|$$

for all $x > 0$, as this equation just says

$$1 \cdot x \leq 2x \leq 2 \cdot x.$$

In terms of the graphs of these functions, this says that the graph of $g(x) = 2x$ lies between the graphs of $f(x) = x$ and $2f(x) = 2x$.

A harder example of functions that grow at the same rate

Let $f(x) = x^2 - 3, g(x) = 4x^2$. It's pretty clear that for x large enough, the graph of g lies between two scaled versions of f, but we'll show this in detail.

To find C_1 : We have $|f(x)| \leq |g(x)|$ for all $x \gg 0$.

Proof: $|x^2 - 3| < x^2 + 3$ for $x > \sqrt{3}$ (as then $|x^2 - 3| = x^2 - 3 < x^2 + 3$). Also, $x^2 + 3 < 4x^2$ for all $x > 0$ (as $x^2 + 3 < 4x^2 \Leftrightarrow -3 < 3x^2$, which is true). Therefore

$$|f(x)| = |x^2 - 3| < x^2 + 3 < 4x^2 = |4x^2| = |g(x)|.$$

So far, we can take $C_1 = 1$ and $x_0 = \sqrt{3}$.

To find C_2: If we try $C_2 = 4$, we want to show $4x^2 \leq 4|x^2 - 3|$. The quadratic terms look ok, but the constant term -3 causes problems. So let's bump C_2 up a little, say $C_2 = 5$. For $x > \sqrt{3}$, we have $|f(x)| = |x^2 - 3| = x^2 - 3$, so

$$4x^2 < 5(x^2 - 3) \Leftrightarrow 15 < x^2 \Leftrightarrow x > \sqrt{15}.$$

So we get the estimate we want for $C_2 = 5, x_0 = \sqrt{15}$.

Since we have two choices for x_0, if we take the larger choice, we will get the estimates we want in both cases. In summary, for $C_1 = 1, C_2 = 2, x_0 = \sqrt{15}$, we get

$$C_1 \cdot |f(x)| \leq |g(x)| \leq C_2 \cdot |f(x)|$$

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An example of functions that do not grow at the same rate

Let $f(x) = 4x, g(x) = x^2$. It's again pretty clear that $g(x)$ is growing faster – there is no way to sandwich the parabola between scaled versions of the linear function $f(x)$.

To prove this, assume that for any fixed choice of $x_0 > 0$, there exists a C_2 such that $g(x) \leq C_2 f(x)$ for all $x > x_0$. (These functions are positive, so we can drop the absolute value signs.) This means that $x^2 < 4C_2 x$ for all $x > x_0$. But

$$x^2 < 4C_2 x \Leftrightarrow x^2 - 4C_2 x < 0 \Leftrightarrow x(x - 4C_2) < 0 \Leftrightarrow x - 4C_2 < 0 \Leftrightarrow x < 4C_2.$$

(Here I used that $x > 0$, so $x(x - 4C_2) < 0 \Leftrightarrow x - 4C_2 < 0$.) Thus the inequality we need, $x^2 < 4C_2 x$, can only hold for $x < 4C_2$, so it cannot hold for all $x > x_0$. This contradiction shows that these functions do not grow at the same rate.