ST112, Problem Set 1
Spring 2011

In all problems, be sure to show all your final work.

Notation: $\mathbb{N} = \{1, 2, 3, ...\}, \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}, \mathbb{R}$ is the set of all real numbers, $\mathbb{Z}_d = \{0, 1, 2, ..., d\}$. $\mathbb{N}$ is called the set of *natural numbers*, and $\mathbb{Z}$ is called the set of *integers*.

Exploration: Solving equations in various number systems

P1. Consider the following sets with the usual addition and multiplication operations: $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{Z}_2, \mathbb{Z}_5, \mathbb{Z}_8, \mathbb{Z}_{31}$. In which of these systems does the equation $x + 5 = 4$ have a unique solution? In which of these systems does the equation $2x + 5 = 4$ have a solution? In which of these systems does the equation $2x + 4 = 8$ have a unique solution?

P2. For fixed $a, b, c, d \in \mathbb{N}$, make a precise conjecture about when the equation $ax + b = c$ has a unique solution in $\mathbb{Z}_d$. How much of your conjecture can you prove?

P3. Find all the solutions of $x^2 = 4$ for each of the sets in P1. Find all the solutions of $x^2 = 0$ for each of these sets. Find all the solutions of $x^2 = -6$ for each of these sets.

P4. Which is easier to solve: (a) $x^3+2x^2-3x+5 = 0$ in $\mathbb{R}$, or (b) $x^5-9x^4+2x^3-5x^2+9x-7 = 0$ in $\mathbb{Z}_{31}$? Explain your answer, but don't look for the solutions!

Arithmetic

P5. Make up a word problem, suitable for ten year olds, whose solution is given by solving $\frac{3}{4} \times \frac{2}{3}$. Make up another word problem, again suitable for ten year olds, whose solution is given by solving $\frac{3}{4} \div \frac{2}{3}$.

P6. Write a clear explanation, suitable for ten year olds, for the "invert and multiply" algorithm for dividing a fraction by a fraction. Try to do this in at most 200 words.

P7. I don't believe $\frac{1}{3} = .3333333....$, because $3 \times \frac{1}{3} = 1$, but $3 \times .3333333... = .9999999....$ Write a convincing explanation, suitable for adults, that $\frac{1}{3} = .3333333...$

Estimation

P8. Estimate the number of raindrops that fall on a deck that is 10 feet by 12 feet during a steady rain that lasts one hour. Show the assumptions you make to get your estimate, and use them to give a reasonable estimate of the range of error in your answer.

P9. Find an upper and lower estimate for the number of non-overlapping squares with sides 1/10 inch that you can pack inside a circle of radius 6 inches. Repeat this for squares with sides 1/100 inch, and then for squares with sides of 1/1000 inch. What conjectures can you make?

P10. According to Wikipedia[1], there are 19,100,000 people in the greater New York City area and 4,400,000 people in the greater Boston area. Are there more piano tuners in greater New York City or cabdrivers in greater Boston? Give a convincing explanation for your answer.

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[1] http://en.wikipedia.org/wiki/New_York_City, http://en.wikipedia.org/wiki/Greater_Boston

The UN Game

P11. Let's say that two people are *compatriots* if they are citizens of the same country. Denote people by $x, y, z, ...$, and write $x \sim y$ if x and y are compatriots. Which of the following rules hold? (Warnings: Some people have dual citizenships. Some people are stateless citizens, i.e. they are citizens of no recognized country.)

(i) For all x, $x \sim x$.

(ii) For all x, y, $x \sim y$ iff [i.e.if and only if] $y \sim x$.

(iii) For all x, y, z, if $x \sim y$ and $y \sim z$, then $x \sim z$.

If there are no dual citizens and no stateless citizens, show that rules (i) – (iii) hold.

P12. Assuming there are no dual citizens and no stateless citizens, we form a UN by picking one person from each country. If there are N countries labeled $A_1, A_2, ..., A_N$ and A_1 has n_1 citizens, A_2 has N_2 citizens, etc., how many possible UNs can be formed?

P13. Let's say that two real numbers x, y are *compatriots* iff $x - y \in \mathbb{Z}$, and write $x \sim y$ if x and y are compatriots. Check that rules (i) – (iii) hold. For a fixed $x_0 \in \mathbb{R}$, let $\mathcal{C}_x$ denote the set of all the compatriots of a fixed $x \in \mathbb{R}$, and call $\mathcal{C}_x$ the *country* of x. Show that every $x \in \mathbb{R}$ belongs to a unique country. What are $\mathcal{C}_7, \mathcal{C}_9, \mathcal{C}_\pi$? A set $A \subset \mathbb{R}$ is called a UN if there is exactly one element in A from each country. How many possible UNs are there? Write down three different UNs explicitly.

P14. Now say that $x, y \in \mathbb{R}$ are compatriots if 5 divides $x - y$ (with no remainder). For example, π and $15 + \pi$ are compatriots. What are $\mathcal{C}_7, \mathcal{C}_9, \mathcal{C}_\pi$? How many possible UNs are there? Write down three different UNs explicitly

P15. Now sat that $x, y \in \mathbb{R}$ are compatriots if $x - y$ is rational (i.e. $x - y = \frac{a}{b}$ for some integers a, b). What are $\mathcal{C}_7, \mathcal{C}_9, \mathcal{C}_\pi$? How many possible UNs are there? Write down three different UNs explicitly.

Challenge Problem

P16. In my basketball game, all ordinary baskets count for 5 points and foul shots count for 3 points. What are the possible total scores I can get at the end of a game? What if ordinary baskets count for 7 points and foul shots count for 4 points? What if ordinary baskets count for m points and foul shots count for n points, where m, n are positive integers?