ST112, Problem Set 2
Spring 2011

Exploration: Picture proofs

P1. We often teach children that $x \cdot y = y \cdot x$, at least for $x, y \in \mathbb{N}$, by drawing an $x \times y$ box with grid lines, and explaining that we can calculate the area either by taking x rows of y boxes or y rows of x boxes. This seems to work well in the classroom. Do you accept this as a proof that $x \cdot y = y \cdot x$ for $x, y \in \mathbb{N}$? Why or why not?

P2. Can you draw a picture proof that $x \cdot y = y \cdot x$ for $x, y \in \mathbb{Q}$? Can you prove this algebraically, assuming $x \cdot y = y \cdot x$ for $x, y \in \mathbb{N}$? Watch out for the case when (at least) one of x or y is zero or negative. If you assume $x \cdot y = y \cdot x$ for $x, y \in \mathbb{Q}$, can you prove $x \cdot y = y \cdot x$ for $x, y \in \mathbb{R}$?

P3. Draw a picture proof that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

P4. Draw a picture proof that $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$.

P5. Draw a picture proof that $1 + 3 + 5 + ... + (2n-1) = n^2$.

Sets

X is a *subset* of Z, written $X \subset Z$, if every element of X is an element of Z. For example $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. For $X \subset Z$, the *complement* of X in Z, written X^c, is $\{y \in Z : y \notin X\}$.

P6. What is the complement of $\mathbb{N}$ in $\mathbb{Z}$? What is the complement of $\mathbb{Z}$ in $\mathbb{Q}$? What is the complement of $\mathbb{Q}$ in $\mathbb{R}$? Find specific elements in each of these complements, and prove that these elements really are in the complements.

P7. Say $X, Y \subset Z$. Prove that $(X \cup Y)^c = X^c \cap Y^c$. (Hint: To prove an equality of sets like $A = B$, typically we prove $A \subset B$ and $B \subset A$. To show $A \subset B$, you have to show that every element of A is an element of B; a similar proof is needed to show $B \subset A$.)

P.8 State and prove a formula similar to P7 for $(X \cap Y)^c$.

P9. Which of the "distributive laws" $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ is valid? Prove your answer. (This means give a proof if the result is correct, and find a counterexample if the result is not always correct.)

Functions

A function $f : X \to Y$ is *injective* or *into* or *one-to-one* if different inputs yield different outputs: formally, f is injective if $f(x_1) = f(x_2)$ iff $x_1 = x_2$. f is *surjective* or *onto* if every possible output really is the output for some input: formally, f is surjective if for all $y \in Y$, there exists an $x \in X$ with $f(x) = y$. Finally, f is *bijective* if it is both injective and surjective.

P10. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is injective but not surjective. (If you can't find an explicit example, draw a convincing picture of the graph of such a function.) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is surjective but not injective. Give an example of a function which is neither injective nor surjective. Give an example of a function which is bijective.

P11. Prove or disprove and salvage if possible (PODASIP): every linear function $f : \mathbb{R} \to \mathbb{R}$ (so $f(x) = mx + b$ for some $m, b \in \mathbb{R}, m \neq 0$) is bijective.

P12. PODASIP: every quadratic function $f\mathbb{R} \to \mathbb{R}$ (so $f(x) = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}, a \neq 0$) is bijective.

Do one of the following two problems.

P13a. Let $f(x) = 2x - 3, g(x) = \sqrt{x^2 + 5}, h(x) = \frac{x}{x^2+1}$, with all domains and ranges equal to $\mathbb{R}$. Compute $(f \circ g) \circ h$ and $f \circ (g \circ h)$. Compute $(h \circ f) \circ g$ and $h \circ (f \circ g)$.

P13b For any functions $f : X \to Y, g : Y \to Z, h : Z \to W$, show that $(h \circ g) \circ f = h \circ (g \circ f)$. (Composition of functions is always associative!)

Challenge Problems

P14. Fix $n \in \mathbb{N}$, and say X has n elements. How many subsets does X have? (Hint: the answer works out better if (i) you remember that $X \subset X$, and (ii) we assume the existence of the *emptyset* $\emptyset$, the set with no elements. We have $\emptyset \subset X$, since we cannot find an element of $\emptyset$ which is not in X.)

P15. PODASIP: (a) If $Y \subset X$ and $f : X \to Y$ is injective, then $X = Y$. (b) If $Y \subset X$ and $f : Y \to X$ is surjective, then $X = Y$.

P16. Let X have n elements and Y have m elements. How many distinct functions are there from X to Y? How many injective functions are there? How many surjective functions are there? How many bijections are there?

*P17. PODASIP: There does not exist a surjective function $f : \mathbb{R} \to \mathbb{R}^2$.