ST112, Problem Set 3
Spring 2011

On many of these problems, you'll have to use a calculator. Be sure to state your formulas clearly, with citations as needed. Be sure to clearly state which results are exact (up to calculator round off error), and which are best guesses as a result of calculator estimates.

Exploration: Decay rates and shopping

P1. True story: a friend went shopping at a department store with a bunch of 10% off coupons. She bought 6 items. At the register, the clerk totaled her items, and then hit the 10% off button 6 times. Why was this a mistake? Assuming the clerk had only a high school education, how would you succinctly point out his error?

P2. Let's say the total bill was $100, and the clerk kept hitting the 10% off button. How many times would it take before my friend owed less than a penny? How many times until she truly owed the store nothing? Would the store ever owe her money?

P3. No longer a true story: The person behind my friend in line said to her, "Listen, I have the same coupons, and I want to buy 5 items. If you don't point out the clerk's error, I'll give you $100 and I'll still get my stuff for $100 cheaper than if the clerk rang up the bill correctly." What was the minimum dollar value of this person's purchases?

P4. Let's say my friend bought k items, each costing $\$x$, and that the clerk hit the 10% off button k times. Is my friend better off buying one item, ten items, or a million items? Your answer may depend on the value of x.

Exploration: Growth rates and loans

P5. You need to borrow $10,000 to pay for your executive box at the Super Bowl. Your bank will offer you a loan at 5% for 10 years with the interest compounded monthly. What is the amount of interest you'll owe after 10 years? (You can look up the formula for computing interest, but be sure to cite your source.)

P6. Your bank offers another loan plan: "half the interest, twice as long to pay off!" In other words, you can get an interest rate of 2.5% on a loan for 20 years, compounded monthly. Is this a better deal?

P7. Your bank offers the Math Afficionado Loan Plan: the interest rate is 5/x% on a loan period of 10x years, compounded monthly, where you get to choose x. What x should you choose to minimize your interest payments?

More on growth and decay rates

P8. I read that health care costs have grown exponentially over the past 30 years, but an economist friend says that the costs have been actually growing like a quadratic polynomial. Who is right?

P9. Draw a rough but overall accurate graph of $f(x) = \log x$ for $\log = \log_{10}$. Do the same for $f^2(x)$, i.e. $(f \circ f)(x) = f(f(x))$. Do the same for $f^{100}(x)$. Is there a function which grows to infinity as $x \to \infty$ "as slowly as possible"?

P10. What is the inverse function of $f^2(x)$ in P9? What is the inverse function of $f^{100}(x)$? Is there a function which grows to infinity as $x \to \infty$ "as quickly as possible"?

P11. Let's say that two functions $f, g : \mathbb{R}^+ \to \mathbb{R}$ *grow at the same rate* if there are positive constants C_1, C_2 with

$$C_1 \cdot |f(x)| \leq |g(x)| \leq C_2 \cdot |f(x)| \tag{1}$$

for all $x \gg 0$. (This means "for all x large enough." More precisely, this means that there is an $x_0 > 0$ such that for all $x > x_0$, equation (1) holds.) Let $f(x) = x^2 - x$. Write an explicit function g that grows at the same rate as f. Write an explicit function h that does not grow at the same rate as f. Prove that your functions have these properties.

P12. Let $g(x) = x$ and $h(x) = x^2$ for $x \in \mathbb{R}^+$. Find a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(x) \leq f(x) \leq h(x)$, where f does not grow at the same rate as g and f does not grow at the same rate at h.

P13. Which of the following functions grow at the same rate? (a) $f(x) = \sin(x)$, (b) $g(x) = \frac{1}{x}$, (c) $h(x) = 2^x$, (d) $j(x) = -5 \cdot 10^6 x^2 - 10^9 x$, (e) $k(x) = 10^{-10} x^2$, (f) $\ell(x) = 16x + \frac{1}{x}$, (g) $m(x) = \frac{\sin(x)}{x^2+1}$, (h) $n(x) = 2^{2x}$, (i) $p(x) = \frac{1}{x^2}$, (j) $q(x) = -6x + 3$, (k) $r(x) = 2^{\sqrt{x}}$.

Challenge Problems: Growth/decay rates of functions

P14. Say that a function $f(x)$ grows faster than polynomially if for any polynomial $p(x)$, $p(x) \leq f(x)$ for $x \gg 0$ (i.e. for all x sufficiently large). Show that 2^x grows faster than polynomially.

P15. Say that a function $g(x)$ grows subexponentially if for any $b > 1$, $g(x) \leq b^x$ for $x \gg 0$. Show that any polynomial grows subexponentially.

P16. Find a function that grows faster than polynomially but subexponentially.

P17. Given two functions $f(x), g(x)$ with $0 \leq f(x) < g(x)$ and with f and g not growing at the same rate, show that there exists a function $h(x)$ with $f(x) < h(x) < g(x)$ and with h not growing at the same rate as f and h not growing at the same rate as g. Conclude that there is no infinite list of functions $h_1(x), h_2(x), h_3(x), ...$ with the property that for any $f : \mathbb{R}^+ \to \mathbb{R}$, f grows at the same rate as one of the h's.