ST112, Problem Set 5
Spring 2011

Drawing good calculus pictures

P1. The link `Problem 1 Graph` on the course webpage gives the graph of the position $y = p(x)$ of a particle moving on a line as a function of time. (Here time is labeled x, and position is labeled y.) Draw, as accurately as possible, a graph of the velocity function $y = v(x)$ and a graph of the acceleration function $y = a(x)$.

P2. The link `Problem 2 Graph` gives a graph of three functions. One is the position function $y = p(x)$ of a particle moving on a line, one is the velocity function $y = v(x)$, and one is the acceleration function $y = a(x)$. Which is which? Explain how you got your answer.

P3. The link `Problem 3 Graph` gives the graph of the velocity function $y = v(x)$ of a particle moving on a line. Draw, as accurately as possible, the graph of the position function $y = p(x)$ for the particle. There are many possible correct answers. How do the different correct answers relate to each other?

P4. The link `Problem 4 Graph` gives the graph of the acceleration function $y = a(x)$ of a particle moving on a line. Draw, as accurately as possible, the velocity functions $y = v(x)$ and the position functions $y = p(x)$ for this particle.

P5. Repeat P4 for the graph in `Problem 5 Graph`. Be sure that your particle travels on a continuous path! There are many possible correct answers. How do the different correct answers relate to each other?

Continuity arguments

P6. Draw a closed up curve in the plane which doesn't cross itself (e.g. a circle or the outline of a fattened letter C). Show, as rigorously as possible, that for any fixed angle θ, there is a line ℓ_θ that cuts the x-axis with angle θ and that divides the interior of the curve precisely in half. Hint: fix θ, and take a line with angle θ that is completely "to the left" of the curve. Now move the line, keeping θ the same, until it's completely "to the right."

P7. Now assume you have two such curves, and assume that the interiors of each curve is convex. (This means that for any two points in the interior of one curve, the line segment joining those points stays in the interior. So the interior of a circle is convex, but the interior of the fattened letter C isn't.) Show as rigorously as possible that there is a line in the plane that cuts both interiors in half. Hint: Argue that ℓ_θ in P6 for the first curve can be chosen to move continuously as you move θ. Now find a θ_0 such that ℓ_{θ_0} is completely "to the left" of the second curve and a θ_1 such that ℓ_{θ_1} is completely "to the right." Now increase or decrease θ from θ_0 to θ_1.

P8. Given three convex shapes in space, prove that there is a plane that cuts all three in half.

P9. Assume the earth is a perfect sphere. Call two points v_0, v_1 on the earth's surface *antipodal* if a straight line through the earth's center cuts the sphere at v_0 and v_1. Let C be a great circle going through the north and south pole. Prove that at any fixed time there exist a pair of antipodal points on C with exactly the same temperature.

P10. Prove that there exist antipodal points having exactly the same temperature and relative humidity. Hint: Argue that the points you find in P9 move continuously as you move C continuously, so you get a closed curve γ_T of points "going around the earth" consisting of points with the same temperature as their antipodes. Now do the same for relative humidity for great circles C' going through your choice of east and west poles to produce a curve γ_{RH}. Now argue that γ_T and γ_{RH} must intersect.

P11. Prove that P10 still holds if we only assume that the earth is convex (e.g. no natural stone arches).

Differentiability and continuity

P12. For $x \in [0,1]$, $f_n(x) = x^n$ is a differentiable and hence continuous function for $n \in \mathbb{N}$. Set $f : [0,1] \to \mathbb{R}$ by $f(x) = \lim_{n\to\infty} x^n$. Is f continuous? Why or why not? (Are you surprised that the limit of continuous function isn't necessarily continuous?)

P13. Let $g_n : [0, 2\pi] \to \mathbb{R}$ be given by $g_n(\theta) = \sin(n\theta)$. Set $g(\theta) = \lim_{n\to\infty} g_n(\theta)$. What can you say (informally and then rigorously) about g?

Challenge Problems

P14. What is $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\dots}}}}$? Hint: Call this number y. Show that formally $y^2 = 2 + y$. Now solve for y. Justify as carefully as possible that this procedure isn't cheating by showing that

$$\lim_{n\to\infty} \overbrace{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\dots+\sqrt{2}}}}}}^{n \text{ two's}}$$

exists. Subhint: Show that this sequence of real numbers is increasing, and each term is definitely less than e.g. 10. Conclude from an obvious but deep property of $\mathbb{R}$ that the limit exists.

P15. Fix $x > 0$. Compute $\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\dots}}}}$ formally and then rigorously. Set $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\dots}}}}$. Is f continuous?

P16. When $x = 0$, "obviously" $\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\dots}}}} = 0$. Justify this. Using your answer from P15, note that $\lim_{x\to 0} \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\dots}}}} = 1$. In other words,

$$\lim_{x\to 0} f(x) \neq f(0).$$

Thus the definition of f extends to a function $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, but f is not continuous at 0. Why not? Hint: Note that the last equation says

$$\lim_{x\to 0}\lim_{n\to\infty} \overbrace{\sqrt{x+\sqrt{x+\sqrt{x+\dots+\sqrt{x}}}}}^{n\ x's} \neq \lim_{n\to\infty}\lim_{x\to 0} \overbrace{\sqrt{x+\sqrt{x+\sqrt{x+\dots+\sqrt{x}}}}}^{n\ x's}$$

Now look up references to *uniform convergence.* The moral is that you can't always switch limits.

P17. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \in \mathbb{Q} \text{ in lowest terms,} \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Prove that f is continuous at all positive irrational numbers and discontinuous at all positive rational numbers.