ST112, Problem Set 6
Spring 2011

Understanding graphs qualitatively

P1. The link `Sine and a Taylor polynomial approximation` on the course webpage shows a degree 13 polynomial whose graph appears to coincide with the graph of the sine function on the interval $[-\pi, \pi]$. Can these graphs actually coincide on this interval, or must they be slightly different? Explain your answer.

P2. Say you are a graphing calculator programmer, and you want users to be able to graph $y = \sin x$ and variants like $y = \sin(2x - \frac{\pi}{3})$ on the interval $[-\pi, \pi]$. Would it make more sense for you to store 1000 values of $\sin x$ like $\sin(-\pi), \sin(-\pi + \frac{2\pi}{1000}), \sin(-\pi + 2\frac{2\pi}{1000}), \ldots$ in the calculators memory, or should you make the calculator calculate the degree 13 polynomial from P1 1000 times? (The first way uses up more memory, the second takes more calculations. But remember that you want to be able to graph all sorts of variants of sine.)

P3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with a horizontal asymptote as $x \to \infty$. What does the graph of $F(x) = \int_0^x f(t)dt$ look like as $x \to \infty$?

P4. True or false: if the integral $F(x) = \int_0^x f(t)dt$ has $\lim_{x\to\infty} F(x) = 0$, then $\lim_{x\to\infty} f(x) = 0$. Draw some pictures to justify your answer. Don't forget that $\lim_{x\to\infty} f(x)$ might not exist.

Discrete versions of continuity problems

P5. One hundred people are sitting around a very large table. Each person is either the same age, one year older, or one year younger than each of his/her immediate neighbors. Can each person be a different age? (Age is measured in one year increments as usual.)

P6. Suppose now that the one hundred people differs in age from both of his/her immediate neighbors. If Mr. X is 52 and his antipode Ms. Y (the fiftieth person to his left) is 34, must their exist at least one pair of antipodes the same age?

P7. Repeat P6 if Ms. Y is 33.

Limits and integrals

P8. Let $f_n : [0, 1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} n & x \in [0, \frac{1}{n}] \\ 0, & x \in (\frac{1}{n}, 1]. \end{cases}$$

Show that

$$\lim_{n\to\infty} \int_0^1 f_n(t)dt = 1 \neq 0 = \int_0^1 \lim_{n\to\infty} f_n(t)dt = 0.$$

Therefore we have to be careful exchanging limits and integrals.

P9. Find a sequence of functions $g_n : [0, \pi] \to [-1, 1]$ such that $\int_0^\pi g_n(t)dt = 1$ for all n, but $\lim_{n\to\infty} \int_0^\pi g_n'(t)dt = \infty$. Therefore the integral of a function doesn't control the integral of the derivative.