ST112, SOLUTIONS FOR PS 1

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When you write down your answer, you should try to CONVINCE your readers (Chan-Ho, Ross, and Prof. Rosenberg) by writing your argument carefully. Don't forget this UHC is a WRITING class. Please "try this at home" before turning in your work – i.e. don't just write down the first thing that occurs to you and turn it in.

Problem 1

First, we need to specify "the addition" in each circumstance: For $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, we use the usual arithmetic. For $\mathbb{Z}_2, \mathbb{Z}_5, \mathbb{Z}_8, \mathbb{Z}_{31}$, we use the modular arithmetic.

Consider the given equations:

$$\begin{aligned} x+5 &= 4 \\ 2x+5 &= 4 \\ 2x+4 &= 8 \end{aligned}$$

Naively, we want to say the solutions are (respectively):

$$\begin{aligned} \text{“}x &= -1\text{”} \\ \text{“}x &= -\frac{1}{2}\text{”} \\ \text{“}x &= 2\text{”} \end{aligned}$$

but we have to be careful when we deal with the modular arithmetic. We will find a (unique) element satisfying the given equation in each arithmetic system.

The Equation $x + 5 = 4$.

- In $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, we have the unique answer $x = -1$ using the usual arithmetic, but $-1 \notin \mathbb{N}$. Thus, the equation $x + 5 = 4$ has no solution in $\mathbb{N}$ and has a unique solution in $\mathbb{Z}, \mathbb{R}$.
- In $\mathbb{Z}_2$,
$$x + 5 = 4$$
is the same as
$$x + 1 = 0. \text{ (why?)}$$
Thus, $x = 1(= -1)$ in $\mathbb{Z}_2$, so the solution is uniquely determined. Note that we can find the answer by comparing the parities (even and odd) of the LHS (left hand side) and RHS (right hand side) in this case.
- In $\mathbb{Z}_5$,
$$x + 5 = 4$$
is the same as
$$x = 4, \text{ so the solution is uniquely determined.}$$

- In $\mathbb{Z}_8$,
$$x + 5 = 4$$
is as same as
$$x = -1 = 7 \text{ in } \mathbb{Z}_8,$$
so the solution is uniquely determined. Note that $\mathbb{Z}_8$ is $\{0, 1, \cdots, 7\}$ as a set.
- In $\mathbb{Z}_{31}$,
$$x + 5 = 4$$
is the same as
$$x = -1 = 30 \text{ in } \mathbb{Z}_{31},$$
so the solution is uniquely determined.

The Equation $2x + 5 = 4$.

- In $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, we have the unique answer $x = -\frac{1}{2}$ using the usual arithmetic, but $x \notin \mathbb{N}$ and $x \notin \mathbb{Z}$. Thus, the equation $2x + 5 = 4$ has no answer in $\mathbb{N}$ and $\mathbb{Z}$ and has a unique answer in $\mathbb{R}$.
- In $\mathbb{Z}_2$,
$$2x + 5 = 4$$
is as same as
$$1 = 0$$
comparing the parity of each side, so there is no solution.
- In $\mathbb{Z}_5$,
$$2x + 5 = 4$$
is as same as
$$2x = 4.$$
Hence, we can deduce that there is a unique solution $x = 2$ in $\mathbb{Z}_5$. Note that "division by 2" is possible in $\mathbb{Z}_5$.
- In $\mathbb{Z}_8$,
$$2x + 5 = 4$$
is as same as
$$2x = -1 = 7 \text{ in } \mathbb{Z}_8.$$
Since the two sides have the different parities, there is no solution. You can also find that there is no solution by substituting all the elements of $\mathbb{Z}_8$ in the equation. Can you figure out why the lack of a solution in $\mathbb{Z}_2$ implies the same for $\mathbb{Z}_8$? This case says that "division by 2" is impossible in $\mathbb{Z}_8$.
- In $\mathbb{Z}_{31}$,
$$2x + 5 = 4$$
is as same as
$$2x = -1 = 30 \text{ in } \mathbb{Z}_{31},$$
so the solution is $x = 15$ and is uniquely determined in $\mathbb{Z}_{31}$.

The Equation $2x + 4 = 8$**.**

- In $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, we have the unique answer $x = 2$ using the usual arithmetic. Thus, the equation has the unique answer in $\mathbb{N}, \mathbb{Z}, \mathbb{R}$.
- In f $\mathbb{Z}_2$,
$$2x + 4 = 8$$
is the same as
$$0 = 0$$
Thus, all elements in $\mathbb{Z}_2$ are solutions, so the solution is not unique.
- In $\mathbb{Z}_5$,
$$2x + 4 = 8$$
is as same as
$$2x = 4.$$
Hence, we can deduce that there is a unique solution $x = 2$ in $\mathbb{Z}_5$ (by "division by 2.")
- In $\mathbb{Z}_8$,
$$2x + 4 = 8$$
is as same as
$$2x = 4,$$
and has two solutions $x = 2$ and 6, so the solution is not unique.
- In $\mathbb{Z}_{31}$,
$$2x + 4 = 8$$
is the same as
$$2x = 4 \text{ in } \mathbb{Z}_{31},$$
so the solution is $x = 2$ and is uniquely determined in $\mathbb{Z}_{31}$.

Problem 2

Given the equation
$$ax + b = c \text{ in } \mathbb{Z}_d$$
for $a, b, c, d \in \mathbb{N}$, we want to find conditions on a, b, c, and d so that the equation has a unique solution in $\mathbb{Z}_d$. Let $e = c - b$. Then we can write
$$ax = e \text{ in } \mathbb{Z}_d$$
Solving this equation reduces to the question: when is "division by a" possible in $\mathbb{Z}_d$? I will sketch the argument, and please try to fill in the gaps.

You may try to prove this first.

Claim 0.1. Let $e, f, g, h \in \mathbb{N}$. If g and h are coprime (= have no common divisor) and
$$eg = fg \text{ in } \mathbb{Z}_h,$$
then
$$e = f \text{ in } \mathbb{Z}_h.$$
In other words, in this case you can "cancel g,"

Using this claim, you can prove

Claim 0.2. If a and d have no common divisor, then all elements in $a\mathbb{Z}_d = \{0, a, a \cdot 2, a \cdot 3, \cdots, a \cdot (d-1)\}$ are pairwise distinct in the $\mathbb{Z}_d$..

Finally, we have

Claim 0.3. If a and d have no common divisor, then

$$ax = e \text{ in } \mathbb{Z}_d$$

has a unique solution.

Try to prove all the claims by yourself!

PROBLEM 3

Consider the given equations:

$$\begin{aligned} x^2 &= 4 \\ x^2 &= 0 \\ x^2 &= -6 \end{aligned}$$

Thus, we want to find the elements (respectively):

$$\begin{aligned} \text{``}x &= \pm 2\text{''} \\ \text{``}x &= 0\text{''} \\ \text{``}x &= \pm\sqrt{-6}\text{''} \end{aligned}$$

in each arithmetic system.

The Equation $x^2 = 4$.

- In $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, we have the answer $x = \pm 2$ using the usual arithmetic. Thus, $\mathbb{N}$ has only one solution $x = 2$, and $\mathbb{Z}, \mathbb{R}$ have ± 2.
- In the case of $\mathbb{Z}_2$,

  $$x^2 = 4$$

  is as same as

  $$x^2 = 0$$

  Thus, $x = 0$ in $\mathbb{Z}_2$ is the unique solution.
- In the case of $\mathbb{Z}_5$, one can solve

  $$x^2 = 4$$

  by

  $$(x+2)(x-2) = 0$$

  and -2 = 3 in $\mathbb{Z}_5$, so 2 and 3 are solutions.
- In the case of $\mathbb{Z}_8$, the factorization

  $$(x+2)(x-2) = 0$$

  shows and 2 and -2 = 6 are solutions.
- In the case of $\mathbb{Z}_{31}$, the factorization

  $$(x+2)(x-2) = 0$$

  shows and 2 and -2 = 29 are solutions.

The Equation $x^2 = 0$. You may want to say all solutions are just $x = 0$ in all the cases! But it's not true!!

- In $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, $x = 0$ is the only possible solution, but note that $0 \notin \mathbb{N}$.
- In the case of $\mathbb{Z}_2$, $x = 0$ is the unique solution.
- In the case of $\mathbb{Z}_5$, $x = 0$ is the unique solution.
- In the case of $\mathbb{Z}_8$, $x = 0, 4$ are the solutions.
- In the case of $\mathbb{Z}_{31}$, $x = 0$ is the unique solution.

The Equation $x^2 = -6$.

- In $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, there are no solutions.
- In $\mathbb{Z}_2$, we have $-6 = 0$, so $x = 0$ as above.
- In $\mathbb{Z}_5$, we have $-6 = 4$, so $x = 2, 3$ as above.
- In f $\mathbb{Z}_8$, we have $-6 = 2$. You can see that there is no solution, since $1^2 = 3^2 = 5^2 = 7^2 = 1, 2^2 = 6^6 = 4$, and $0^2 = 4^4$.
- In t $\mathbb{Z}_{31}$,
$$x^2 = -6 = 25$$
shows that $x = 5$ and $-5 = 26$ are the solutions in $\mathbb{Z}_{31}$.

Problem 4

Equation (b) is easier to solve because you can do it by just substituting in the 31 possible solutions and checking whether or not you get a solution. In contrast, it is very hard to solve cubics exactly in $\mathbb{R}$.

Problem 5

Alice bought a pizza for dinner and gave $\frac{3}{4}$ of it to Bob, who ate $\frac{2}{3}$ of what Alice gave him. What fraction of the whole pizza did Bob eat?

To make a batch of cookies, Bob needs $\frac{2}{3}$ of a cup of sugar. He has $\frac{3}{4}$ of a cup of sugar, and he wants to use all of it. How many batches of cookies can Bob make?

Problem 6

Since we already know how to multiply fractions, we would also like to know how to divide fractions. Let's figure out how to solve
$$\frac{a}{b} \div \frac{c}{d}$$
Notice that we can write this same division problem as
$$\begin{aligned} \frac{a}{b} \div \frac{c}{d} &= \left(\frac{a}{b} \times 1\right) \div \frac{c}{d} \\ &= \frac{a}{b} \times \left(1 \div \frac{c}{d}\right) \end{aligned}$$
Because we know how to do the multiplication, if we can figure out how to divide 1 by $\frac{c}{d}$, we will know how to divide all fractions. If we think of 1 as being a whole object, then the division problem says "how many $\frac{c}{d}$'s are there in 1?" To figure this out, we can start by dividing the whole object into d pieces. Then we compare the two sizes. We have d pieces of the whole and c pieces of d on the other side. So we are trying to

find the number of c's in d, which means the same as $\frac{d}{c}$. Now we can figure out our original problem.

$$\begin{aligned}\frac{a}{b} \div \frac{c}{d} &= \frac{a}{b} \times \left(1 \div \frac{c}{d}\right) \\ &= \frac{a}{b} \times \frac{d}{c} \\ &= \frac{a \times d}{b \times c}\end{aligned}$$

Notice the pattern here. To divide $\frac{a}{b}$ by $\frac{c}{d}$, we flip the numerator and denominator of the second fraction, also called "inverting" this fraction, and then multiply the two resulting fractions together. We now have an easy way to divide fractions, "invert and multiply."

Problem 7

To make an argument that $\frac{1}{3} = 0.3333333...$ convincing to an adult, we need a rigorous mathematical proof. Start by letting $x = 0.\bar{3}$. At this point, I want to multiply both sides of the equation by 10. To do this, I need to know how to multiply repeating decimals. We need to think of $0.\bar{3}$ as the limit of the sequence $\{a_n\} = \{0.3, 0.33, 0.333, \ldots\}$. Thus, we have

$$\begin{aligned}10 \cdot 0.\bar{3} &= 10 \cdot \lim_{n\to\infty} a_n \\ &= \lim_{n\to\infty} (10 \cdot a_n) \\ &= \lim_{n\to\infty} \{3, 3.3, 3.33, \ldots\} \\ &= 3.\bar{3}\end{aligned}$$

Then,

$$10x = 3.\bar{3}$$

$$\begin{aligned}10x - x &= 3.\bar{3} - x \\ 9x &= 3.\bar{3} - 0.\bar{3}\end{aligned}$$

Again, we need to be careful about subtracting repeating decimals.

$$\begin{aligned}3.\bar{3} - 0.\bar{3} &= \lim_{n\to\infty} \{3, 3.3, 3.33 \ldots\} - \lim_{n\to\infty} \{0, 0.3, 0.33, 0.333 \ldots\} \\ &= \lim_{n\to\infty} \{3, 3, 3, 3, \ldots\} \\ &= 3\end{aligned}$$

Continuing our proof with the above in mind, we have

$$\begin{aligned}9x &= 3 \\ x &= \frac{1}{3} \text{ by the definiton of } \frac{1}{3}\end{aligned}$$

We are being a little cavalier here, since we have not defined limits or derived their properties. With the correct definition, you can check that the two properties of limits that we used in this proof are valid. Also, note that in the subtraction step, I used a slightly different sequence for $0.\bar{3}$. Think about this, and justify this change to yourself.

Problem 8

The estimate we need to make in this problem is the number of raindrops falling on some unit of area per some unit of time. The smaller we can make this scales, the easier our estimate will be. A reasonable choice would be number of raindrops in a square inch per second. I think a moderate rain would fall at a rate of 2 drops per square inch per second. The deck is $10 \cdot 12 = 120$ inches wide, by $12 \cdot 12 = 144$ inches long, so therefore has an area of $120 \cdot 144 = 17,280$ square inches. There are 3,600 seconds in an hour, so to find the number of raindrops falling on the deck in an hour, we have

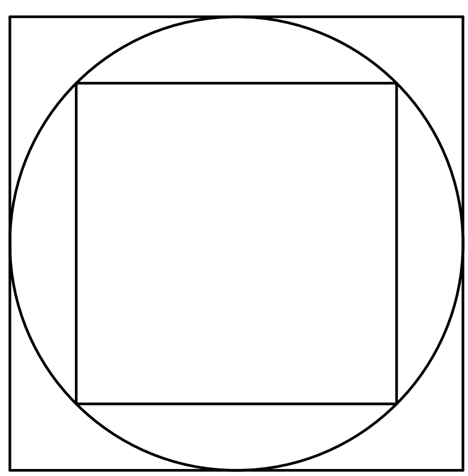
$$2 \cdot 17,280 \cdot 3,600 = 124,416,000$$

We only made one assumption in this problem, so our error should be relatively easy to calculate. The value of 2 drops per square inch per second could be between 1 and 3. Thus, with error, our calculated value of raindrops is

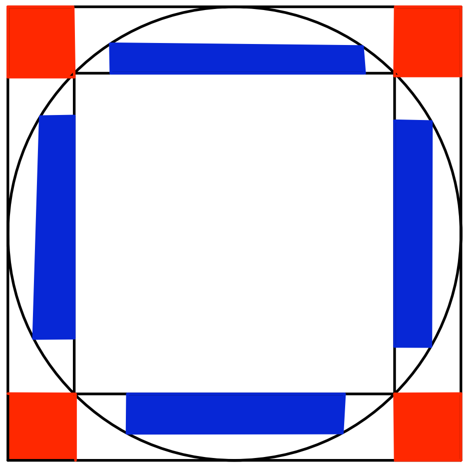
$$124,416,000 \pm 62,208,000$$

Problem 9

For the first step in estimating the number of squares of dimension $1/10$ inch approximating the circle we will use squares. Build a square that the circle is inscribed in, with sides of length 12 inches. The upper limit will be given by the number of small squares that fit in this large square. In this case, the upper limit is $120 \cdot 120 = 14,400$ squares. For the lower limit, we can inscribe a square inside the circle. By the Pythagorean theorem, the sides of the square must have length $6\sqrt{2} \approx 8.49$ inches. We can't quite fill up this square with the small squares, but we can get at most 84 per side, for a total of 7,056. A picture of the squares for the upper and lower estimates is shown below.



By using small squares with sides of length $1/100$ inch, we should be able to make a more accurate estimate. We will do this by subtracting squares from each corner of the large square, and adding in rectangles to the smaller square. This is shown below, with the red squares as the ones to be subtracted and the blue rectangles to be added.



We won't explicitly find the number of squares with sides of length 1/100 inch that will fit in either of these shapes. You can calculate this on your own. A similar process of subtracting regions on the larger shape and adding regions to the smaller shape will give a more accurate estimate for squares with sides of length 1/1000 inch. The point of these estimates is that with each successive estimation we get a better approximation of the area of the circle. We can conjecture that if we take the difference of the number of squares in the upper estimate and the number of squares in the lower estimate, this number will approach zero as the sides of the small squares approaches zero. In other words, our upper bound shape and lower bound shape converge to the circle.

Problem 10

To estimate the number of piano tuners in New York City and the number of cab drivers in Boston, it would first be helpful to estimate the demand for these positions. Since New York City homes are typically small, pianos are expensive, and pianos are typically owned by families, I estimate that 1/100 of the population in New York City owns a piano. This gives 191,000 pianos in the metro area. Assuming 245 workdays in a year, and estimating that the average piano tuner can tune 5 pianos per day, we get that there is a demand for 156 piano tuners in the New York City metro area.

On any given day in the Boston area, I estimate that 1/20 of the population will use a taxi. I will also assume that an average cab driver will give carry 25 passengers during a shift. Therefore, the demand for cab drivers in the Boston area on a given day is 8,800. However, not every cab driver works every day, so I will assume that on a given day, 1/5 of the taxi fleet is off duty. This gives a total of 10,560 cab drivers in the Boston area. This number is greater than 156, so my estimates show that there are more cab drivers in greater Boston than piano tuners in greater New York City.

Now all of my assumptions and estimates here have a range of error, but 10,560 and 156 are far enough away to discount any reasonable margin of error. You can work this out to convince yourself.

Problem 11

Basically, only (ii) holds in general.

If there are stateless citizens, (i) does not hold, because we define the compatriot relation only for citizens for *some* country.

For a counterexample to (iii), if we assume that x has dual citizenship in Canada and the US, y has dual citizenships in Canada and Germany, and z has dual citizenships in France and Germany, then $x \sim y$ and $y \sim z$ but $x \not\sim z$.

If there are no stateless citizens, you can see that (i) holds easily. If there are no dual citizens, the relation (iii) holds because x, y, z all have the same citizenship.

Problem 12

For each $i = 1, \cdots, N$, the country A_i has n_i possible choices to pick their representative. Thus, all number of possible ways of forming a UN is

$$n_1 \cdot n_2 \cdots n_{N-1} \cdot n_N$$

Problem 13

You can check the rules as follows: (i) For all $x \in \mathbb{R}$, we have

$$x - x = 0 \in \mathbb{Z}$$

(ii) If $x - y \in \mathbb{Z}$, then

$$y - x = -(x - y) \in \mathbb{Z}$$

(iii) If $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$, then

$$(x - y) + (y - z) = x - z \in \mathbb{Z}$$

Does every element $x \in \mathbb{R}$ belong to a unique country? First, any $x \in \mathbb{R}$ belongs to a country (= there are no stateless citizens) because $x \in \mathcal{C}_x$. Now we prove the uniqueness. Suppose that $x \in \mathcal{C}_y$ and $x \in \mathcal{C}_z$ for some $y, z \in \mathbb{R}$. Then we have $x - y \in \mathbb{Z}$ and $x - z \in \mathbb{Z}$. This implies that $(x - y) - (x - z) = z - y \in \mathbb{Z}$. Thus y and z are compatriots, and their countries are same. ($\mathcal{C}_x = \mathcal{C}_y = \mathcal{C}_z$)

Explicit descriptions of $\mathcal{C}_7, \mathcal{C}_9$, and $\mathcal{C}_\pi$:

$$\begin{aligned} \mathcal{C}_7 &= \{x \in \mathbb{R} : x \sim 7\} \\ &= \{x \in \mathbb{R} : x - 7 \in \mathbb{Z}\} \\ &= \{x \in \mathbb{R} : x \in \mathbb{Z}\} \\ &= \mathbb{Z} \end{aligned}$$

Similarly,

$$\mathcal{C}_9 = \mathbb{Z} = \mathcal{C}_7$$

Finally,

$$\begin{aligned} \mathcal{C}_\pi &= \{x \in \mathbb{R} : x \sim \pi\} \\ &= \{x \in \mathbb{R} : x - \pi \in \mathbb{Z}\} \\ &= \{n + \pi \in \mathbb{R} : n \in \mathbb{Z}\} \end{aligned}$$

Three explicit examples of UNs: one UN is $[0, 1) = \{x : 0 \leq x < 1\}$; another is $[5, 6)$; a third is $[0, 1/2] \cup (3/2, 2)$. So we see that a UN must contain “an interval’s worth of numbers,” and there are infinitely many possible UNs.

Problem 14

We will look at C_7 first. By definition,

$$\begin{aligned} C_7 &= \{x \in \mathbb{R} : x \sim 7\} \\ &= \{x \in \mathbb{R} : x - 7 = 5k; k \in \mathbb{Z}\} \\ &= \{x \in \mathbb{R} : x = 7 + 5k; k \in \mathbb{Z}\} \\ &= \{\ldots, -8, -3, 2, 7, 12, 17, \ldots\} \end{aligned}$$

Likewise, we can see that

$$C_9 = \{x \in \mathbb{R} : x = 9 + 5k; k \in \mathbb{Z}\}$$

and that

$$C_\pi = \{x \in \mathbb{R} : x = \pi + 5k; k \in \mathbb{Z}\}$$

Notice that unlike Problem 13, $C_7 \neq C_9 \neq C_\pi \neq C_7$, but we do have identities such as $C_7 = C_{12} = C_{-8}$.

Three explicit UNs are: $[0, 5)$, $[-6, -1)$, and $[0, 3] \cup (8, 10)$. As above, a UN must contain "an interval's worth of numbers," but here the interval must have length 5. Likewise, we get an infinite number of possible UNs.

Problem 15

Again, we will begin by looking at C_7.

$$\begin{aligned} C_7 &= \{x \in \mathbb{R} : x \sim 7\} \\ &= \{x \in \mathbb{R} : x - 7 = \frac{a}{b}; a, b \in \mathbb{Z}, b \neq 0\} \\ &= \{x \in \mathbb{R} : x = 7 + \frac{a}{b}; a, b \in \mathbb{Z}, b \neq 0\} \\ &= \{x \in \mathbb{R} : x = \frac{c}{b}; c, b \in \mathbb{Z}, b \neq 0\} \\ &= \mathbb{Q} \end{aligned}$$

A similar analysis gives that

$$C_9 = \mathbb{Q} = C_7$$

We can also see that

$$C_\pi = \{x \in \mathbb{R} : x = \pi + \frac{a}{b}; a, b \in \mathbb{Z}, b \neq 0\}$$

You can convince yourself that $C_{a/b} = \mathbb{Q}$ for any rational number $\frac{a}{b} \in \mathbb{Q}$, and further, that $C_\pi \cap C_{a/b} = \emptyset$ for **every** rational number $\frac{a}{b} \in \mathbb{Q}$.

Now, let's look at possible UNs. First, note that there will be an infinite number of UNs, since each country has an infinite number of citizens. Let's try to build a UN explicitly. Based on what we did in Problems 13 and 14, we will try to build a UN as an interval. However, **any** interval $[s, t)$ in $\mathbb{R}$ contains an infinite number of rational numbers. In other words, there is **no** interval in $\mathbb{R}$ that contains a single point of $\mathbb{Q}$, so a UN cannot be made out of intervals. This seems like a huge problem, and indeed it is. In fact, we cannot explicitly build a UN in this problem. We cannot write down an algorithm for constructing a UN. Think about this! To build a UN in this case, we have to use special functions called "choice functions." Do such functions even exist?