# ST112, SOLUTIONS FOR PS 2

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When you write down your answer, you should try to CONVINCE your readers (Chan-Ho, Ross, and Prof. Rosenberg) by writing your argument carefully. Don't forget this UHC is a WRITING class. Please "try this at home" before turning in your work – i.e. don't just write down the first thing that occurs to you and turn it in.

## Problem 1

I do not accept this picture for a rigorous proof of the commutativity of multiplication of natural numbers. Perhaps the largest issue is the fact that a picture can only show a single case of this identity. For a proof, the result should follow for all possible combinations of natural numbers, and this single picture cannot encapsulate all of the possibilities. We are limited by our ability to draw only a finite number of pictures, so we must conclude that this picture does not demonstrate a proof. In fact, we will end up taking this result as an axiom of multiplication on $\mathbb{N}$.

Note that in general, pictures are a fantastic way to obtain an intuitive understanding of a result, but they rarely suffice for a rigorous proof. So draw lots of pictures, but be critical and remember that an over-simplified picture can lead you astray.

## Problem 2

First, note that this problem is asking for a "picture proof," not a rigorous proof, so we do not necessarily run into the same difficulties we had in P1.

Unfortunately, a picture proof for commutativity of multiplication in $\mathbb{Q}$ is unreliable. Previously, the picture proof for $\mathbb{N}$ relied on adding up boxes. However, $\mathbb{Q}$ includes negative rational numbers, as well as 0, and the concept of negative, or zero, length is not one a picture is well-equipped to deal with. If we happen to restrict ourselves to positive rational numbers, however, there is a picture we can draw. Try this for yourself! Be careful, as rational numbers can be larger than one. How should your picture deal with this?

Recall that multiplication of rational numbers is defined as follows

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Assume that multiplication on $\mathbb{N}$ is commutative. Then, if all of $a, b, c, d > 0$, by our assumption,

$$\begin{aligned} \frac{a}{b} \cdot \frac{c}{d} &= \frac{a \cdot c}{b \cdot d} \\ &= \frac{c \cdot a}{d \cdot b} \\ &= \frac{c}{d} \cdot \frac{a}{b} \end{aligned}$$

So for positive rational numbers, commutativity of multiplication follows from commutativity of multiplication on $\mathbb{N}$. However, if either rational number is zero or negative, we cannot apply our assumption. If we want to mirror the argument we used above, we need to know that multiplication on $\mathbb{Z}$ is commutative. To prove this, we need a definition for multiplication on $\mathbb{Z}$. What follows is a candidate definition, justify to yourself that this is a good definition.

**Definition 0.1.** Let $x, y \in \mathbb{Z}$. Then,

1. If $x, y > 0$, then $x \cdot y$ is defined by multiplication in $\mathbb{N}$
2. If $x < 0$, write $x = -x_0$ for $x_0 \in \mathbb{N}$ by the definition of $\mathbb{Z}$. If $y > 0$, define $x \cdot y = -(x_0 \cdot y)$.
3. If $y < 0$, write $y = -y_0$ for $y_0 \in \mathbb{N}$ by the definition of $\mathbb{Z}$. If $x > 0$, define $x \cdot y = -(x \cdot y_0)$.
4. If $x, y < 0$, write $x = -x_0, = -y_0$ as above. Define $x \cdot y = x_0 \cdot y_0$.

Now that we have a definition for multiplication in $\mathbb{Z}$, work out the proof for yourself that multiplication in $\mathbb{Z}$ is commutative. Note that for zero, the identity follows the proof we did in class. Then, applying the argument above, we see that multiplication in $\mathbb{Q}$ is commutative.

Assume that multiplication in $\mathbb{Q}$ is commutative. To prove commutativity in $\mathbb{R}$, first we need a definition for multiplication in $\mathbb{R}$. Recall that if $x \in \mathbb{R}$, then $x = \lim_{n\to\infty} a_n$ where $\{a_n\}$ is a converging sequence of rational numbers.

**Definition 0.2.** Let $x, y \in \mathbb{R}$ be given by limits of $\{a_n\}$ and $\{b_n\}$, respectively. Then define

$$x \cdot y = \lim_{n\to\infty} (a_n \cdot b_n)$$

where the multiplication $a_n \cdot b_n$ is multiplication in $\mathbb{Q}$.

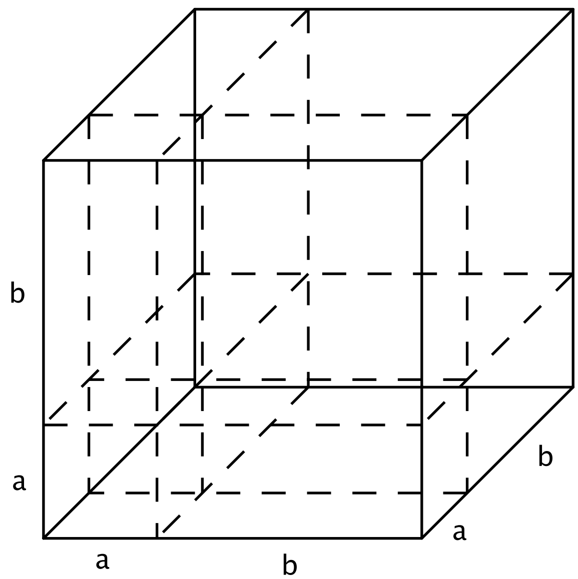
Now, this definition is a bit more tricky, since a given real number can be written as the limit of many, in fact infinitely many, sequences of rational numbers. So you should justify to yourself that this product does not depend on the choice of sequence. Further, we haven't proven any properties about limits, so justify to yourself that it makes sense to move limits through products like this. With this definition, we should be in a good position to prove commutativity of multiplication.

$$\begin{aligned} x \cdot y &= \lim_{n\to\infty} (a_n \cdot b_n) \\ &= \lim_{n\to\infty} (b_n \cdot a_n) \\ &= y \cdot x \end{aligned}$$

Here, the first and third equalities are our definition of multiplication in $\mathbb{R}$, and the second is commutativity of multiplication in $\mathbb{Q}$.

## Problem 3

The exponent 3 in this identity is suggestive of volume, so we draw a cube with sides of length $a + b$. Further, we divide each side into segments of length $a$ and $b$, then connect each segment to obtain the following picture.



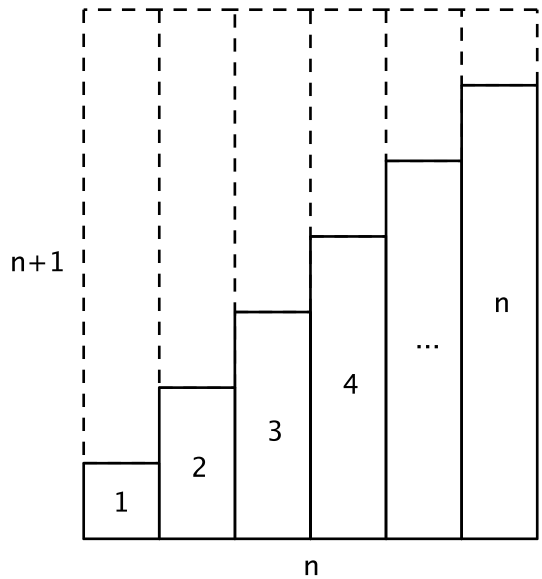

Note that this has divided the cube into eight boxes. There is one box with dimensions $a \times a \times a$, one of $b \times b \times b$, three of $a \times b \times a$, and three of $b \times a \times b$. The equation for the volume of a rectangular prism is length times width times height, so these boxes have volumes $a^3, b^3, a^2b$, and $ab^2$, respectively. From the picture, you can see that none of the boxes overlap, so the sum of their volumes equals the volume of the large cube. Thus,

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

## Problem 4

Draw a series of boxes where each column has height corresponding to a term in the sum in the identity. This is shown below with solid lines. Take the same picture, reflect across a diagonal line, and place on top of the original picture. This second piece is shown with dotted lines. These fit together nicely to form a rectangle with dimensions $n \times (n+1)$. The sum on the left hand side of the identity is the area of the region with solid lines. From the picture, we can see that this is half of the area of the rectangle. Therefore,

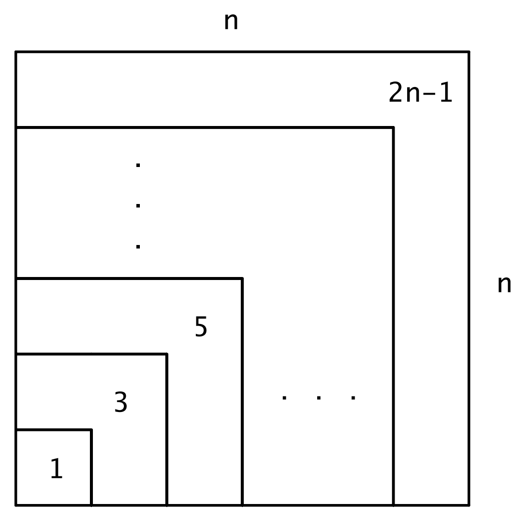
$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$




PROBLEM 5

Start with a single box corresponding to the 1 in the sum. Corresponding to the 3, add 3 blocks to the previous picture in an upside-down "L" shape. This creates a square with sides of length 2. Repeat this process with each term in the sum to obtain a square with sides of length $n$. Then, the sum must be equal to the area of the cube.

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$




PROBLEM 6

- The complement of $\mathbb{N}$ in $\mathbb{Z}$ is the set of "non-positive" integers, i.e. the set consisting of zero and negative integers.

$$\mathbb{N}^c = \{0, -1, -2, \cdots\}$$

Note that any negative integer of zero gives an example.
- The complement of $\mathbb{Z}$ in $\mathbb{Q}$ is the set of "non-integers" in the rational numbers; we can write down this set more precisely.

$$\mathbb{Z}^c = \{a/b \in \mathbb{Q} : a, b \in \mathbb{Z} \text{ with } a \neq 0, b \neq 0, \pm 1, \text{and } gcd(a, b) = 1\}$$

Note that any nonzero reduced fraction with the denominator $\neq \pm 1$ gives an example. (A fraction $a/b \in \mathbb{Q}$ $(a, b \in \mathbb{Z})$ is called *reduced* if $gcd(a, b) = 1$.)
- The complement of $\mathbb{Q}$ in $\mathbb{R}$ is the set of "non-rational" numbers in the real numbers. You may write down as follows:

$$\{x \in \mathbb{R} : x \text{ cannot be expressed as any fraction. }\}$$

Considering this problem, you may have this big question: what is the real definition of the real numbers? Have you thought of this? The possible examples are $\pi, e, \sqrt{\text{any square-free positive integer}}$, but it seems not so easy to prove all they are really not rational numbers. Here is the standard proof of the irrational property of $\sqrt{2}$. Assume that $\sqrt{2}$ is a rational number. Then $\sqrt{2} = a/b$ for some $a, b \in \mathbb{Z}$. Without loss of generality, we may assume that $gcda, b = 1$. Squaring the both sides, we have

$$2 = \frac{a^2}{b^2},$$

so we obtain

$$2b^2 = a^2$$

Since LHS has 2 as factor, the both sides are even. This imples that 2 divides $a^2$, so 2 also divides $a$. Hence RHS is divisible by 4. This implies that $b$ is also divisible by 2. However, we assumed $gcd(a, b) = 1$, so contradiction. Q.E.D.

## Problem 7

Let $X, Y$ be subsets in a set $Z$. The claim is that

$$(X \cup Y)^c = X^c \cap Y^c.$$

We first prove

$$(X \cup Y)^c \subseteq X^c \cap Y^c.$$

Let $a \in (X \cup Y)^c$. Then

$$a \notin X \cup Y,$$

so $a$ cannot be in $X$ and also cannot be in $Y$. This implies

$$a \notin X \text{ and } a \notin Y$$

This is equivalent to

$$a \in X^c \text{ and } a \in Y^c$$

Finally, we have

$$a \in X^c \cap Y^c$$

Thus, any element in $(X \cup Y)^c$ is also in $X^c \cap Y^c$; we have the one inclusion.

The other direction is as follows: Let $b \in X^c \cap Y^c$. Then $b \in X^c$ and $b \in Y^c$ This is as same as

$$b \notin X \text{ and } b \notin Y.$$

Thus, $b$ cannot be an element in $X$ and also cannot be an element in $Y$, so $b$ cannot be an element in $X \cup Y$. This is just

$$b \notin X \cup Y,$$

so we have

$$b \in (X \cup Y)^c.$$

Therefore, we have the other inclusion, so we have the conclusion.

## Problem 8

We use the same notation as Problem 7. The claim is as follows:

*Claim* 0.3.

$$(X \cap Y)^c = X^c \cup Y^c$$

We first prove $(X \cap Y)^c \subseteq X^c \cup Y^c$. Let $a \in (X \cap Y)^c$. Then $a \notin X \cap Y$. Thus, $a \notin X$ or $a \notin Y$. We have $a \in X^c$ or $a \in Y^c$, so $a \in X^c \cup Y^c$.

Let's consider the other direction. Let $b \in X^c \cup Y^c$. Then $b \in X^c$ or $b \in Y^c$, so $b \notin X$ or $b \notin Y$. Then $b \notin X \cap Y$, and we have $b \in (X \cap Y)^c$

## Problem 9

Our claims are

*Claim* 0.4.

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \end{aligned}$$

Consider the first one: Let $a \in A \cap (B \cup C)$. Then $a \in A$ and $a \in B \cup C$. Thus, $a \in A \cap B$ or $a \in A \cap C$. We have $a \in (A \cap B) \cup (A \cap C)$. The other direction: Let $b \in (A \cap B) \cup (A \cap C)$. Then $b \in (A \cap B)$ or $b \in (A \cap C)$. Thus, $b \in A$ and $b \in B \cup C$. We have $b \in A \cap (B \cup C)$.

Move to the second one: Let $c \in A \cup (B \cap C)$. Then $c \in A$ or $c \in B \cap C$. Thus, $c \in A \cup B$ and $c \in A \cup C$. We have $c \in (A \cup B) \cap (A \cup C)$. The other direction: Let $d \in (A \cup B) \cap (A \cup C)$. Then $d \in (A \cup B)$ and $d \in (A \cup C)$. Thus, $d \in A$ or $d \in B \cup C$. We have $d \in A \cup (B \cap C)$.

## Problem 10

First, recall the definitions of injective and bijective functions. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Say $f$ is *injective* if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Say $f$ is *surjective* if for any $y \in \mathbb{R}$ there exist $x \in \mathbb{R}$ such that $y = f(x)$.

- Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is injective but not surjective.

  $$f(x) = e^x$$

  can be an example because if we have

  $$e^{x_1} = e^{x_2}$$

  then we have

  $$\begin{aligned} e^{x_1} &= e^{x_2} \\ log_e(e^{x_1}) &= log_e(e^{x_2}) \\ x_1 &= x_2 \end{aligned}$$

  so the exponential function is injective. But it is not surjective since $e^x$ is positive for all $x \in \mathbb{R}$,i.e. it cannot have negative values.

  $$f(x) = arctan(x)$$

  is also an example.
- Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is surjective but not injective.

  $$f(x) = x(x-1)(x+1)$$

  For any given $y_0 \in \mathbb{R}$, there exists $x_0 \in \mathbb{R}$ such that $y_0 = x_0(x_0-1)(x_0+1)$ using the vertical line test. But it is not injective because all $x$=0, 1, and -1 map to $f(x) = 0$.
- Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is neither injective nor surjective.

  $$f(x) = x^2$$

  This function cannot have negative values, so not surjective. Moreover, $\pm a$ have the same value $a^2$ , so not injective.

- Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is bijective:

$$f(x) = x$$

  We will prove this in the next problem.

## PROBLEM 11

Let $f : \mathbb{R} \to \mathbb{R}$ be a linear function. Then it can be written as

$$f(x) = mx + b$$

for some $m, b \in \mathbb{R}$ with $m \neq 0$. We first prove the injectivity of $f$. Let $x_1$ and $x_2$ be different numbers. Assume the values $f(x_1)$ and $f(x_2)$ are same. (We will derive contradiction with this assumption.) Then

$$\begin{aligned} f(x_1) &= f(x_2) \\ mx_1 + b &= mx_2 + b \\ mx_1 &= mx_2 \\ x_1 &= x_2 \text{ (Here we used } m \neq 0.) \end{aligned}$$

so contradition. To prove the surjectivity, we do as follows. Let $y \in \mathbb{R}$ and consider the number $\frac{y-b}{m} \in \mathbb{R}$. Then

$$f(\frac{y-b}{m}) = y,$$

so surjective.

## PROBLEM 12

We already gave a counterexample:

$$f(x) = x^2$$

is neither injective nor bijective.

## PROBLEM 13A

Let $f(x) = 2x - 3, g(x) = \sqrt{x^2 + 5}, h(x) = \frac{x}{x^2+1}$. Then we compute the compositions as follows

$$\begin{aligned} f \circ g &= 2\sqrt{x^2+5} - 3 \\ (f \circ g) \circ h &= 2\sqrt{\left(\frac{x}{x^2+1}\right)^2 + 5} - 3 \\ g \circ h &= \sqrt{\left(\frac{x}{x^2+1}\right)^2 + 5} \\ f \circ (g \circ h) &= 2\sqrt{\left(\frac{x}{x^2+1}\right)^2 + 5} - 3 \end{aligned}$$

So we see that $(f \circ g) \circ h = f \circ (g \circ h)$.

$$\begin{aligned} h \circ f &= \frac{2x-3}{(2x-3)^2+1} \\ (h \circ f) \circ g &= \frac{2\sqrt{x^2+5}-3}{(2\sqrt{x^2+5}-3)^2+1} \\ f \circ g &= 2\sqrt{x^2+5}-3 \\ h \circ (f \circ g) &= \frac{2\sqrt{x^2+5}-3}{(2\sqrt{x^2+5}-3)^2+1} \end{aligned}$$

Again, we have associativity $(h \circ f) \circ g = h \circ (f \circ g)$.

## Problem 13b

Let $x \in X$. Then we use the definition of composition, following order of operations defined by the parenthesis.

$$\begin{aligned} ((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) \\ &= h(g(f(x))) \end{aligned}$$

For the other case, we have

$$\begin{aligned} (h \circ (g \circ f))(x) &= h((g \circ f)(x)) \\ &= h(g(f(x))) \end{aligned}$$

Thus, we see that $((h \circ g) \circ f)(x) = (h \circ (g \circ f))(x)$. However, since we picked an arbitrary $x \in X$, this must hold for all $x \in X$. Since the functions agree on every point in the domain, they must be the same. Convince yourself of this! Therefore, $(h \circ g) \circ f = h \circ (g \circ f)$.