ST112, SOLUTIONS FOR PS 3

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When you write down your answer, you should try to CONVINCE your readers (Chan-Ho, Ross, and Prof. Rosenberg) by writing your argument carefully. Don't forget this UHC is a WRITING class. Please "try this at home" before turning in your work – i.e. don't just write down the first thing that occurs to you and turn it in.

Problem 1

Suppose the items my friend purchased had costs $x_1, \ldots, x_6$. The coupons she had were for 10% off each item, so the total should have been calculated as

$$0.9x_1 + 0.9x_2 + \ldots + 0.9x_6 = 0.9(x_1 + x_2 + \ldots + x_6)$$

In other words, the correct calculation should have been done by adding the costs of the six items and then taking 10% off the total. Instead, the clerk applied 10% off the total cost six times, or

$$(0.9)^6(x_1 + x_2 + \ldots + x_6)$$

As $0.9 \neq (0.9)^6$, the clerk's method was incorrect. I would explain to the clerk that 10% off each item by the distributive law is the same as 10% off of the total price, not 10% off of the total six times.

Problem 2

Consider the function $f(x)$ on $\mathbb{R}$ defined by

$$f(x) = 100 \cdot (0.9)^x$$

Then this function represents the money my friend will pay after hitting 10% off button n times if $x = n$.

- The first question can be interpreted as follows: Find the minimal $n \in \mathbb{N}$ such that

  $$100 \cdot (0.9)^n < 0.01$$

  .

  (Brutal Force) By computing the exponent using computer or calculator, you can see

  $$\begin{aligned} 100 * (0.9)^{87} &\approx 0.0104495676331779 \\ 100 * (0.9)^{88} &\approx 0.00940461086986007 \end{aligned}$$

  We also note that $f(x)$ is decreasing for all $x \in \mathbb{R}$ since it is an exponential function with base $0.9 < 1$.

  (a bit more theoretical) We can talk more using log. We have inequality

  $$(0.9)^n < 0.0001,$$

and, taking $\log_{0.9}$,

$$n > \log_{0.9}(0.0001) \approx 87.4173813071313$$

(Why is the inequality sign changed? The key idea is $0.9 < 1$.)

Then we can conclude that it takes 88 times that my friend owes less than a penny.

- The next question asks the following: Does there exist $n \in \mathbb{N}$ such that

$$100 \cdot (0.9)^n = 0?$$

It is as same as finding n such that $(0.9)^n = 0$.

Claim 0.1. There is no $n \in \mathbb{N}$ such that

$$(0.9)^n = 0.$$

Proof. Suppose not. Then there exists $n_0 \in \mathbb{N}$ such that $(0.9)^{n_0} = 0$. Then

$$(0.9)^{n_0-1} \cdot 0.9 = 0$$

Thus, dividing both sides by 0.9, we have

$$(0.9)^{n_0-1} = 0$$

Now we can iterate this process. Then we have

$$0.9 = 0,$$

so contradiction. □

- The last question can be translated as: Does there exist $n \in \mathbb{N}$ such that

$$100 \cdot (0.9)^n < 0?$$

It is as same as finding n such that $(0.9)^n < 0$. The same argument as the above works changing $=$ by $<$.

If you draw the graph of $f(x)$ and look at the behavior using computer, you can see what should happen. This will give you more intuition and the right direction. But I recommend you to try to give a proof and explain your proof, not your picture.

Problem 3

The second person in line claims that the incorrect method will yield a price that is at least \$200 less than the price calculated correctly. If this is exactly correct, then we can figure out the total pre-discount cost of her items as follows. Assume that the pre-discount total of her items is x. Note that since she has five items, we can assume she will use five coupons. Then we can set up and solve the following equation:

$$\begin{aligned} (0.9)^5 x &= 0.9x - 200 \\ 0.9x - (0.9)^5 x &= 200 \\ (0.9 - (0.9)^5)x &= 200 \\ x &= \frac{200}{0.9 - (0.9)^5} \\ x &\approx 646.18 \end{aligned}$$

Thus, the minimum pre-discount cost of her items is \$646.18.

Problem 4

The total cost depends on k and x and can be written as a function on $k \in \mathbb{N}$:

$$f(k) = k \cdot x \cdot (0.9)^k$$

Note that x is a constant here. Then we have

- Buying one item, the total cost is $f(1) = 0.9x$.
- Buying ten items, the total cost is

$$\begin{aligned} f(10) &= 10 \cdot (0.9)^1 0 \cdot x \\ &\approx 3.48678440100000 \cdot x \end{aligned}$$

- Buying a million items, the total cost is

$$\begin{aligned} f(10^6) &= 10^6 \cdot (0.9)^{10^6} \cdot x \\ &= (\text{a VERY small number but not 0}) \cdot x \end{aligned}$$

  I want to emphasize $10^6 \cdot (0.9)^{10^6}$ is NOT zero because if you pick a very big x like \$ $(1.2)^{10^{100}}$, then the total cost can be far from \$0.

We can easily see that buying millions is the best way in this situation. However, we can also see that the function $f(k)$ is NOT always decreasing comparing these values. ($f(10^6) < f(1) < f(10)$!!) Thus, we cannot say that buying millions is the best way "because" the "limit" goes to 0.

Problem 5

Note that this computation involves the *monthly* compound interest. the compound interest formula can be written as

$$y = P(1 + r/n)^{nt},$$

where P is the initial investment (or loan), r is the interest rate as a decimal, n is the number of times interest is compounded per year, t is the amount of time in years, and y is the final amount. Then the formula for this problem is given by

$$10000 \cdot (1 + 0.05/12)^{12 \cdot 10}$$

If we compute this,

$$10000 \cdot (1 + 0.05/12)^{12 \cdot 10} \approx 16470.0949769028$$

Thus, the total interest is about \$ 6470.

Problem 6

The similar computation to the above says

$$10000 \cdot (1 + 0.025/12)^{12 \cdot 20} \approx 16478.6397545968$$

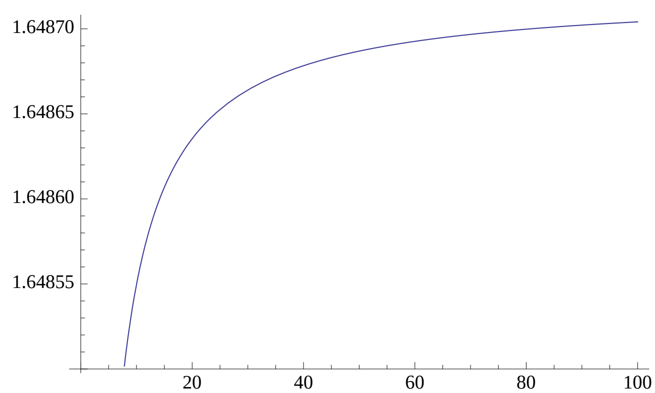
Thus, the total interest is about \$ 6479. So the former deal is better.

Problem 7

We will use the compound interest formula $y = P(1 + \frac{r}{n})^{nt}$, where P is the initial investment, r is the interest rate as a decimal, n is the number of times interest is compounded per year, t is the amount of time in years, and y is the final amount. To make things look better, we can assume that $P = 1$. Then we have $r = \frac{5}{100x}$, $n = 12$, and $t = 10x$. Therefore, we have the following equation:

$$y = \left(1 + \frac{1}{240x}\right)^{120x}$$

We want to find the value of x that inimizes y, the total amount we have to pay. Now, if you've seen calculus, you know that derivatives give a fantastic way of analyzing functions. In particular, we can use them to minimize functions. If you haven't seen calculus, this problem would be much harder to do rigorously. So without using calculus, our best option is to look at the graph of this function and try to interpret it. The following graph was created in Mathematica (see the note at the end about computer graphs):



First, note the scale on the y-axis. The point at which the x-axis and y-axis intersect is not (0,0) in this picture. The scale is set up this way to give a better sense of the shape of the function. You can play around with the range values yourself to see what happens near zero. It seems reasonable from this graph to conclude that the interest function is always increasing. Thus, as x increases, we will always be paying more interest than before. So we conclude that the best option is to pay everything up front, i.e. $x = 0$.

Now let's do the same analysis, but rigorously, using calculus. Note that for any value of x, $y > 0$. By our rationale above, we would like to prove that the function is always increasing. In calculus terms, this means we would like to show that the derivative is always positive. To find the derivative of this function, we need to use logarithmic differentiation.

$$\begin{aligned}
y &= \left(1+\frac{1}{240x}\right)^{120x} \\
\ln(y) &= \ln\left(\left(1+\frac{1}{240x}\right)^{120x}\right) \\
\ln(y) &= (120x)\ln\left(1+\frac{1}{240x}\right) \\
\frac{y'}{y} &= 120\ln\left(1+\frac{1}{240x}\right)+(120x)\frac{-\frac{1}{240x^2}}{1+\frac{1}{240x}} \\
\frac{y'}{y} &= 120\ln\left(1+\frac{1}{240x}\right)-\frac{120}{240x+1} \\
y' &= \left(1+\frac{1}{240x}\right)^{120x}\left(120\ln\left(1+\frac{1}{240x}\right)-\frac{120}{240x+1}\right)
\end{aligned}$$

Since the first term in y' is always positive, we would like to show that the second term in the parenthesis is always positive. Call this term

$$u = 120\ln\left(1+\frac{1}{240x}\right)-\frac{120}{240x+1}$$

It seems difficult to show algebraically that u is always positive, so again we turn to calculus. Note that

$$\lim_{x\to\infty}\left(120\ln\left(1+\frac{1}{240x}\right)-\frac{120}{240x+1}\right)=0$$

Thus, if we can show that u' is always negative, we will know that the graph of u can never cross the x-axis, as it must then approach $u=0$ from above. This will give us that u is always positive.

$$\begin{aligned}
u' &= 120\frac{-\frac{1}{240x^2}}{1+\frac{1}{240x}}+\frac{120(240)}{(240x+1)^2} \\
&= \frac{-\frac{120}{x}}{240x+1}+\frac{28800}{(240x^2+1)^2} \\
&= \frac{-\frac{120}{x}(240x+1)+28800}{(240x+1)^2} \\
&= \frac{28800-28800-\frac{120}{x}}{(240x+1)^2} \\
&= -\frac{120}{x(240x+1)^2}
\end{aligned}$$

Therefore, we see that $u'<0$ for all $x>0$. We can then conclude that y' is positive for all $x>0$, which tells us that y is strictly increasing. To minimize y, we have to make x as small as possible. Unfortunately, this function is undefined at zero, so to understand the behavior of the function there, we have to use limits.

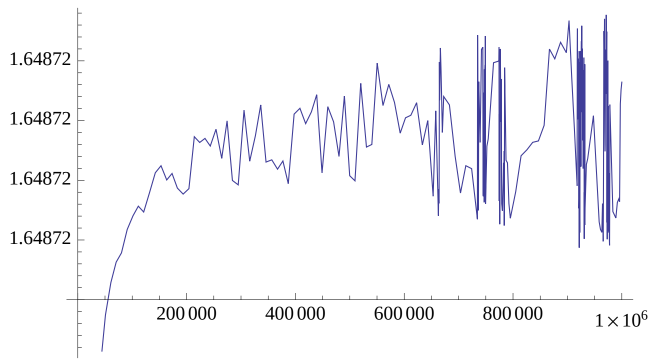
$$\lim_{x\to 0^+}\left(1+\frac{1}{240x}\right)^{120x}=1$$

Now we can redefine y as

$$y = \begin{cases} \left(1 + \frac{1}{240x}\right)^{120x} & : x > 0 \\ 1 & : x = 0 \end{cases}$$

This makes sense, as the original y was a continuous function on $(0, \infty)$, so that our new function is continuous on $[0, \infty)$. We then see that the minimum value of y is 1, i.e. the principal value we invested, and this occurs when $x = 0$. It turns out our calculus-free analysis was correct, and that the minimum payment is incurred when we pay everything up front.

Note Calculator and computer graphs can be a great source of intuition about a problem. But they have their limits. The largest problem is accuracy. Say I wanted to convince myself of the behavior of the graph of the function by taking larger and larger domains. When I try the domain $[0, 1000000]$, I get the following picture:

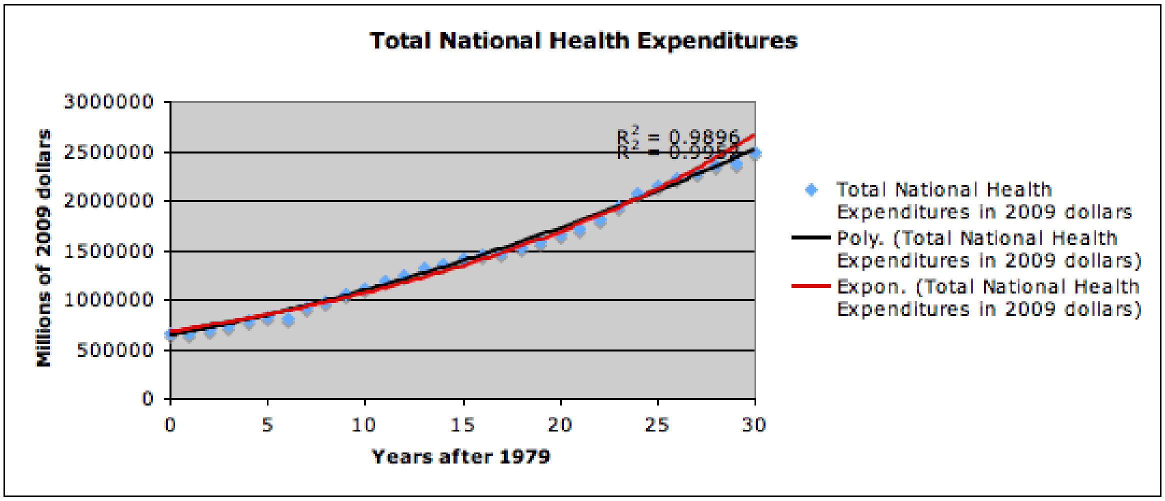


There is no way this graph is correct. We know the function is "smooth," so this picture cannot be accurate. Even though this graph was created with computer graphing software, it was easy to find the bounds of its accuracy in this case. In the end, use calculators and computers, but think about what you input and be critical of the output.

Problem 8

To make any sort of determination, we need to find appropriate data to analyze. The data used in this problem is from the Centers for Medicare and Medicaid Services (CMS) division of the U.S. Department of Health and Human Services [1]. A good measure of the cost of health care in the United States is given by the total national health care expenditures. This includes both public and private spending over a 30 year period from 1979 to 2009. However, the data given by the CMS is not adjusted for inflation, which will certainly skew the results of any analysis. To fix this, the data was run through an inflation calculator [2] to normalize costs to 2009 dollars.

To determine if that data is better fit by a quadratic or exponential function, a scatter plot is formed in Microsoft Excel, from which quadratic and exponential regressions can be run. The plot, along with the two regressions is shown below.



To measure the effectiveness of these regressions, consider the coefficient of determination for each regression. This is a measure of how well each regression fits the data. Denoted R^2, it takes a value between 0 and 1, with 0 indicating no correlation and 1 indicating a perfect fit. For the quadratic regression, $R^2 = 0.9952$, and for the exponential, $R^2 = 0.9896$. Thus, both regressions are quite good at representing the data. In fact, it seems unlikely that one can be favored over another, considering errors in obtaining the data.

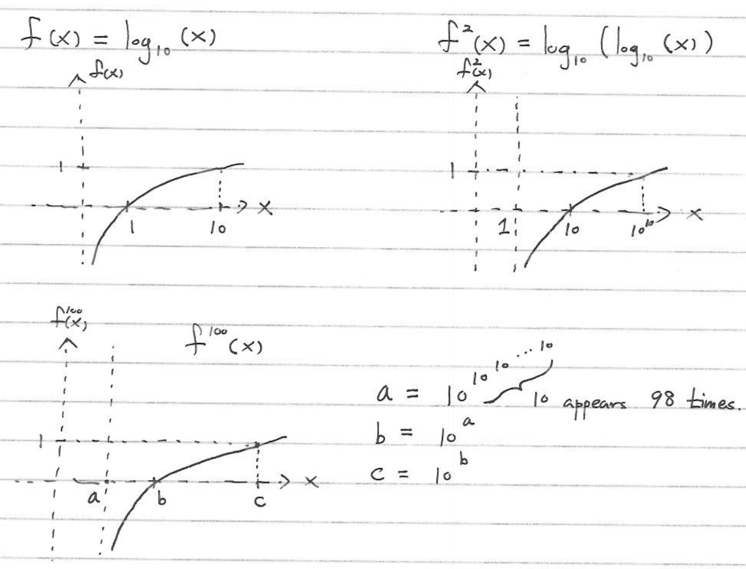
There is a good mathematical reason for not preferring one function over another. When mathematicians talk about growth rates of functions, this means behavior of the function as x is very large and increasing. With this definition, it is not hard to see that exponential functions grow at a faster rate than quadratic functions. However, in this problem, we are looking only at an interval of length 30. Over any finite interval, a quadratic function can appear to grow much faster than an exponential function. A general quadratic function is $f(x) = ax^2 + bx + c$, and a general exponential function is $g(x) = C \cdot r^x$. If a is extremely large and r is very close to 1 (assume $C = 1$), then the growth of f appears to be much faster than g over a finite interval. Likewise, over a finite interval, these functions can be made to appear to grow at the same rate. This is the case in our analysis. There is an exponential function and a quadratic function that fit the data very well, and therefore appear to grow at the same rate. Our conclusion is that in modeling historical health care costs, neither quadratic nor exponential models should be preferred.

[1] "National Health Expenditures by type of service and source of funds, CY 1960-2009." *Centers for Medicare & Medicaid Services*. 20 Jan. 2011. Web. 23 Feb. 2011. `https://www.cms.gov/nationalhealthexpenddata/02_nationalhealthaccountshistorical.asp`

[2] "CPI Inflation Calculator." *Bureau of Labor Statistics*. Web. 23 Feb. 2011. `http://www.bls.gov/data/inflation_calculator.htm`

Problem 9

First, look at the following graphs.



We can see the graphs of f^2 and f^{100} looks very similar. This means the graphs could be (very) "rough." But we can make them precice by scaling. To make them precise, we need (at least) the following data:

- the vertical asymptotes (to determine the domain of the functions)
- the x-intercepts (to determine whether the values are positive or negative)
- the x-coordinates with $(x, 1)$ on the graphs (to investigate how much the functions grow comparing with the x-intercepts)

Thus, I described them on the graphs, and for example, we can see the graph of f^{100} grows much slower than f^2 because $c - b$ is much bigger than $10^{10} - 1$.

Drawing these graphs, we can observe that for any increasing function $f(x)$, taking $\log(f(x))$ make its growth slower (but it still grows.)

This observation shows that there is no "slowest increasing" function. Let $f(x)$ be such a function. Then $\log_{10}(f(x))$ is still increasing, but it increases slower than $f(x)$.

Problem 10

In order to think of the inverse function, we should first check $f^2(x)$ and $f^{100}(x)$ is bijective functions on some sources and some targets. We can see that

$$f^2 : \{x \in \mathbb{R} : x > 1\} \to \mathbb{R}$$
$$f^{100} : \{x \in \mathbb{R} : x > a\} \to \mathbb{R}$$

are bijective, so we can think of their inverses. First compute the inverse of $f^2(x)$ Since $f^2(x) = \log_{10}(\log_{10}(x))$, put $y = \log_{10}(\log_{10}(x))$. Then

$$\begin{aligned} 10^y &= \log_{10}(x) \\ 10^{10^y} &= x \end{aligned}$$

Thus, the inverse of f^2 can be written as 10^{10^x}.

Before computing the inverse of f^{100}, let's do more experiment. Then you will see the inverse of $f^3(x)$ is given by

$$10^{10^{10^x}}$$

and the inverse of $f^4(x)$ is given by

$$10^{10^{10^{10^x}}}$$

Now can you see the pattern? The inverse of $f^{100}(x)$ is given by $10^{(\cdots)^{10^x}}$ where 10 appears 98 times are in the $(\cdots)$.

By the same principle, there is no "quickest increasing function" since $10^{f(x)}$ increases quicker than $f(x)$ for any increasing function $f(x)$.

Problem 11

Let $f(x) = x^2 - x$ and pick two functions

$$g(x) = x^2$$
$$h(x) = 1$$

Then

Claim 0.2.
- $f(x)$ and $g(x)$ grow at the same rate.
- $f(x)$ and $h(x)$ do not grow at the same rate.

Proof.
- $f(x)$ and $g(x)$ grow at the same rate?
  Note that
  $$\mid x^2 - x \mid = x^2 - x$$
  for $x > 1$. Then we can see that
  $$\mid x^2 - x \mid \leq \mid x^2 \mid$$
  for all $x > 1$. ($C_1 = 1$.) Consider the other inequality. Thus, (a part of) the inequality can also be written as
  $$x^2 \overset{?}{\leq} C_2 \cdot (x^2 - x)$$
  for $x > 1$. If we take $C_2 = 2$, then the inequality reduces to $x > 2$. Thus, we have
  $$\mid x^2 - x \mid \leq \mid x^2 \mid \leq 2 \cdot \mid x^2 - x \mid$$
  for $x > 2$, so they grow at the same rate.

- $f(x)$ and $h(x)$ do not grow at the same rate?
  The function $f(x)$ cannot be bounded by the constant function $h(x)$, i.e. there is no constant C such that

  $$\mid f(x) \mid \leq C \cdot \mid h(x) \mid (= C)$$

  since $f(x)$ is an increasing function after $x > 1$. Thus, they cannot grow at the same rate.

□

Problem 12

We have $g(x) = x$ and $h(x) = x^2$. To find a function $f(x)$ that does not grow at the same rate as g and h, it seems reasonable to start with an exponential function whose exponent is between 1 and 2. An easy choice to work with is $f(x) = x^{3/2}$. Note that since $1 \leq x^{1/2}$ for $x \geq 1$, multiplying by x gives $x \leq x^{3/2}$. Likewise, multiplying by $x^{3/2}$ gives $x^{3/2} \leq x^2$. Thus, for $x \geq 1$,

$$g(x) \leq f(x) \leq h(x)$$

We can now try to show that f does not grow at the same rate as g. By the definition of growth rates, if f and g grow at the same rate, there are positive constants C_1 and C_2 with

$$C_1|f(x)| \leq |g(x)| \leq C_2|f(x)|$$

Both f and g are positive when $x > 0$, so the absolute value signs are not necessary. So we have

$$C_1 x^{3/2} \leq x \leq C_2 x^{3/2}$$

Consider the first inequality.

$$\begin{aligned} C_1 x^{3/2} &\leq x \\ C_1 x^{1/2} &\leq 1 \\ x^{1/2} &\leq \frac{1}{C_1} \\ x &\leq \frac{1}{{C_1}^2} \end{aligned}$$

But this is clearly false. No matter how small C_1 is made, there will be values of x that exceed $1/{C_1}^2$. Therefore, f does not grow at the same rate as g.

We do the same analysis with f and h. Again, note that h is positive, so absolute value signs are not needed. The system of inequalities is

$$C_1 x^{3/2} \leq x^2 \leq C_2 x^{3/2}$$

Now look at the inequality on the right.

$$\begin{aligned} x^2 &\leq C_2 x^{3/2} \\ x^{1/2} &\leq C_2 \\ x &\leq {C_2}^2 \end{aligned}$$

Again, this is not possible. No matter how large ${C_2}^2$ is made, there will always be values of x that surpass it. Thus, f does not grow at the same rate as h.

Problem 13

First, note that as x approaches infinity, some of the functions in this list grow (the value of the function approaches $\pm\infty$), some decay (the value of the function approaches 0), and one does neither. We list these below.

$$\begin{aligned}
\text{Grow}: &\quad (c), (d), (e), (f), (h), (j), (k)\\
\text{Decay}: &\quad (b), (g), (i)\\
\text{Neither}: &\quad (a)
\end{aligned}$$

Justify to yourself that functions from different groups above cannot grow at the same rate (even (a) and (g)!). Instead, we have to look within each group. In the group that grows, we can separate these into three rough groups. Quadratic functions ((d) and (e)), exponential-like functions((c), (h), and (k)), and linear-like functions ((f) and (j)). Check yourself that functions from different groups here cannot grow at the same rate. We will actually work out the case of (d) and (e). For positive constants C_1 and C_2, we consider the inequalities

$$C_1|-5\cdot 10^6x^2-10^9x| \leq |10^{-10}x^2| \leq C_2|-5\cdot 10^6x^2-10^9x|$$

For $x>0$, this can be rewritten as

$$C_1(5\cdot 10^6x^2+10^9x) \leq 10^{-10}x^2 \leq C_2(5\cdot 10^6x^2+10^9x)$$

We will work first with the left inequality.

$$\begin{aligned}
C_1(5\cdot 10^6x^2+10^9x) &\leq 10^{-10}x^2\\
(5\cdot 10^6C_1-10^{-10})x^2 &\leq -10^9C_1x\\
\left(5\cdot 10^6-\frac{10^{-10}}{C_1}\right)x &\leq -10^9
\end{aligned}$$

Now we need to choose a value of C_1 such that $(5\cdot 10^6-10^{-10}/C_1)$ is negative. This will then give a lower bound for x for the original inequality to hold. Let $C_1=\frac{1}{5}\cdot 10^{-17}$. Then the inequality above becomes

$$\begin{aligned}
\left(5\cdot 10^6-\frac{10^{-10}}{\frac{1}{5}\cdot 10^{-17}}\right)x &\leq -10^9\\
(5\cdot 10^6-5\cdot 10^7)x &\leq -10^9\\
x &\geq \frac{10^9}{5\cdot 10^7-5\cdot 10^6}\\
x &\geq \frac{200}{9}
\end{aligned}$$

Thus, for our choice of C_1, when $x\geq 200/9$, the inequality holds.

Now we turn our attention to the inequality on the right.

$$\begin{aligned}
10^{-10}x^2 &\leq C_2(5\cdot 10^6x^2+10^9x)\\
-10^9C_2x &\leq (5\cdot 10^6C_2-10^{-10})x^2\\
-10^9 &\leq \left(5\cdot 10^6-\frac{10^{-10}}{C_2}\right)x
\end{aligned}$$

Now choose $C_2 = 1$ and note that then $(5 \cdot 10^6 - 10^{-10}) > 0$. Solving the inequality above for x gives

$$\frac{-10^9}{5 \cdot 10^6 - 10^{-10}} \leq x$$

This is obviously satisfied for and positive x, as the number on the left in negative. Therefore, the functions in (d) and (e) grow at the same rate.

In the "Growth" group, the only other pair functions that grow at the same rate is the pair of functions in (f) and (j), as you can check.

For the "Decay" group, consider the functions in (g) and (i). Then we need to look at the system of inequalities

$$C_1\left|\frac{\sin x}{x^2+1}\right| \leq \left|\frac{1}{x^2}\right| \leq C_2\left|\frac{\sin x}{x^2+1}\right|$$

which can be rewritten as

$$C_1\frac{|\sin x|}{x^2+1} \leq \frac{1}{x^2} \leq C_2\frac{|\sin x|}{x^2+1}$$

The function in the middle, $1/x^2$ is never equal to zero. However, on the right hand side of the system of inequalities, the function $C_2|\sin x|/(x^2+1)$ is zero when $x = k\pi$ for $k \in \mathbb{N}$. Thus, there is no way the second inequality can hold. Therefore, the functions in (g) and (i) cannot decay at the same rate. As you can check for yourself, there are no pairs of functions in the "Decay" group that decay at the same rate.

We can now conclude that the only pairs of functions that grow/decay at the same rate are the pairs of functions in (d) and (e) as well as (f) and (j).