ST112, SOLUTIONS FOR PS 4

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When you write down your answer, you should try to CONVINCE your readers (Chan-Ho, Ross, and Prof. Rosenberg) by writing your argument carefully. Don't forget this UHC is a WRITING class. Please "try this at home" before turning in your work – i.e. don't just write down the first thing that occurs to you and turn it in.

Problem 1

Note that the instructions at the beginning of this section ask you to make a guess for the solution first. Do this on your own before starting the problem.

To estimate how many jelly beans fit in the jar, we need to make a couple of assumptions. First, we need to approximate the jelly bean by some shape whose volume is easy to calculate. Then, since jelly beans do not pack evenly, we need to approximate the amount of empty space left over.

It seems reasonable to approximate the jelly bean with a cylinder, so we need to estimate the radius and height. Now there is some possibility for disagreement here, as jelly beans from different manufacturers range in size. For our purposes, let's assume that the cylinders have radius $r = 0.2$ inches and height $h = 0.7$ inches. Then, each jelly bean has an approximate volume of

$$\begin{aligned} V_{bean} &= \pi r^2 h \\ &= \pi(0.2)^2(0.7) \\ &\approx 0.09 \text{ in}^3 \end{aligned}$$

We will also assume that jelly beans will fill 75% of the volume of the container. Call this factor $P_{bean} = 0.75$.

The total volume of the container is

$$\begin{aligned} V_{jar} &= \pi R^2 H \\ &= \pi(5)^2(12) \\ &\approx 942.48 \text{ in}^3 \end{aligned}$$

However, the jelly beans only fill 75% of this volume, which is

$$0.75(942.48) = 706.86 \text{ in}^3$$

Thus, the approximate number of jelly beans that will fit in the jar is

$$N_{bean} = \frac{706.86}{0.09} = 7854$$

Problem 2

Our approach will be the same as in Problem 1. We can approximate a grain of rice with a cylinder of radius $r = 0.04$ inches and height $h = 0.25$ inches. Then, the approximate volume of a grain of rice is

$$\begin{aligned} V_{rice} &= \pi r^2 h \\ &= \pi(0.04)^2(0.25) \\ &\approx 0.001 \text{ in}^3 \end{aligned}$$

Because of its shape, rice will pack more efficiently than jelly beans, so we can estimate that it will fill 85% of the container. Call this factor $P_{rice} = 0.85$.

The size of the container is the same as in Problem 1, so the rice will fill

$$0.85(942.48) = 801.11 \text{ in}^3$$

Therefore, the approximate number of grains of rice that will fit in the jar is

$$N_{rice} = \frac{801.11}{0.001} = 801,111$$

Problem 3

Dividing the approximate number of grains of rice to fill the jar in Problem 2 by the approximate number of jelly beans in the jar in Problem 1 will not give an accurate estimation of the number of grains of rice in a jelly bean. This is due to the different factors we have for the amount of space taken up by each object. We have

$$\begin{aligned} \frac{N_{rice}}{N_{bean}} &= \frac{\frac{P_{bean}V_{jar}}{V_{bean}}}{\frac{P_{rice}V_{jar}}{V_{rice}}} \\ &= \frac{P_{rice}V_{bean}}{P_{bean}V_{rice}} \end{aligned}$$

However, to estimate the number of grains of rice that would fit inside a jelly bean, we would need to calculate just

$$\frac{P_{rice}V_{bean}}{V_{rice}}$$

These two quantities are clearly not the same, so we cannot assume that such a calculation would give an accurate approximation.

Problem 4

The circumference of Earth is approximately $C_{Earth} = 40,075.017$ km [1]. Recall that circumference of a circle is related to the radius of the circle by

$$C = 2\pi r$$

The radius of the steel band is $r_{band} = r_{Earth} + 1$. Thus,

$$\begin{aligned} C_{band} = 2\pi r_{band} & \\ &= 2\pi(r_{Earth} + 1) \\ &= 2\pi r_{Earth} + 2\pi \\ &= C_{Earth} + 2\pi \end{aligned}$$

Thus, the length of the steel band is only 2π feet longer than the circumference of Earth.

The circumference of the moon is approximately $C_{moon} = 10,921$ km [2]. Notice that the calculation we did above holds for the moon as well. Therefore, a steel band placed one foot above the surface of the moon has length 2π feet longer than the circumference of the moon as well.

[1] `http://en.wikipedia.org/wiki/Earth`
[2] `http://en.wikipedia.org/wiki/Moon`

Problem 5

We want to compare Professor's salary and her parking costs under the time change.

- Salary : \$ $75000 \cdot (1 + \frac{2.5}{100})^x$
- Parking Cost : \$ $12 \cdot 100 \cdot (1 + \frac{4}{100})^x$

where x means time (year). Note that she pays \$ 100 *per month* for parking. We want to find x such that

$$75000 \cdot (1 + \frac{2.5}{100})^x = 12 \cdot 100 \cdot (1 + \frac{4}{100})^x$$

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Then we can compute

$$\begin{aligned}
75000 \cdot (1 + \frac{2.5}{100})^x &= 12 \cdot 100 \cdot (1 + \frac{4}{100})^x \\
\frac{75000}{1200} \cdot (1.025)^x &= (1.04)^x \\
62.5 \cdot (1.025)^x &= (1.04)^x \\
62.5 &= \frac{(1.04)^x}{(1.025)^x} \\
\ln(62.5) &= \ln(\frac{(1.04)^x}{(1.025)^x}) \\
\ln(62.5) &= x \cdot \ln(\frac{1.04}{1.025})
\end{aligned}$$

Thus, we have

$$x = \frac{\ln(62.5)}{\ln(1.04/1.025)}$$

Now we use a calculator to find this value. I use SAGE (`http://wwwsagemath.org`) here. Here is somewhat delicate approximation problem. Since

$$1.04/1.025 \sim 1.01463414634146$$

only gives an approximated value, I get two *different* values

$$\begin{aligned}
\frac{\ln(62.5)}{\ln(1.0146)} &\sim 285.293174628356 \\
\frac{\ln(62.5)}{\ln(1.04) - \ln(1.025)} &\sim 284.632291663742
\end{aligned}$$

To find a more precise value for x, we compare the following values

$$\begin{aligned}
75000 \cdot (1.025)^{284} &\sim 8.32988555530650 \times 10^7 \\
1200 \cdot (1.04)^{284} &\sim 8.25371762739528 \times 10^7 \\
75000 \cdot (1.025)^{285} &\sim 8.53813269418917 \times 10^7 \\
1200 \cdot (1.04)^{285} &\sim 8.58386633249110 \times 10^7
\end{aligned}$$

Thus, we have

$$\begin{aligned} 75000 \cdot (1.025)^{284} &> 1200 \cdot (1.04)^{284} \\ 75000 \cdot (1.025)^{285} &< 1200 \cdot (1.04)^{285} \end{aligned}$$

Thus, the parking costs will become lager than her salary after 285 years and the second approximation is more precise. (In fact, we approximated the value twice for the first one.)

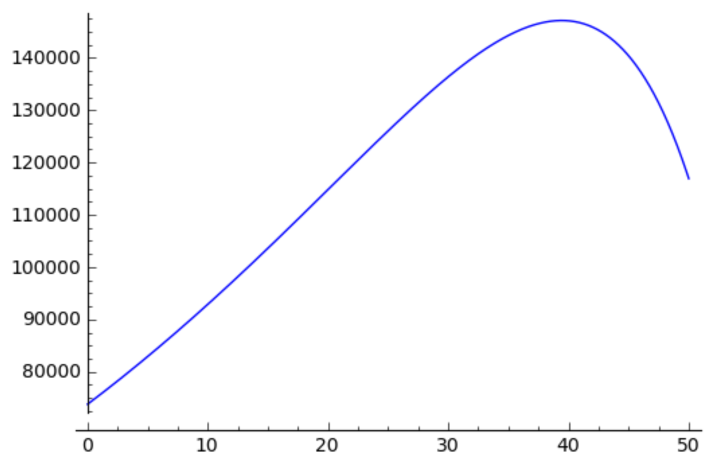
Problem 6

We want to find the value x which makes

$$75000 \cdot (1 + \frac{2.5}{100})^x - 12 \cdot 100 \cdot (1 + \frac{10}{100})^x$$

maximum. (It is a sort of cheating to use calculus for this problem.) I made a graph for the function

$$f(x) = 75000 \cdot (1 + \frac{2.5}{100})^x - 12 \cdot 100 \cdot (1 + \frac{10}{100})^x$$



This graph show that her salary minus her parking costs will be at a maximum after 39. SAGE complutation also says

$$x \sim 39.43118059866336$$

Problem 7

We compare two loan plans:

Bank A : \$ 5000 × $(1.025)^x$
Bank B : \$ 5000 + 500 (fee) before 10 years
\$ 5000 × $(1.04)^{(x-10)}$ + 500 after 10 years

where x means time (year).

Here is the table with approximated values.

x (years)	Bank A (\$)	Bank B (\$)
2	5253.12500000000	5500
10	6400.42272098178	5500
20	8193.08220145197	7901.22142459172
25	9269.72049161076	9504.71752753458

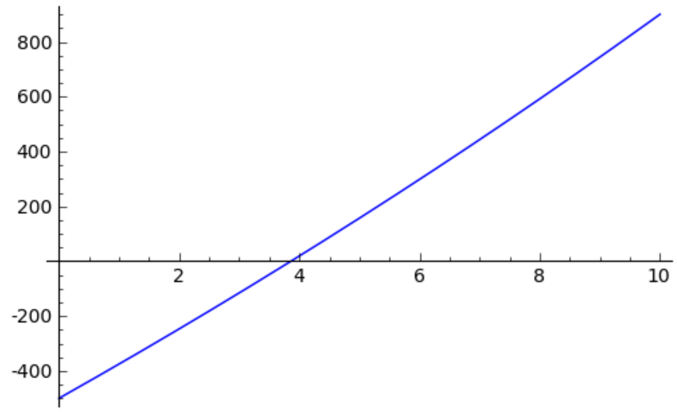
Now we find the values x such that

$$5000 \times (1.025)^x - 5500 = 0$$

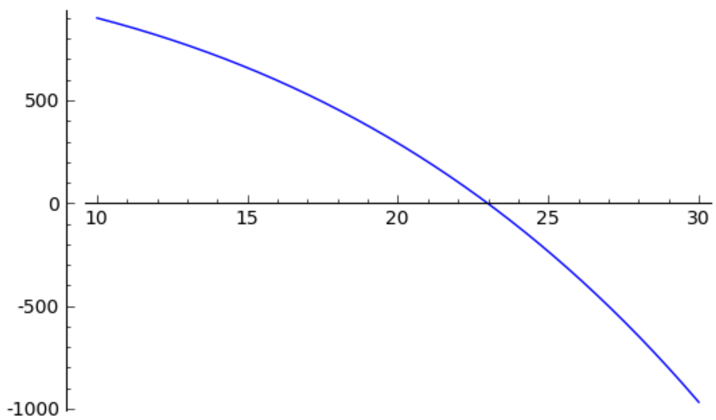
in $0 \leq x \leq 10$, or

$$5000 \times (1.025)^x - 5000 \times (1.04)^{(x-10)} - 500 = 0$$

in $10 < x$. If we observe the above table, we can guess we have at least one value in each range. The graph in the first range is given as follows:



Here is the graph in the second range:



We observe that this function decreases after $x > 10$, thus, there will be only one x satisfying

$$5000 \times (1.025)^x - 5000 \times (1.025)^{(x-10)} - 500 = 0$$

in the second range. Now we compute them:

$(0 \leq x \leq 10)$

$$\begin{aligned} 5000 \times (1.025)^x - 5500 &= 0 \\ (1.025)^x &= \frac{5500}{5000} \\ (1.025)^x &= 1.1 \\ x &= \frac{\ln(1.1)}{\ln(1.025)} \\ x &\sim 3.8598661626226618000000000000 \end{aligned}$$

$(10 < x)$

$$5000 \times (1.025)^x - 5000 \times (1.04)^{(x-10)} - 500 = 0$$

SAGE computation gives the value

$$x \sim 22.978688069699274$$

Thus, at these two times, the two loan plans are equally good.

Problem 8

a. The circumference of this circle is 2π feet. Since

$$\frac{100}{2\pi} \sim 15.915494309189533576888376340$$

the wheel turns around 15 times (and a bit more) to wind up the hose.

b. The angle is determined by the "a bit more" part:

$$0.9154943091895335768883763$$

If we use degree as unit, then

$$360 - 0.9154943091895335768883763 \cdot 360 = 30.42204869176791232018452(^\circ)$$

If we use radian as unit, then

$$2\pi - 0.9154943091895335768883763 \cdot 2\pi = 0.1690113816209328462232473\pi$$

Problem 9

First, we need to rescale the unit of the radius of hose.

$$1/2(\text{inch}) = 1/24(\text{feet})$$

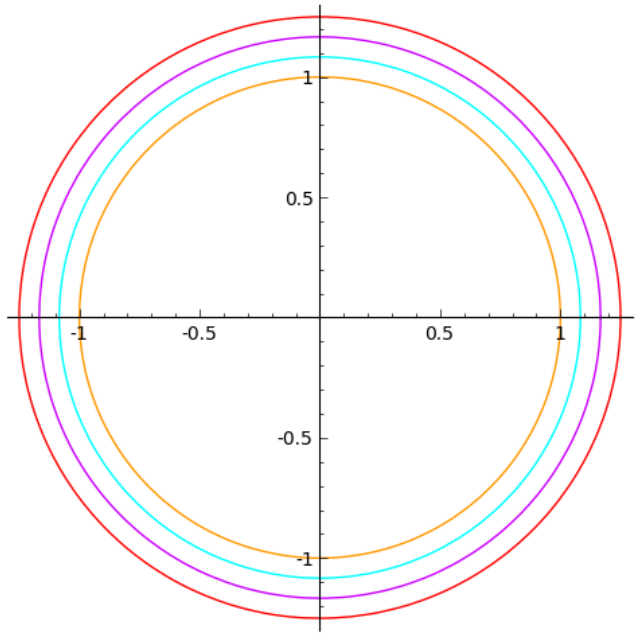
Then the radius of the wheel increases

$$1/12(\text{feet})$$

per each 5 time hose winding. Then the radius of the wheel is given by

- 1 (feet) for the first 5 windings
- 1 + 1/12 (feet) for the second 5 windings
- 1 + 1/12 + 1/12 (feet) for the third 5 windings

Here is the picture:



Now we calculate

$$\begin{aligned}
100 - 5 \cdot 2 \cdot \pi &\sim 68.584073464102067615373566 17\\
100 - 5 \cdot 2 \cdot \pi - 5 \cdot 2 \cdot (1 + 1/12) \cdot \pi &\sim 34.550153050212640865361596 18\\
100 - 5 \cdot 2 \cdot \pi - 5 \cdot 2 \cdot (1 + 1/12) \cdot \pi &- 5 \cdot 2 \cdot (1 + 1/12 + 1/12) \cdot \pi\\
&\sim -2.1017612416682802500359099 58\\
100 - 5 \cdot 2 \cdot \pi - 5 \cdot 2 \cdot (1 + 1/12) \cdot \pi &- 4 \cdot 2 \cdot (1 + 1/12 + 1/12) \cdot \pi\\
&\sim 5.2286216167079039730435912 71
\end{aligned}$$

This calculation indicates that the wheel turns around 14 times (and a bit more) to wind up the hose. To determine the angle, we look at the value 5.228621616707903973043591271. Since

$$2 \cdot (1 + 1/12 + 1/12) \cdot \pi \sim 7.330382858376184223079501228$$

the ratio determines the angle.

$$\frac{5.228621616707903973043591271}{7.330382858376184223079501228} \sim 0.713280836448171637332894003$$

If we use degree as unit, then

$$0.713280836448171637332894003 \cdot 360 - 180 = 76.781101121341789439841841(^\circ)$$

If we use radian as unit, then

$$0.713280836448171637332894003 \cdot 2\pi - \pi = 0.426561672896343274665788801\pi$$

Problem 9'

I measure the inner diameter of the CD to be about $d_{in} = 9/16$ inches, and the outer diameter to be about $d_{out} = 19/4$ inches. To find the speed of a point moving on the inner edge, we need to find how far that point has moved in a minute, based on our knowledge that the disk rotates at 500 rpm when the reader is positioned there.

$$\begin{aligned} C_{in} &= \pi d_{in} \\ &= \frac{9\pi}{16} \\ &\approx 1.77 \text{ in} \end{aligned}$$

Then, to find the speed, we multiply the circumference by the revolutions per minute, and then convert to miles per hour.

$$\begin{aligned} S_{in} &\approx (1.77\,\text{in})\left(\frac{500\,\text{rev}}{1\,\text{min}}\right)\left(\frac{60\,\text{min}}{1\,\text{hr}}\right)\left(\frac{1\,\text{ft}}{12\,\text{in}}\right)\left(\frac{1\,\text{mi}}{5280\,\text{ft}}\right) \\ &\approx 0.84 \text{ mph} \end{aligned}$$

We can do a similar calculation for the speed of a point on the outer edge.

$$\begin{aligned} C_{out} &= \pi d_{out} \\ &= \frac{19\pi}{4} \\ &\approx 14.92 \text{ in} \end{aligned}$$

Then the speed is approximately

$$\begin{aligned} S_{out} &\approx (14.92\,\text{in})\left(\frac{200\,\text{rev}}{1\,\text{min}}\right)\left(\frac{60\,\text{min}}{1\,\text{hr}}\right)\left(\frac{1\,\text{ft}}{12\,\text{in}}\right)\left(\frac{1\,\text{mi}}{5280\,\text{ft}}\right) \\ &\approx 2.83 \text{ mph} \end{aligned}$$

Problem 10

First, note that since the piston is constrained to move in a cylinder, it only has vertical motion. Also, the piston goes smoothly up from its lowest position to its highest position, and vice versa. The piston head is at its lowest point when the connection between the piston arm and the crankshaft is at the bottom, and is at its highest when the connection is at the top. Thus, over the course of half a revolution, the piston head moves a distance equal to the diameter, which is 8 inches. So in an entire revolution, the piston head moves 16 inches. The total distance traveled in one second can be calculated as follows.

$$\begin{aligned} D &= (16\,\text{in})\left(\frac{18,000\,\text{rev}}{1\,\text{min}}\right)\left(\frac{1\,\text{min}}{60\,\text{sec}}\right) \\ &= 4800 \text{ in} \end{aligned}$$

Thus, the distance traveled is 4800 inches, or 400 feet.