

RIEMANNIAN GEOMETRY ON LOOP SPACES

YOSHIAKI MAEDA, STEVEN ROSENBERG, AND FABIÁN TORRES-ARDILA

ABSTRACT. A Riemannian metric on a manifold M induces a family of Riemannian metrics on the loop space LM depending on a Sobolev space parameter s . In Part I, we compute the Levi-Civita connection for these metrics for $s \in \mathbb{Z}^+$. The connection and curvature forms take values in pseudodifferential operators (Ψ DOs), and we compute the top symbols of these forms. In Part II, we develop a theory of Wodzicki-Chern-Simons classes $CS_K^W \in H^{2k-1}(LM^{2k-1})$, for $K = (k_1, \dots, k_\ell)$ a partition of $2k-1$, using the Wodzicki residue on Ψ DOs. We use CS_5^W to prove that certain actions on $S^2 \times S^3$ are not smoothly homotopic, and that $\pi_1(\text{Diff}(S^2 \times S^3))$ is infinite.

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1. Introduction

The loop space LM of a manifold M appears frequently in mathematics and mathematical physics. In this paper, using an infinite dimensional version of Chern-Simons theory, we develop a computable theory of secondary characteristic classes on the tangent bundle to loop spaces. We apply these secondary classes to distinguish circle actions on $S^2 \times S^3$, and we prove that $\pi_1(\text{Diff}(S^2 \times S^3))$ is infinite.

The theory of primary characteristic classes on certain infinite rank bundles was treated via Chern-Weil theory in [20], and depends on the choice of a trace on the algebra of nonpositive order classical pseudodifferential operators (Ψ DOs). While these classes can be nonzero for some traces [16], the Chern classes vanish on mapping spaces $\text{Maps}(N, M)$ for the trace used in this paper. Thus we are forced to consider secondary classes.

In finite dimensions, a characteristic class on a manifold M can give rise to two types of secondary classes: (i) if a characteristic form of degree $2k$ vanishes for a specific connection ∇ , there is an associated “absolute” Chern-Simons form $CS(\nabla) \in H^{2k-1}(M, \mathbb{R}/\mathbb{Z})$; (ii) if the characteristic forms are equal for two connections ∇_0, ∇_1 , there is an associated “relative” Chern-Simons class $CS(\nabla_0, \nabla_1) \in H^{2k-1}(M, \mathbb{R})$. For technical reasons, it is easier to work with the relative classes in our context.

Since Chern-Weil and Chern-Simons theory are geometric, it is necessary to understand connections and curvature on loop spaces. We work with Sobolev spaces of highly differentiable loops, as this is a Hilbert manifold, although we could work with the Fréchet manifold of smooth loops. A Riemannian metric g on M induces a family of metrics g^s on LM parametrized by a Sobolev space parameter $s \geq 0$, where $s = 0$ gives the usual L^2 metric, and the smooth case is a kind of limit as $s \rightarrow \infty$. Thus we think of s as a regularizing parameter, and pay attention to the parts of the theory which are independent of s .

In Part I, we compute the connection and curvature for the Levi-Civita connection for g^s for $s \in \mathbb{Z}^+$. These forms take values in zeroth order pseudodifferential operators (Ψ DOs) acting on a trivial bundle over S^1 , as first shown by Freed for loop groups [10]. We calculate the principal and subprincipal symbols for the connection and curvature forms. In contrast, there is no Levi-Civita connection for nonintegral values of s in a precise sense, but we can define a family of modified connections which vary continuously in s .

In Part II, we develop a theory of Chern-Simons classes on loop spaces. The structure group for the Levi-Civita connection for (LM, g^s) is the set of invertible zeroth order Ψ DOs, so we need invariant polynomials on the corresponding Lie algebra. The naive choice is the standard polynomials $\text{Tr}(\Omega^k)$ of the curvature $\Omega = \Omega^s$, where Tr is the operator trace. However, Ω^k is zeroth order and hence not trace class, and in any case the operator trace is impossible to compute in general. Instead, as in [20] we use the Wodzicki residue, the only trace on the full algebra of Ψ DOs. Following Chern-Simons [6] as much as possible, we build a theory of Wodzicki-Chern-Simons (WCS)

classes, which gives classes in $H^{2k-1}(LM^{2k-1})$ associated to partitions of k . The main difference from the finite dimensional theory is the absence of a Narasimhan-Ramanan universal connection theorem. As a result, we do not have a theory of differential characters [4]. In contrast to the operator trace, the Wodzicki residue is locally computable, so we can write explicit expressions for the WCS classes. In particular, we can see how the WCS classes depend on the Sobolev parameter s , and hence define a “regularized” or s -independent WCS classes. The local expression also yields some vanishing results for WCS classes. More importantly, we produce a non-vanishing WCS class on $L(S^2 \times S^3)$. This leads to the topological results described in the first paragraph.

The paper is organized as follows. Part I treats the family of metrics g^s on LM associated to (M, g) . §2 discusses connections associated to g^s . After some preliminary material, we compute the Levi-Civita connection for $s = 0$ (Lemma 2.1), $s = 1$ (Theorem 2.2), and $s \in \mathbb{Z}^+$ (Theorem 2.10). In contrast, the Levi-Civita connection does not exist in a precise sense for nonintegral s (Theorem 2.12). Nevertheless, by removing the “bad terms” from this calculation, we can construct a family of connections, called the H^s connections, which depend continuously on s (Proposition 2.13). These connections allow us to track how the geometry of LM depends on s .

Both the Levi-Civita and H^s connections have connection and curvature forms taking values in Ψ DOs of order zero. In §3, we compute the symbols of these forms needed in Part II. In §4, we compare our results to Freed’s on loop groups [10].

Part II covers Wodzicki-Chern-Simons classes. In §5, we review the finite dimensional construction of Chern and Chern-Simons classes, and use the Wodzicki residue to define Wodzicki-Chern (WC) and WCS classes (Definition 5.1). We prove the necessary vanishing of the WC classes for mapping spaces (and in particular for LM) in Proposition 5.2. In Theorem 5.5, we give the explicit local expression for the relative WCS class $CS_{2k-1}^W(g) \in H^{2k-1}(LM^{2k-1})$ associated to the trivial partition of k . We then define the regularized or s -independent WCS class. In Theorem 5.6, we give a vanishing result for WCS classes.

In particular, the WCS class which is the analogue of the classical dimension three Chern-Simons class vanishes on loop spaces of 3-manifolds, so we look for nontrivial examples on 5-manifolds. In §6, we use a Sasaki-Einstein metric constructed in [12] to produce a nonzero WCS class $CS_5^W \in H^5(L(S^2 \times S^3))$. We prove $CS_5^W \neq 0$ by an exact computer calculation showing $\int_{[a^L]} CS_5^W \neq 0$, where $[a^L] \in H_5(LM)$ is a cycle associated to a simple circle action on $S^2 \times S^3$. From this nonvanishing, we conclude both that the circle action is not smoothly homotopic to the trivial action and the $\pi_1(\text{Diff}(S^2 \times S^3))$ is infinite. We expect other similar results in the future.

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Part I. The Levi-Civita Connection on the Loop Space LM

In this part, we compute the Levi-Civita connection on LM associated to a Riemannian metric on M and a Sobolev parameter $s \in \mathbb{Z}^+$. In §2, the main results are Lemma 2.1, Theorem 2.2, and Theorem 2.10, which compute the Levi-Civita connection for $s = 0$, $s = 1$, and $s \in \mathbb{Z}^+$, respectively. For $s \notin \mathbb{Z}^+$, the Levi-Civita connection does not exist (Theorem 2.12), but there are closely related connections (Proposition 2.13) which depend continuously on s . In §3, we compute the relevant symbols of the connection one-forms and the curvature two-forms. In §4, we compare our results with work of Freed [10] on loop groups.

2. The Levi-Civita Connection for Sobolev Parameter $s \geq 0$

This section covers background material and computes the Levi-Civita connection on LM for integer Sobolev parameter s . In §2.1, we review material on LM , and in §2.2 we review pseudodifferential operators and the Wodzicki residue. In §2.3, we give the main computation for the Levi-Civita connections for $s = 0, 1$. In §2.4, we extend this computation to $s \in \mathbb{Z}^+$. In §2.5, we show that the Levi-Civita connection does not exist for $s \notin \mathbb{Z}^+$. In §2.6, we modify the Levi-Civita connection for $s \in \mathbb{Z}^+$ to a torsion free connection, called the H^s connection, which extends to all $s \geq 0$. In this sense, the H^s connection behaves better than the Levi-Civita connection. In §2.7, we discuss how the geometry of LM forces an extension of the structure group of LM from a gauge group to a group of bounded invertible Ψ DOs.

2.1. Preliminaries on LM .

Let $(M, \langle \cdot, \cdot \rangle)$ be a closed oriented Riemannian n -manifold with loop space $LM = C^\infty(S^1, M)$ of smooth loops. LM is a smooth infinite dimensional Fréchet manifold, but it is technically simpler to work with the smooth Hilbert manifold of loops in some Sobolev class $s \gg 0$, as we now recall. For $\gamma \in LM$, the formal tangent space $T_\gamma LM$ is $\Gamma(\gamma^*TM)$, the space of smooth sections of the pullback bundle $\gamma^*TM \rightarrow S^1$. For some $s > 1/2$, we complete $\Gamma(\gamma^*TM \otimes \mathbb{C})$ with respect to the Sobolev inner product

$$\langle X, Y \rangle_s = \frac{1}{2\pi} \int_0^{2\pi} \langle (1 + \Delta)^s X(\alpha), Y(\alpha) \rangle_{\gamma(\alpha)} d\alpha, \quad X, Y \in \Gamma(\gamma^*TM).$$

Here $\Delta = D^*D$, with $D = D/d\gamma$ the covariant derivative along γ . (We use this notation instead of the classical D/dt to keep track of γ .) We need the complexified pullback bundle $\gamma^*TM \otimes \mathbb{C}$, denoted from now on just as γ^*TM , in order to apply the pseudodifferential operator $(1 + \Delta)^s$. The construction of $(1 + \Delta)^s$ is reviewed in §2.2. We denote this completion by $H^s(\gamma^*TM)$.

A small real neighborhood U_γ of the zero section in $H^s(\gamma^*TM)$ is a coordinate chart near γ in the space of H^s loops via the pointwise exponential map

$$\exp_\gamma : U_\gamma \rightarrow LM, \quad X \mapsto (\alpha \mapsto \exp_{\gamma(\alpha)} X(\alpha)). \quad (2.1)$$

The differentiability of the transition functions $\exp_{\gamma_1}^{-1} \cdot \exp_{\gamma_2}$ is proved in [7] and [11, Appendix A]. Here γ_1, γ_2 are close loops in the sense that a geodesically convex

neighborhood of $\gamma_1(\theta)$ contains $\gamma_2(\theta)$ and vice versa for all θ . Since γ^*TM is (non-canonically) isomorphic to the trivial bundle $\mathcal{R} = S^1 \times \mathbb{C}^n \longrightarrow S^1$, the model space for LM is the set of H^s sections of this trivial bundle.

The complexified tangent bundle TLM has transition functions $d(\exp_{\gamma_1}^{-1} \circ \exp_{\gamma_2})$. Under the isomorphisms $T_{\gamma_1}LM \simeq \mathcal{R} \simeq T_{\gamma_2}LM$, the transition functions lie in the gauge group $\mathcal{G}(\mathcal{R})$, so this is the structure group of TLM .

2.2. Review of Ψ DO Calculus.

We recall the construction of classical pseudodifferential operators (Ψ DOs) on a closed manifold M from [13, 22], assuming knowledge of Ψ DOs on \mathbb{R}^n .

A linear operator $P : C^\infty(M) \longrightarrow C^\infty(M)$ is a Ψ DO of order d if for every open chart $U \subset M$ and functions $\phi, \psi \in C_c^\infty(U)$, $\phi P \psi$ is a Ψ DO of order d on \mathbb{R}^n , where we do not distinguish between U and its diffeomorphic image in \mathbb{R}^n . Let $\{U_i\}$ be a finite cover of M with subordinate partition of unity $\{\phi_i\}$. Let $\psi_i \in C_c^\infty(U_i)$ have $\psi_i \equiv 1$ on $\text{supp}(\phi_i)$ and set $P_i = \psi_i P \phi_i$. Then $\sum_i \phi_i P_i \psi_i$ is a Ψ DO on M , and P differs from $\sum_i \phi_i P_i \psi_i$ by a smoothing operator, denoted $P \sim \sum_i \phi_i P_i \psi_i$. In particular, this sum is independent of the choices up to smoothing operators. All this carries over to Ψ DOs acting on sections of a bundle over M .

An example is the Ψ DO $(1 + \Delta - \lambda)^{-1}$ for Δ a positive order nonnegative elliptic Ψ DO and λ outside the spectrum of $1 + \Delta$. In each U_i , we construct a parametrix P_i for $A_i = \psi_i(1 + \Delta - \lambda)\phi_i$ by formally inverting $\sigma(A_i)$ and then constructing a Ψ DO with the inverted symbol. By [1, App. A], $B = \sum_i \phi_i P_i \psi_i$ is a parametrix for $(1 + \Delta - \lambda)^{-1}$. Since $B \sim (1 + \Delta - \lambda)^{-1}$, $(1 + \Delta - \lambda)^{-1}$ is itself a Ψ DO. For $x \in U_i$, by definition

$$\sigma((1 + \Delta - \lambda)^{-1})(x, \xi) = \sigma(P)(x, \xi) = \sigma(\phi P \phi)(x, \xi),$$

where ϕ is a bump function with $\phi(x) = 1$ [13, p. 29]; the symbol depends on the choice of (U_i, ϕ_i) .

The operator $(1 + \Delta)^s$ for $\text{Re}(s) < 0$, which exists as a bounded operator on $L^2(M)$ by the functional calculus, is also a Ψ DO. To see this, we construct the putative symbol σ_i of $\psi_i(1 + \Delta)^s\phi_i$ in each U_i by a contour integral $\int_\Gamma \lambda^s \sigma[(1 + \Delta - \lambda)^{-1}] d\lambda$ around the spectrum of $1 + \Delta$. We then construct a Ψ DO Q_i on U_i with $\sigma(Q_i) = \sigma_i$, and set $Q = \sum_i \phi_i Q_i \psi_i$. By arguments in [22], $(1 + \Delta)^s \sim Q$, so $(1 + \Delta)^s$ is a Ψ DO.

Recall that the *Wodzicki residue* of a Ψ DO P on sections of a bundle $E \longrightarrow M^n$ is

$$\text{res}^w(P) = \int_{S^*M} \text{tr} \sigma_{-n}(P)(x, \xi) d\xi dx, \quad (2.2)$$

where S^*M is the unit cosphere bundle for some metric. The Wodzicki residue is independent of choice of local coordinates, and up to scaling is the unique trace on the algebra of Ψ DOs if $\dim(M) > 1$ (see e.g. [9] in general and [21] for the case $M = S^1$).

The Wodzicki residue will be used in Part II to define characteristic classes on LM . In our particular case, the operator P will be an Ψ DO of order -1 acting on

sections of a bundle over S^1 (see (5.10)), so $\sigma_{-1}(P)$ is globally defined. Of course, $\int_{S^*S^1} \text{tr } \sigma_{-1}(P) d\xi d\theta = 2 \int_{S^1} \text{tr } \sigma_{-1}(P) d\theta$. It is easy to check that this integral, which strictly speaking involves a choice of cover of S^1 and a partition of unity, equals the usual $2 \int_0^{2\pi} \text{tr } \sigma_{-1}(P) d\theta$.

2.3. The Levi-Civita Connection for $s = 0, 1$.

The H^s metric makes LM a Riemannian manifold. The corresponding Levi-Civita connection on LM , if it exists, is determined by the six term formula

$$\begin{aligned} 2\langle \nabla_Y^s X, Z \rangle_s &= X\langle Y, Z \rangle_s + Y\langle X, Z \rangle_s - Z\langle X, Y \rangle_s \\ &\quad + \langle [X, Y], Z \rangle_s + \langle [Z, X], Y \rangle_s - \langle [Y, Z], X \rangle_s. \end{aligned} \quad (2.3)$$

In particular, the Levi-Civita connection exists if for fixed X, Y , each term T_i on the right hand side defines a *continuous* linear functional $T_i : H^s(\gamma^*TM) \rightarrow \mathbb{C}$ in Z . If so, $T_i(Z) = \langle T'_i(X, Y), Z \rangle_s$ for a unique $T'(X, Y) \in H^s(\gamma^*TM)$, and $\nabla_Y^s X = \frac{1}{2} \sum_i T'_i$.

We first discuss local coordinates on LM . For motivation, recall that

$$[X, Y]^a = X(Y^a)\partial_a - Y(X^a)\partial_a \equiv \delta_X(Y) - \delta_Y(X) \quad (2.4)$$

in local coordinates on a finite dimensional manifold. Note that $X^i \partial_i Y^a = X(Y^a) = (\delta_X Y)^a$ in this notation.

Let Y be a vector field on LM , and let X be a tangent vector at $\gamma \in LM$. The local variation $\delta_X Y$ of Y in the direction of X at γ is defined as usual: let $\gamma(\varepsilon, \theta)$ be a family of loops in M with $\gamma(0, \theta) = \gamma(\theta)$, $\frac{d}{d\varepsilon}|_{\varepsilon=0} \gamma(\varepsilon, \theta) = X(\theta)$. Fix θ , and let (x^a) be coordinates near $\gamma(\theta)$. We call these coordinates *manifold coordinates*. Then

$$\delta_X Y^a(\gamma)(\theta) \stackrel{\text{def}}{=} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} Y^a(\gamma(\theta)).$$

Note that $\delta_X Y^a = (\delta_X Y)^a$ by definition.

Remark 2.1. Having (x^a) defined only near a fixed θ is inconvenient. We can find coordinates that work for all θ as follows. For fixed γ , there is an ε such that for all θ , $\exp_{\gamma(\theta)} X$ is inside the cut locus of $\gamma(\theta)$ if $X \in T_{\gamma(\theta)}M$ has $|X| < \varepsilon$. Fix such an ε . Call $X \in H^s(\gamma^*TM)$ *short* if $|X(\theta)| < \varepsilon$ for all θ . Then

$$U_\gamma = \{\theta \mapsto \exp_{\gamma(\theta)} X(\theta) \mid X \text{ is short}\} \subset LM$$

is a coordinate neighborhood of γ parametrized by $\{X : X \text{ is short}\}$.

We know $H^s(\gamma^*TM) \simeq H^s(S^1 \times \mathbb{R}^n)$ noncanonically, so U_γ is parametrized by short sections of $H^s(S^1 \times \mathbb{R}^n)$ for a different ε . In particular, we have a smooth diffeomorphism β from U_γ to short sections of $H^s(S^1 \times \mathbb{R}^n)$.

Put coordinates (x^a) on \mathbb{R}^n , which we identify canonically with the fiber \mathbb{R}_θ^n over θ in $S^1 \times \mathbb{R}^n$. For $\eta \in U_\gamma$, we have $\beta(\eta) = (\beta(\eta)^1(\theta), \dots, \beta(\eta)^n(\theta))$. As with finite dimensional coordinate systems, we will drop β and just write $\eta = (\eta(\theta)^a)$. These coordinates work for all η near γ and for all θ . The definition of $\delta_X Y$ above carries over to exponential coordinates.

We will call these coordinates *exponential coordinates*.

(2.4) continues to hold for vector fields on LM , in either manifold or exponential coordinates. To see this, one checks that the coordinate-free proof that $L_X Y(f) = [X, Y](f)$ for $f \in C^\infty(M)$ (e.g. [24, p. 70]) carries over to functions on LM . In brief, the usual proof involves a map $H(s, t)$ of a neighborhood of the origin in \mathbb{R}^2 into M , where s, t are parameters for the flows of X, Y , resp. For LM , we have a map $H(s, t, \theta)$, where θ is the loop parameter. The usual proof uses only s, t differentiations, so θ is unaffected. The point is that the Y^i are local functions on the (s, t, θ) parameter space, whereas the Y^i are not local functions on M at points where loops cross or self-intersect.

We first compute the L^2 ($s = 0$) Levi-Civita connection invariantly and in manifold coordinates.

Lemma 2.1. *Let ∇^{LC} be the Levi-Civita connection on M . Let $\text{ev}_\theta : LM \rightarrow M$ be $\text{ev}_\theta(\gamma) = \gamma(\theta)$. Then $D_X Y(\gamma)(\theta) \stackrel{\text{def}}{=} (\text{ev}_\theta^* \nabla^{LC})_X Y(\gamma)(\theta)$ is the L^2 Levi-Civita connection on LM . In manifold coordinates,*

$$(D_X Y)^a(\gamma)(\theta) = \delta_X Y^a(\gamma)(\theta) + \Gamma_{bc}^a(\gamma(\theta)) X^b(\gamma)(\theta) Y^c(\gamma)(\theta). \quad (2.5)$$

Proof. $\text{ev}_\theta^* \nabla^{LC}$ is a connection on $\text{ev}_\theta^* TM \rightarrow LM$. We have $\text{ev}_{\theta,*}(X) = X(\theta)$. If U is a coordinate neighborhood on M near some $\gamma(\theta)$, then on $\text{ev}_\theta^{-1}(U)$,

$$\begin{aligned} (\text{ev}_\theta^* \nabla^{LC})_X Y^a(\gamma)(\theta) &= (\delta_X Y)^a(\gamma)(\theta) + ((\text{ev}_\theta^* \omega_X^{LC}) Y)^a(\theta) \\ &= (\delta_X Y)^a(\gamma)(\theta) + \Gamma_{bc}^a(\gamma(\theta)) X^b(\gamma)(\theta) Y^c(\gamma)(\theta) \end{aligned}$$

Since $\text{ev}_\theta^* \nabla^{LC}$ is a connection, for each fixed θ, γ and $X \in T_\gamma LM, Y \mapsto (\text{ev}_\theta^* \nabla^{LC})_X Y(\gamma)$ has Leibniz rule with respect to functions on LM . Thus D is a connection on LM .

D is torsion free, as from the local expression, $D_X Y - D_Y X = \delta_X Y - \delta_Y X = [X, Y]$.

To show that $D_X Y$ is compatible with the L^2 metric, first recall that for a function f on LM , $D_X f = \delta_X f = \frac{d}{d\varepsilon}|_{\varepsilon=0} f(\gamma(\varepsilon, \theta))$ for $X(\theta) = \frac{d}{d\varepsilon}|_{\varepsilon=0} \gamma(\varepsilon, \theta)$. (Here f depends

only on γ .) Thus (suppressing the partition of unity, which is independent of ε)

$$\begin{aligned}
D_X \langle X, Y \rangle_0 &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_{S^1} g_{ab}(\gamma(\varepsilon, \theta)) Y^a(\gamma(\varepsilon, \theta)) Z^b(\gamma(\varepsilon, \theta)) d\theta \\
&= \int_{S^1} \partial_c g_{ab}(\gamma(\varepsilon, \theta)) X^c Y^a(\gamma(\varepsilon, \theta)) Z^b(\gamma(\varepsilon, \theta)) d\theta \\
&\quad + \int_{S^1} g_{ab}(\gamma(\varepsilon, \theta)) (\delta_X Y)^a(\gamma(\varepsilon, \theta)) Z^b(\gamma(\varepsilon, \theta)) d\theta \\
&\quad + \int_{S^1} g_{ab}(\gamma(\varepsilon, \theta)) Y^a(\gamma(\varepsilon, \theta)) (\delta_X Z)^b(\gamma(\varepsilon, \theta)) d\theta \\
&= \int_{S^1} \Gamma_{ca}^e g_{eb} X^c Y^a Z^b + \Gamma_{cb}^e g_{ae} X^c Y^a Z^b \\
&\quad + g_{ab} (\delta_X Y)^a Z^b + g_{ab} Y^a (\delta_X Z)^b d\theta \\
&= \langle D_X Y, Z \rangle_0 + \langle Y, D_X Z \rangle_0.
\end{aligned}$$

□

Remark 2.2. The local expression for $D_X Y$ also holds in exponential coordinates. More precisely, let $(e_1(\theta), \dots, e_n(\theta))$ be a global frame of $\gamma^* TM$ given by the trivialization of $\gamma^* TM$. Then $(e_i(\theta))$ is also naturally a frame of $T_X T_{\gamma(\theta)} M$ for all $X \in T_{\gamma(\theta)} M$. We use $\exp_{\gamma(\theta)}$ to pull back the metric on M to a metric on $T_{\gamma(\theta)} M$:

$$g_{ij}(X) = (\exp_{\gamma(\theta)}^* g)(e_i, e_j) = g(d(\exp_{\gamma(\theta)})_X(e_i), d(\exp_{\gamma(\theta)})_X(e_j))_{\exp_{\gamma(\theta)} X}.$$

Then the Christoffel symbols $\Gamma_{bc}^a(\gamma(\theta))$ are computed with respect to this metric. For example, the term $\partial_\ell g_{bc}$ means $e_\ell g(e_a, e_b)$, etc. The proof that $D_X Y$ has the local expression (2.5) then carries over to exponential coordinates.

The $s = 1$ Levi-Civita connection on LM is given as follows.

Theorem 2.2. *The $s = 1$ Levi-Civita connection $\nabla = \nabla^1$ on LM is given at the loop γ by*

$$\begin{aligned}
\nabla_X Y &= D_X Y + \frac{1}{2}(1 + \Delta)^{-1} [-\nabla_{\dot{\gamma}}(R(X, \dot{\gamma})Y) - R(X, \dot{\gamma})\nabla_{\dot{\gamma}}Y \\
&\quad - \nabla_{\dot{\gamma}}(R(Y, \dot{\gamma})X) - R(Y, \dot{\gamma})\nabla_{\dot{\gamma}}X \\
&\quad + R(X, \nabla_{\dot{\gamma}}Y)\dot{\gamma} - R(\nabla_{\dot{\gamma}}X, Y)\dot{\gamma}].
\end{aligned}$$

We prove this in a series of steps.

Proposition 2.3. *The Levi-Civita connection for the $s = 1$ metric is given by*

$$\nabla_X^1 Y = D_X Y + \frac{1}{2}(1 + \Delta)^{-1} [D_X, 1 + \Delta] Y + \frac{1}{2}(1 + \Delta)^{-1} [D_Y, 1 + \Delta] X + A_X Y,$$

where we assume that $A_X Y$ is well-defined by

$$-\frac{1}{2}\langle [D_Z, 1 + \Delta]X, Y \rangle_0 = \langle A_X Y, Z \rangle_1.$$

Proof. By Lemma 2.1,

$$\begin{aligned} X\langle Y, Z \rangle_1 &= X\langle (1 + \Delta)Y, Z \rangle_0 = \langle D_X((1 + \Delta)Y), Z \rangle_0 + \langle (1 + \Delta)Y, D_X Z \rangle_0 \\ Y\langle X, Z \rangle_1 &= \langle D_Y((1 + \Delta)X), Z \rangle_0 + \langle (1 + \Delta)X, D_Y Z \rangle_0 \\ -Z\langle X, Y \rangle_1 &= -\langle D_Z((1 + \Delta)X), Y \rangle_0 - \langle (1 + \Delta)X, D_Z Y \rangle_0 \\ \langle [X, Y], Z \rangle_1 &= \langle (1 + \Delta)(\delta_X Y - \delta_Y X), Z \rangle_0 = \langle (1 + \Delta)(D_X Y - D_Y X), Z \rangle_0 \\ \langle [Z, X], Y \rangle_1 &= \langle (1 + \Delta)(D_Z X - D_X Z), Y \rangle_0 \\ -\langle [Y, Z], X \rangle_1 &= -\langle (1 + \Delta)(D_Y Z - D_Z Y), X \rangle_0. \end{aligned}$$

The six terms on the left hand side sum up to twice the $s = 1$ Levi-Civita connection $\nabla = \nabla^1$. After some cancellations, we get

$$\begin{aligned} 2\langle \nabla_X Y, Z \rangle_1 &= \langle D_X((1 + \Delta)Y), Z \rangle_0 + \langle D_Y((1 + \Delta)X), Z \rangle_0 \\ &\quad + \langle (1 + \Delta)(D_X Y - D_Y X), Z \rangle_0 - \langle D_Z((1 + \Delta)X), Y \rangle_0 \\ &\quad + \langle (1 + \Delta)D_Z X, Y \rangle_0 \\ &= \langle (1 + \Delta)D_X Y, Z \rangle_0 + \langle [D_X, 1 + \Delta]Y, Z \rangle_0 \\ &\quad + \langle (1 + \Delta)D_Y X, Z \rangle_0 + \langle [D_Y, 1 + \Delta]X, Z \rangle_0 \\ &\quad + \langle (1 + \Delta)(D_X Y - D_Y X), Z \rangle_0 - \langle [D_Z, 1 + \Delta]X, Y \rangle_0 \\ &= 2\langle D_X Y, Z \rangle_1 + \langle (1 + \Delta)^{-1}[D_X, 1 + \Delta]Y, Z \rangle_1 \\ &\quad + \langle (1 + \Delta)^{-1}[D_Y, 1 + \Delta]X, Z \rangle_1 - \langle A_X Y, Z \rangle_1. \end{aligned}$$

□

Now we compute the bracket terms in the Proposition. We have $[D_X, 1 + \Delta] = [D_X, \Delta]$. Also,

$$0 = \dot{\gamma}\langle X, Y \rangle_0 = \langle \nabla_{\dot{\gamma}} X, Y \rangle_0 + \langle X, \nabla_{\dot{\gamma}} Y \rangle_0,$$

so

$$\Delta = \nabla_{\dot{\gamma}}^* \nabla_{\dot{\gamma}} = -\nabla_{\dot{\gamma}}^2. \quad (2.6)$$

Lemma 2.4. $[D_X, \nabla_{\dot{\gamma}}]Y = R(X, \dot{\gamma})Y$.

Proof. Note that $\gamma^\nu, \dot{\gamma}^\nu$ are locally defined functions on $S^1 \times LM$. Let $\tilde{\gamma} : [0, 2\pi] \times (-\varepsilon, \varepsilon) \rightarrow M$ be a smooth map with $\tilde{\gamma}(\theta, 0) = \gamma(\theta)$, and $\frac{d}{d\tau}|_{\tau=0} \tilde{\gamma}(\theta, \tau) = Z(\theta)$. Since (θ, τ) are coordinate functions on $S^1 \times (-\varepsilon, \varepsilon)$, we have

$$\begin{aligned} Z(\dot{\gamma}^\nu) &= \delta_Z(\dot{\gamma}^\nu) = \partial_\tau^Z(\dot{\gamma}^\nu) = \frac{\partial}{\partial \tau} \Big|_{\tau=0} \left(\frac{\partial}{\partial \theta}(\tilde{\gamma}(\theta, \tau)^\nu) \right) \\ &= \frac{\partial}{\partial \theta} \frac{\partial}{\partial \tau} \Big|_{\tau=0} \tilde{\gamma}(\theta, \tau)^\nu = \partial_\theta Z^\nu \equiv \dot{Z}^\nu. \end{aligned} \quad (2.7)$$

We compute

$$\begin{aligned}
(D_X \nabla_{\dot{\gamma}} Y)^a &= \delta_X (\nabla_{\dot{\gamma}} Y)^a + \Gamma_{bc}^a X^b \nabla_{\dot{\gamma}} Y^c \\
&= \delta_X (\dot{\gamma}^j \partial_j Y^a + \Gamma_{bc}^a \dot{\gamma}^b Y^c) + \Gamma_{bc}^a X^b (\dot{\gamma}^j \partial_j Y^c + \Gamma_{ef}^c \dot{\gamma}^e Y^f) \\
&= \dot{X}^j \partial_j Y^a + \dot{\gamma}^j \partial_j \delta_X Y^a + \partial_M \Gamma_{bc}^a X^m \dot{\gamma}^b Y^c + \Gamma_{bc}^a \dot{X}^b Y^c + \Gamma_{bc}^a \dot{\gamma}^b \delta_X Y^c \\
&\quad + \Gamma_{bc}^a X^b \dot{\gamma}^j \partial_j Y^c + \Gamma_{bc}^a \Gamma_{ef}^c X^b \dot{\gamma}^e Y^f. \\
(\nabla_{\dot{\gamma}} D_X Y)^a &= \dot{\gamma}^j (\partial_j (D_X Y)^a + \Gamma_{bc}^a \dot{\gamma}^b (D_X Y)^c) \\
&= \dot{\gamma}^j \partial_j (\delta_X Y^a + \Gamma_{bc}^a X^b Y^c) + \Gamma_{bc}^a \dot{\gamma}^b (\delta_X Y^c + \Gamma_{sf}^c X^s Y^f) \\
&= \dot{\gamma}^j \partial_j \delta_X Y^a + \dot{\gamma}^j \partial_j \Gamma_{bc}^a X^b Y^c + \Gamma_{bc}^a \dot{X}^b Y^c + \Gamma_{bc}^a X^b \dot{Y}^c + \Gamma_{bc}^a \dot{\gamma}^b \delta_X Y^c \\
&\quad + \Gamma_{bc}^a \Gamma_{ef}^c \dot{\gamma}^b X^e Y^f.
\end{aligned}$$

Therefore

$$\begin{aligned}
(D_X \nabla_{\dot{\gamma}} Y - \nabla_{\dot{\gamma}} D_X Y)^a &= \partial_m \Gamma_{bc}^a X^m \dot{\gamma}^b Y^c - \partial_j \Gamma_{bc}^a \dot{\gamma}^j X^b Y^c + \Gamma_{bc}^a \Gamma_{ef}^c X^b \dot{\gamma}^e Y^f \\
&\quad - \Gamma_{bc}^a \Gamma_{ef}^c \dot{\gamma}^b X^e Y^f \\
&= (\partial_j \Gamma_{bc}^a - \partial_b \Gamma_{jc}^a + \Gamma_{je}^a \Gamma_{bc}^e - \Gamma_{be}^a \Gamma_{jc}^e) \dot{\gamma}^b X^j Y^c \\
&= R_{jbc}{}^a X^j \dot{\gamma}^b Y^c,
\end{aligned}$$

so

$$D_X \nabla_{\dot{\gamma}} Y - \nabla_{\dot{\gamma}} D_X Y = R(X, \dot{\gamma})Y.$$

□

Corollary 2.5. *At the loop γ , $[D_X, \Delta]Y = -\nabla_{\dot{\gamma}}(R(X, \dot{\gamma})Y) - R(X, \dot{\gamma})\nabla_{\dot{\gamma}}Y$. In particular, $[D_X, \Delta]$ is a zeroth order operator.*

Proof.

$$\begin{aligned}
[D_X, \Delta]Y &= (-D_X \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} + \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} D_X)Y \\
&= -(\nabla_{\dot{\gamma}} D_X \nabla_{\dot{\gamma}} Y + R(X, \dot{\gamma})\nabla_{\dot{\gamma}} Y) + \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} D_X Y \\
&= -(\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} D_X Y + \nabla_{\dot{\gamma}}(R(X, \dot{\gamma})Y) + R(X, \dot{\gamma})\nabla_{\dot{\gamma}} Y) + \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} D_X Y \\
&= -\nabla_{\dot{\gamma}}(R(X, \dot{\gamma})Y) - R(X, \dot{\gamma})\nabla_{\dot{\gamma}} Y.
\end{aligned}$$

□

Now we complete the proof of Theorem 2.2. In the proof, we justify that $A_X Y$ in Proposition 2.3 exists.

Proof of Theorem 2.2. By Proposition 2.3 and Corollary 2.5, we have

$$\begin{aligned}
\nabla_X Y &= D_X Y + \frac{1}{2}(1 + \Delta)^{-1}[D_X, 1 + \Delta]Y + (X \leftrightarrow Y) + A_X Y \\
&= D_X Y + \frac{1}{2}(1 + \Delta)^{-1}(-\nabla_{\dot{\gamma}}(R(X, \dot{\gamma})Y) - R(X, \dot{\gamma})\nabla_{\dot{\gamma}} Y) + (X \leftrightarrow Y) + A_X Y,
\end{aligned}$$

where $(X \leftrightarrow Y)$ denotes the previous term with X and Y switched.

The curvature tensor satisfies

$$-\langle Z, R(X, Y)W \rangle = \langle R(X, Y)Z, W \rangle = \langle R(Z, W)X, Y \rangle$$

pointwise, so

$$\begin{aligned} \langle A_X Y, Z \rangle_1 &= -\frac{1}{2} \langle [D_Z, 1 + \Delta] X, Y \rangle_0 \\ &= -\frac{1}{2} \langle (-\nabla_{\dot{\gamma}}(R(Z, \dot{\gamma})X) - R(Z, \dot{\gamma})\nabla_{\dot{\gamma}}X, Y) \rangle_0 \\ &= -\frac{1}{2} \langle R(Z, \dot{\gamma})X, \nabla_{\dot{\gamma}}Y \rangle_0 + \frac{1}{2} \langle R(Z, \dot{\gamma})\nabla_{\dot{\gamma}}X, Y \rangle_0 \\ &= -\frac{1}{2} \langle R(X, \nabla_{\dot{\gamma}}Y)Z, \dot{\gamma} \rangle_0 + \frac{1}{2} \langle R(\nabla_{\dot{\gamma}}X, Y)Z, \dot{\gamma} \rangle_0 \\ &= \frac{1}{2} \langle Z, R(X, \nabla_{\dot{\gamma}}Y)\dot{\gamma} \rangle_0 - \frac{1}{2} \langle Z, R(\nabla_{\dot{\gamma}}X, Y)\dot{\gamma} \rangle_0 \\ &= \frac{1}{2} \langle Z, (1 + \Delta)^{-1}(R(X, \nabla_{\dot{\gamma}}Y)\dot{\gamma} - R(\nabla_{\dot{\gamma}}X, Y)\dot{\gamma}) \rangle_1. \end{aligned}$$

Thus $A_X Y = \frac{1}{2}(1 + \Delta)^{-1}(R(X, \nabla_{\dot{\gamma}}Y)\dot{\gamma} - R(\nabla_{\dot{\gamma}}X, Y)\dot{\gamma})$. For X, Y smooth and in $H^1 = H^1(\gamma^*TM)$, $A_X Y \in H^2 \subset H^1$, since R is zeroth order. It follows that $A_X Y$ takes H^1 to H^1 continuously as a map on either X or Y . Thus taking the H^1 inner product with $A_X Y$ is a continuous linear functional on H^1 . \square

Remark 2.3. Locally on LM , we should have $D_X Y = \delta_X^{LM} Y + \omega_X^{LM}(Y)$. Now $\delta_X^{LM} Y$ can only mean $\frac{d}{d\tau}|_{\tau=0} \frac{d}{d\epsilon}|_{\epsilon=0} \gamma(\epsilon, \tau, \theta)$, where $\gamma(0, 0, \theta) = \gamma(\theta)$, $d\epsilon|_{\epsilon=0} \gamma(\epsilon, 0, \theta) = X(\theta)$, $d\tau|_{\tau=0} \gamma(\epsilon, \tau, \theta) = Y_{\gamma(\epsilon, 0, \cdot)}(\theta)$. In other words, $\delta_X^{LM} Y$ equals $\delta_X Y$. Since $D_X Y^a = \delta_X Y^a + \Gamma_{bc}^a(\gamma(\theta))$, the connection one-form for the L^2 Levi-Civita connection on LM is given by

$$\omega_X^{LM}(Y)^a(\gamma)(\theta) = \Gamma_{bc}^a(\gamma(\theta))X^b Y^c = \omega_X^M(Y)^a(\gamma(\theta)).$$

By this remark, we get

Corollary 2.6. *The connection one-form $\omega = \omega^1$ for ∇^1 in exponential coordinates is*

$$\begin{aligned} \omega_X(Y)(\gamma)(\theta) &= \omega_X^M(Y)(\gamma(\theta)) + \frac{1}{2} \{ (1 + \Delta)^{-1} [-\nabla_{\dot{\gamma}}(R(X, \dot{\gamma})Y) - R(X, \dot{\gamma})\nabla_{\dot{\gamma}}Y \\ &\quad - \nabla_{\dot{\gamma}}(R(Y, \dot{\gamma})X) - R(Y, \dot{\gamma})\nabla_{\dot{\gamma}}X \\ &\quad + R(X, \nabla_{\dot{\gamma}}Y)\dot{\gamma} - R(\nabla_{\dot{\gamma}}X, Y)\dot{\gamma}] \}(\theta). \end{aligned} \tag{2.8}$$

2.4. The Levi-Civita Connection for $s \in \mathbb{Z}^+$.

For $s > 0$, the proof of Prop. 2.3 extends directly to give

Lemma 2.7. *The Levi-Civita connection for the H^s metric is given by*

$$\nabla_X^s Y = D_X Y + \frac{1}{2}(1 + \Delta)^{-s} [D_X, (1 + \Delta)^s] Y + \frac{1}{2}(1 + \Delta)^{-s} [D_Y, (1 + \Delta)^s] X + A_X Y,$$

where

$$-\frac{1}{2}\langle [D_Z, (1 + \Delta)^s]X, Y \rangle_0 = \langle A_X Y, Z \rangle_s, \quad (2.9)$$

provided that for each $X, Y \in H^s$, $Z \mapsto -\frac{1}{2}\langle [D_Z, (1 + \Delta)^s]X, Y \rangle_0$ is a continuous linear functional on H^s .

We now compute the bracket terms.

Lemma 2.8. *For $s \in \mathbb{Z}^+$, at the loop γ ,*

$$[D_X, (1 + \Delta)^s]Y = \sum_{k=1}^s (-1)^k \binom{s}{k} \sum_{j=0}^{2k-1} \nabla_{\dot{\gamma}}^j (R(X, \dot{\gamma}) \nabla_{\dot{\gamma}}^{2k-1-j} Y). \quad (2.10)$$

Proof. The sum over k comes from the binomial expansion of $(1 + \Delta)^s$, so we just need an inductive formula for $[D_X, \Delta^s]$. The case $s = 1$ is Proposition 2.3. For the induction step, we have

$$\begin{aligned} [D_X, \Delta^s] &= D_X \Delta^{s-1} \Delta - \Delta^s D_X \\ &= \Delta^{s-1} D_X \Delta + [D_X, \Delta^{s-1}] \Delta - \Delta^s D_X \\ &= \Delta^s D_X + \Delta^{s-1} [D_X, \Delta] + [D_X, \Delta^{s-1}] \Delta - \Delta^s D_X \\ &= \Delta^{s-1} (-\nabla_{\dot{\gamma}} (R(X, \dot{\gamma}) Y) - R(X, \dot{\gamma}) \nabla_{\dot{\gamma}} Y) \\ &\quad - \sum_{j=0}^{2s-3} (-1)^{s-1} \nabla_{\dot{\gamma}}^j (R(X, \dot{\gamma}) \nabla_{\dot{\gamma}}^{2k-j-1} (-\nabla_{\dot{\gamma}}^2 Y)) \\ &= (-1)^{s-1} (-\nabla_{\dot{\gamma}}^{2s-1} (R(X, \dot{\gamma}) Y) - (-1)^{s-1} \nabla_{\dot{\gamma}}^{2s-2} (R(X, \dot{\gamma}) \nabla_{\dot{\gamma}} Y)) \\ &\quad + \sum_{j=0}^{2s-3} (-1)^s \nabla_{\dot{\gamma}}^j (R(X, \dot{\gamma}) \nabla_{\dot{\gamma}}^{2k-j-1} (-\nabla_{\dot{\gamma}}^2 Y)) \\ &= \sum_{j=0}^{2s-1} (-1)^s \nabla_{\dot{\gamma}}^j (R(X, \dot{\gamma}) \nabla_{\dot{\gamma}}^{2k-1-j} Y). \end{aligned}$$

□

We now show that $A_X Y$ exists for $s \in \mathbb{Z}^+$.

Lemma 2.9. *For $s \in \mathbb{Z}^+$ and fixed $X, Y \in H^s$, the left hand side of (2.9) is a continuous linear map from $Z \in H^s$ to H^s . Thus $A_X Y \in H^s$ is well defined.*

Proof. For smooth $X, Y \in H^s$ and $j, 2k-1-j \in \{0, 1, \dots, 2s-1\}$, a typical term in (2.10) is

$$\begin{aligned}
\langle \nabla_{\dot{\gamma}}^j (R(Z, \dot{\gamma}) \nabla_{\dot{\gamma}}^{2k-1-j} X), Y \rangle_0 &= (-1)^j \langle R(Z, \dot{\gamma}) \nabla_{\dot{\gamma}}^{2k-1-j} X, \nabla_{\dot{\gamma}}^j Y \rangle_0 \\
&= (-1)^j \int_{S^1} g_{i\ell} (R(Z, \dot{\gamma}) \nabla_{\dot{\gamma}}^{2k-1-j} X)^i (\nabla_{\dot{\gamma}}^j Y)^\ell d\theta \\
&= (-1)^j \int_{S^1} g_{i\ell} Z^k R_{krn}{}^{i\dot{\gamma}r} (\nabla_{\dot{\gamma}}^{2k-1-j} X)^n (\nabla_{\dot{\gamma}}^j Y)^\ell d\theta \\
&= (-1)^j \int_{S^1} g_{tm} g^{kt} g_{i\ell} Z^m R_{krn}{}^{i\dot{\gamma}r} (\nabla_{\dot{\gamma}}^{2k-1-j} X)^n (\nabla_{\dot{\gamma}}^j Y)^\ell d\theta \\
&= (-1)^j \langle Z, g^{kt} g_{i\ell} R_{krn}{}^{i\dot{\gamma}r} (\nabla_{\dot{\gamma}}^{2k-1-j} X)^n (\nabla_{\dot{\gamma}}^j Y)^\ell \partial_t \rangle_0 \\
&= (-1)^j \langle Z, R{}^{t\dot{\gamma}r}{}_{rn\ell} (\nabla_{\dot{\gamma}}^{2k-1-j} X)^n (\nabla_{\dot{\gamma}}^j Y)^\ell \partial_t \rangle_0 \\
&= (-1)^{j+1} \langle Z, R_{n\ell r}{}^{t\dot{\gamma}r} (\nabla_{\dot{\gamma}}^{2k-1-j} X)^n (\nabla_{\dot{\gamma}}^j Y)^\ell \partial_t \rangle_0 \\
&= (-1)^{j+1} \langle Z, R(\nabla_{\dot{\gamma}}^{2k-1-j} X, \nabla_{\dot{\gamma}}^j Y) \dot{\gamma} \rangle_0 \\
&= (-1)^{j+1} \langle Z, (1 + \Delta)^{-s} R(\nabla_{\dot{\gamma}}^{2k-1-j} X, \nabla_{\dot{\gamma}}^j Y) \dot{\gamma} \rangle_s.
\end{aligned}$$

(In the integrals and inner products, the local expressions are in fact globally defined one-forms on S^1 , resp. vector fields along γ , so we do not need a partition of unity.) $(1 + \Delta)^{-s} R(\nabla_{\dot{\gamma}}^{2k-1-j} X, \nabla_{\dot{\gamma}}^j Y) \dot{\gamma}$ is of order at least one in either X or Y , so this term is in $H^{s+1} \subset H^s$. Thus the last inner product is well defined. \square

By (2.9), (2.10), we get

$$A_X Y = \sum_{k=1}^s (-1)^k \binom{s}{k} \sum_{j=0}^{2k-1} (-1)^{j+1} (1 + \Delta)^{-s} R(\nabla_{\dot{\gamma}}^{2k-1-j} X, \nabla_{\dot{\gamma}}^j Y) \dot{\gamma}.$$

This gives:

Theorem 2.10. *For $s \in \mathbb{Z}^+$, the Levi-Civita connection for the H^s metric at the loop γ is given by*

$$\begin{aligned}
\nabla_X^s Y(\gamma) &= D_X Y(\gamma) + \frac{1}{2} (1 + \Delta)^{-s} \sum_{k=1}^s (-1)^k \binom{s}{k} \sum_{j=0}^{2k-1} \nabla_{\dot{\gamma}}^j (R(X, \dot{\gamma}) \nabla_{\dot{\gamma}}^{2k-1-j} Y) \\
&\quad + (X \leftrightarrow Y) \\
&\quad + \sum_{k=1}^s (-1)^k \binom{s}{k} \sum_{j=0}^{2k-1} (-1)^{j+1} (1 + \Delta)^{-s} R(\nabla_{\dot{\gamma}}^{2k-1-j} X, \nabla_{\dot{\gamma}}^j Y) \dot{\gamma}.
\end{aligned}$$

2.5. The Levi-Civita Connection for $s \in \mathbb{R}^+ \setminus \mathbb{Z}^+$.

In this subsection, we show that the results for $s \in \mathbb{Z}^+$ do not carry over to nonintegral s .

By Lemma 2.7, we have to examine if $\langle [D_Z, (1 + \Delta)^s]X, Y \rangle_0$ is a continuous linear functional on $Z \in H^s$.

Lemma 2.11. (i) For $\operatorname{Re}(s) \neq 0$, $[D_Z, (1 + \Delta)^s]X$ is a Ψ DO of order $2s - 1$ in X .
(ii) $\sigma_{2s-k}([D_Z, (1 + \Delta)^s])$ has k θ -derivatives of Z .

Proof. (i) For $f : LM \rightarrow \mathbb{C}$, we get $[D_Z, (1 + \Delta)^s]fX = f[D_Z, (1 + \Delta)^s]X$, since $[f, (1 + \Delta)^s] = 0$. Therefore, $[D_Z, (1 + \Delta)^s]X$ depends only on $X|_\gamma$.

By Lemma 2.1, $D_Z = \delta_Z + \Gamma \cdot Z$ in shorthand exponential coordinates. The Christoffel symbol term is zeroth order and $(1 + \Delta)^s$ has scalar leading order symbol, so $[\Gamma \cdot Z, (1 + \Delta)^s]$ has order $2s - 1$.

From the integral expression for $(1 + \Delta)^s$, it is immediate that

$$\begin{aligned} [\delta_Z, (1 + \Delta)^s]X &= (\delta_Z(1 + \Delta)^s)X + (1 + \Delta)^s\delta_ZX - (1 + \Delta)^s\delta_ZX \quad (2.11) \\ &= (\delta_Z(1 + \Delta)^s)X. \end{aligned}$$

$\delta_Z(1 + \Delta)^s$ is a limit of differences of Ψ DOs on bundles isomorphic to γ^*TM . Since the algebra of Ψ DOs is closed in the Fréchet topology of all C^k seminorms of symbols and smoothing terms on compact sets, $\delta_Z(1 + \Delta)^s$ is a Ψ DO.

Assume $\operatorname{Re}(s) < 0$. As in the construction of $(1 + \Delta)^s$, we will compute what the symbol asymptotics of $\delta_Z(1 + \Delta)^s$ should be, and then construct an operator with these asymptotics. From the functional calculus for unbounded operators, we have

$$\begin{aligned} \delta_Z(1 + \Delta)^s &= \delta_Z \left(\frac{i}{2\pi} \int_\Gamma \lambda^s (1 + \Delta - \lambda)^{-1} d\lambda \right) \\ &= \frac{i}{2\pi} \int_\Gamma \lambda^s \delta_Z (1 + \Delta - \lambda)^{-1} d\lambda \quad (2.12) \\ &= -\frac{i}{2\pi} \int_\Gamma \lambda^s (1 + \Delta - \lambda)^{-1} (\delta_Z \Delta) (1 + \Delta - \lambda)^{-1} d\lambda, \end{aligned}$$

where Γ is a contour around the spectrum of $1 + \Delta$, and the hypothesis on s justifies the exchange of δ_Z and the integral. The operator $A = (1 + \Delta - \lambda)^{-1} \delta_Z \Delta (1 + \Delta - \lambda)^{-1}$ is a Ψ DO of order -3 with top order symbol

$$\begin{aligned} \sigma_{-3}(A)(\theta, \xi)_j^\ell &= (\xi^2 - \lambda)^{-1} \delta_k^\ell (-2Z^i \partial_i \Gamma_{\nu\mu}^k \dot{\gamma}^\nu - 2\Gamma_{\nu\mu}^k \dot{Z}^\nu) \xi (\xi^2 - \lambda)^{-1} \delta_j^\mu \\ &= (-2Z^i \partial_i \Gamma_{\nu j}^\ell \dot{\gamma}^\nu - 2\Gamma_{\nu j}^\ell \dot{Z}^\nu) \xi (\xi^2 - \lambda)^{-2}. \end{aligned}$$

Thus the top order symbol of $\delta_Z(1 + \Delta)^s$ should be

$$\begin{aligned} \sigma_{2s-1}(\delta_Z(1 + \Delta)^s)(\theta, \xi)_j^\ell &= -\frac{i}{2\pi} \int_\Gamma \lambda^s (-2Z^i \partial_i \Gamma_{\nu j}^\ell \dot{\gamma}^\nu - 2\Gamma_{\nu j}^\ell \dot{Z}^\nu) \xi (\xi^2 - \lambda)^{-2} d\lambda \\ &= \frac{i}{2\pi} \int_\Gamma s \lambda^{s-1} (-2Z^i \partial_i \Gamma_{\nu j}^\ell \dot{\gamma}^\nu - 2\Gamma_{\nu j}^\ell \dot{Z}^\nu) \xi (\xi^2 - \lambda)^{-1} d\lambda \\ &= s (-2Z^i \partial_i \Gamma_{\nu j}^\ell \dot{\gamma}^\nu - 2\Gamma_{\nu j}^\ell \dot{Z}^\nu) \xi (\xi^2 - \lambda)^{s-1}. \quad (2.13) \end{aligned}$$

Similarly, all the terms in the symbol asymptotics for A are of the form $B_j^\ell \xi^n (\xi^2 - \lambda)^m$ for some matrices $B_j^\ell = B_j^\ell(n, m)$. This produces a symbol sequence $\sum_{k \in \mathbb{Z}^+} \sigma_{2s-k}$, and there exists a Ψ DO P with $\sigma(P) = \sum \sigma_{2s-k}$. (As in §2.2, we first produce operators P_i on a coordinate cover U_i of S^1 , and then set $P = \sum_i \phi_i P_i \psi_i$.) The construction depends on the choice of local coordinates covering γ , the partition of unity and cutoff functions as above, and a cutoff function in ξ ; as usual, different choices change the operator by a smoothing operator. Standard estimates show that $P - \delta_Z(1 + \Delta)^s$ is a smoothing operator, so $\delta_Z(1 + \Delta)^s$ is a Ψ DO of order $2s - 1$.

For $\text{Re}(s) > 0$, motivated by differentiating $(1 + \Delta)^{-s} \circ (1 + \Delta)^s = \text{Id}$, we set

$$\delta_Z(1 + \Delta)^s = -(1 + \Delta)^s \circ \delta_Z(1 + \Delta)^{-s} \circ (1 + \Delta)^s.$$

This is again a Ψ DO of order $2s - 1$.

(ii) The covariant derivative along γ on $Y \in \Gamma(\gamma^*TM)$ is given by

$$\begin{aligned} \frac{DY}{d\gamma} &= (\gamma^* \nabla^M)_{\partial_\theta} (Y) = \partial_\theta Y + (\gamma^* \omega^M)(\partial_\theta)(Y) \\ &= \partial_\theta(Y^i) \partial_i + \dot{\gamma}^t Y^r \Gamma_{tr}^j \partial_j, \end{aligned}$$

where ∇^M is the Levi-Civita connection on M and ω^M is the connection one-form in exponential coordinates on M . For $\Delta = (\frac{D}{d\gamma})^* \frac{D}{d\gamma}$, an integration by parts using the formula $\partial_t g_{ar} = \Gamma_{\ell t}^n g_{rn} + \Gamma_{rt}^n g_{\ell n}$ gives

$$(\Delta Y)^k = -\partial_\theta^2 Y^k - 2\Gamma_{\nu\mu}^k \dot{\gamma}^\nu \partial_\theta Y^\mu - (\partial_\theta \Gamma_{\nu\delta}^k \dot{\gamma}^\nu + \Gamma_{\nu\delta}^k \ddot{\gamma}^\nu + \Gamma_{\nu\mu}^k \Gamma_{\varepsilon\delta}^\mu \dot{\gamma}^\varepsilon \dot{\gamma}^\nu) Y^\delta.$$

Therefore, by (2.7)

$$\begin{aligned} ((\delta_Z \Delta)Y)^k &= (-2Z^i \partial_i \Gamma_{\nu\mu}^k \dot{\gamma}^\nu - 2\Gamma_{\nu\mu}^k \dot{Z}^\nu) \dot{Y}^\mu - \left(\dot{Z}^i \partial_i \Gamma_{\nu\delta}^k \dot{\gamma}^\nu + Z^i \partial_i \dot{\Gamma}_{\nu\delta}^k \dot{\gamma}^\nu + \dot{\Gamma}_{\nu\delta}^k \dot{Z}^\nu \right. \\ &\quad + Z^i \partial_i \Gamma_{\nu\delta}^k \ddot{\gamma}^\nu + \Gamma_{\nu\delta}^k \ddot{Z}^\nu + Z^i \partial_i \Gamma_{\nu\mu}^k \Gamma_{\varepsilon\delta}^\mu \dot{\gamma}^\varepsilon \dot{\gamma}^\nu + \Gamma_{\nu\mu}^k Z^i \partial_i \Gamma_{\varepsilon\delta}^\mu \dot{\gamma}^\varepsilon \dot{\gamma}^\nu \\ &\quad \left. + \Gamma_{\nu\mu}^k \Gamma_{\varepsilon\delta}^\mu \dot{Z}^\varepsilon \dot{\gamma}^\nu + \Gamma_{\nu\mu}^k \Gamma_{\varepsilon\delta}^\mu \dot{\gamma}^\varepsilon \dot{Z}^\nu \right) Y^\delta. \end{aligned} \quad (2.14)$$

Thus $(\delta_Z \Delta)Y$ is a second order differential operators in Z and a first order differential operator in Y . It follows from (2.12) that $\delta_Z(1 + \Delta)^s Y$ is a second order differential operator in Z . As in (2.13), a symbol calculus calculation shows that $\sigma_{-k}(A)$ has $k - 2$ derivatives of Z . It follows that for $\text{Re}(s) < 0$, $\sigma_{2s-k}(\delta_Z(1 + \Delta)^s)$ has at most k derivatives of Z , and the same result then holds for $\text{Re}(s) > 0$. \square

Remark 2.4. (i) For $s \in \mathbb{Z}^+$, $\delta_Z(1 + \Delta)^s$ differs from the usual definition by a smoothing operator.

(ii) For all s , the proof shows that $\sigma(\delta_Z(1 + \Delta)^s) = \delta_Z(\sigma((1 + \Delta)^s))$.

We now determine if there exists $A_X Y \in H^s$ such that $-\frac{1}{2} \langle [D_Z, (1 + \Delta)^s] X, Y \rangle_0 = \langle A_X Y, Z \rangle_s$ for each $X, Y, Z \in H^s$. As above, we write $D_Z = \delta_Z + \Gamma \cdot Z$, and consider each term separately.

For the $\Gamma \cdot Z$ term, we have

$$\langle [\Gamma \cdot Z, (1 + \Delta)^s]X, Y \rangle_0 = \int_{S^1} g_{ab}(\Gamma \cdot Z(1 + \Delta)^s X)^a Y^b - \int_{S^1} g_{ab}((1 + \Delta)^s \Gamma \cdot Z \cdot X)^a Y^b, \quad (2.15)$$

where we have omitted the partition of unity. The first term on the right hand side of (2.15) equals

$$\begin{aligned} \int_{S^1} g_{ab}(\Gamma_{dc}^a Z^d ((1 + \Delta)^s X)^a Y^b &= \int_{S^1} g_{de} Z^d g_{ab} g^{ef} \Gamma_{fc}^a ((1 + \Delta)^s X)^c Y^b \\ &= \langle Z, g_{ab} g^{ef} \Gamma_{fc}^a ((1 + \Delta)^s X)^c Y^b \partial_e \rangle_0 \\ &= \langle Z, (1 + \Delta)^{-s} [g_{ab} g^{ef} \Gamma_{fc}^a ((1 + \Delta)^s X)^c Y^b \partial_e] \rangle_s. \end{aligned} \quad (2.16)$$

Since the components of $(1 + \Delta)^s X$ and Y are in H_{loc}^{-s} and H_{loc}^s respectively, the product $((1 + \Delta)^s X)^c Y^b$ is in H_{loc}^{-s} . Thus $(1 + \Delta)^{-s} [g_{ab} g^{ef} \Gamma_{fc}^a ((1 + \Delta)^s X)^c Y^b \partial_e] \in H^s$, so the inner products in (2.16) are well defined. A similar argument shows that the second term on the right hand side of (2.15) can be rewritten as an H^s inner product with Z .

We now consider the term $[\delta_Z, (1 + \Delta)^s]X = (\delta_Z(1 + \Delta)^s)X$ (by (2.11)). By Lemma 2.11, $\delta_Z(1 + \Delta)^s$ is a Ψ DO with explicitly computable symbol $\sigma \sim \sum_{k \in \mathbb{Z}^+} \sigma_{2s-k}$, and σ_{2s-k} contains at most k derivatives of Z . Fix ℓ and let $P = P_{Z,\ell}$ be a Ψ DO with symbol $\sum_{k=1}^{\ell} \sigma_{2s-k}$. Here $P = \sum_i \phi_i P_i \psi_i$ with respect to some cover $\{(a_i, b_i)\}$ of S^1 . $Q = \delta_Z(1 + \Delta)^s - P$ is an operator of order at most $2s - \ell - 1$. Suppressing the i dependence of the symbols, we have

$$\begin{aligned} &\int_{S^1} g_{ab} \delta_Z (1 + \Delta)^s X^a Y^b \\ &= \int_{S^1} g_{ab} P(X)^a Y^b + \int_{S^1} g_{ab} Q(X)^a Y^b \\ &= \sum_i \int_{(a_i, b_i)} \phi_i g_{ab}^{(i)} \left(\sum_j \int_{T^*(a_j, b_j)} e^{i(\theta - \theta') \cdot \xi} \phi_j(\theta) \right. \\ &\quad \cdot \left. \sum_{k=1}^{\ell} (\sigma_{2s-k})_c^a(\theta, \xi) \psi_j(\theta') X^c(\theta') d\xi d\theta' \right) Y^b d\theta \\ &\quad + \sum_i \int_{(a_i, b_i)} \phi_i g_{ab}^{(i)} (QX)^a Y^b. \end{aligned} \quad (2.17)$$

Strictly speaking, P is given by integrating its symbol plus a smoothing term, which we absorb into Q .

In the next to last term in (2.17), consider a term in a fixed σ_{2s-k_0} of the form $(Z^d)^{(k)} A_{dc}^a$ with $k \leq k_0$ derivatives in Z . In order to write this term as an inner product with Z , we have to perform k integration by parts in θ . For example, if k

θ -derivatives act on ϕ_i , replace this term with $\phi_i \frac{\phi_i^{(k)}}{\phi_i}$, noting that this function extends by zero to $\{\phi_i = 0\}$. This gives the expression

$$\begin{aligned} & \sum_i \int_{(a_i, b_i)} \phi_i \frac{\phi_i^{(k)}}{\phi_i} g_{ab} \left(\sum_j \int_{T^*(a_j, b_j)} e^{i(\theta-\theta') \cdot \xi} Z^d A_{dc}^a(\theta, \xi) \phi_j(\theta) \psi_j(\theta') X^c(\theta') d\xi d\theta' \right) Y^b d\theta \\ &= \sum_i \int_{(a_i, b_i)} \phi_i \frac{\phi_i^{(k)}}{\phi_i} g_{de} g^{er} g_{ab} \\ & \quad \cdot \left(\sum_j \int_{T^*(a_j, b_j)} e^{i(\theta-\theta') \cdot \xi} Z^d A_{rc}^a(\theta, \xi) \phi_j(\theta) \psi_j(\theta') X^c(\theta') d\xi d\theta' \right) Y^b d\theta, \end{aligned}$$

which locally is the L^2 inner product of $Z = Z^d \partial_d$ with

$$\frac{\phi_i^{(k)}}{\phi_i} g^{er} g_{ab} \left(\sum_j \int_{T^*(a_j, b_j)} e^{i(\theta-\theta') \cdot \xi} A_{rc}^a(\theta, \xi) \phi_j(\theta) \psi_j(\theta') X^c(\theta') d\xi d\theta' \right) Y^b \partial_e.$$

This is the (local) H^s inner product of Z with

$$(1 + \Delta)^{-s} \left[\frac{\phi_i^{(k)}}{\phi_i} g^{er} g_{ab} \left(\sum_j \int_{T^*(a_j, b_j)} e^{i(\theta-\theta') \cdot \xi} A_{rc}^a(\theta, \xi) \phi_j(\theta) \psi_j(\theta') X^c(\theta') d\xi d\theta' \right) Y^b \partial_e \right].$$

This term is a Ψ DO of order $-2s$ as an operator on Y .

There are similar expressions if k θ -derivatives act on g_{ab} , on $e^{i\theta \cdot \xi}$, or on $A_{dc}^a(\theta, \xi) \phi_j(\theta)$, and in each case we get terms which are Ψ DOs of order $-2s$ as an operator on Y . Finally, if k θ -derivatives act on Y^b , we get the local H^s inner product of Z with

$$(1 + \Delta)^{-s} \left[g^{er} g_{ab} \left(\sum_j \int_{T^*(a_j, b_j)} e^{i(\theta-\theta') \cdot \xi} A_{rc}^a(\theta, \xi) \phi_j(\theta) \psi_j(\theta') X^c(\theta') d\xi d\theta' \right) Y^{b, (k)} \partial_e \right],$$

which is of order $-2s + k$ as an operator on Y .

In general, each of the k integrations by parts will act as in one of the previous cases, so the final result can be written as an H^s inner product with Z .

In summary, for fixed ℓ , $A_X Y$, if it exists, is the sum of two operators, $(1 + \Delta)^{-s} P'_{X, \ell}$ and $(1 + \Delta)^{-s} Q'_{X, \ell}$. Here $(1 + \Delta)^{-s} P'_{X, \ell}$ is of order at least $-2s + \ell$ and defined by

$$\int_{S^1} g_{ab} P_{Z, \ell}(X)^a Y^b = \int_{S^1} g_{ab} P'_{X, \ell}(Y)^a Z^b = \langle (1 + \Delta)^{-s} P'_{X, \ell}(Y), Z \rangle_s. \quad (2.18)$$

By a similar argument, $(1 + \Delta)^{-s} Q'_{X, \ell}$ is defined by

$$\int_{S^1} g_{ab} Q_{Z, \ell}(X)^a Y^b = \int_{S^1} g_{ab} Q'_{X, \ell}(Y)^a Z^b = \langle (1 + \Delta)^{-s} Q'_{X, \ell}(Y), Z \rangle_s. \quad (2.19)$$

Since $Q = Q_{Z,\ell}$ has symbol $\sum_{k=\ell+1}^{\infty} \sigma_{2s-k}$, $(1+\Delta)^{-s}Q'_{X,\ell}$ has order at least $-2s+\ell+1$. Adding (2.18) and (2.19), we see that

$$-\frac{1}{2}\langle [D_Z, (1+\Delta)^s]X, Y \rangle_0 = \langle (1+\Delta)^{-s}(P' + Q')_{X,\ell}(Y), Z \rangle_s, \quad (2.20)$$

provided the right hand side exists. In particular, $A_X Y$ can only equal $(1+\Delta)^{-s}(P' + Q')_{X,\ell}(Y)$. However, for $\ell > 2s+1$ and $Y \in H^s$, $(1+\Delta)^{-s}(P' + Q')_{X,\ell}(Y) \notin H^s$ in general, so the right hand side of (2.20) is not defined.

This proves that there is no Levi-Civita connection in this case. To be precise, let $H^s(TLM)$ consist of the smooth sections $Y = Y_\gamma$ of TLM such that $Y_\gamma \in H^s(\gamma^*TM)$ for all γ .

Theorem 2.12. *For $s \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, there is no Levi-Civita connection for the H^s metric on LM , in the sense that there exists no map $\nabla^s : H^s(\gamma^*TM) \times H^s(TLM) \rightarrow H^s(TLM)$, $(X, Y) \mapsto \nabla_X^s Y$, which is linear over $C^\infty(LM)$ in X and a derivation in Y , and such that*

$$\begin{aligned} 2\langle \nabla_Y^s X, Z \rangle_s &= X\langle Y, Z \rangle_s + Y\langle X, Z \rangle_s - Z\langle X, Y \rangle_s \\ &\quad + \langle [X, Y], Z \rangle_s + \langle [Z, X], Y \rangle_s - \langle [Y, Z], X \rangle_s. \end{aligned}$$

2.6. Connections on LM that Vary Continuously in s .

We now find a family of connections on LM that depend continuously on s . By the previous subsection, this family cannot consist of Levi-Civita connections for the H^s metric, but they can be used to construct Chern-Simons invariants in Part II.

Lemma 2.13. *The family of operators*

$$\nabla_X^{1,s} Y = D_X Y + \frac{1}{2}(1+\Delta)^{-s}[D_X, (1+\Delta)^s]Y + \frac{1}{2}(1+\Delta)^{-s}[D_Y, (1+\Delta)^s]X$$

are torsion free connections on TLM for each $s > 0$.

$\nabla^{1,s}$ is just the connection in Lemma 2.7 with the $A_X Y$ term dropped.

Proof. For $f \in C^\infty(LM)$, $(1+\Delta)^{-s}[D_{fX}, (1+\Delta)^s]Y = f(1+\Delta)^{-s}[D_X, (1+\Delta)^s]Y$, since $(1+\Delta)^{-s}f = f(1+\Delta)^{-s}$ for fixed γ . Also, $(1+\Delta)^{-s}[D_X, (1+\Delta)^s]fY = f(1+\Delta)^{-s}[D_X, (1+\Delta)^s]Y$, since $(1+\Delta)^{-s}X(f) = X(f)(1+\Delta)^{-s}$. Thus $(X, Y) \mapsto (1+\Delta)^{-s}[D_X, (1+\Delta)^s]Y$ is in $\Lambda^1(LM, \text{End}(TLM))$, as is $(1+\Delta)^{-s}[D_Y, (1+\Delta)^s]X$. $D_X Y$ is a connection on TLM , so adding these elements of $\Lambda^1(LM, \text{End}(TLM))$ produces new connections $\nabla^{1,s}$.

Since $\frac{1}{2}(1+\Delta)^{-s}[D_X, (1+\Delta)^s]Y + \frac{1}{2}(1+\Delta)^{-s}[D_Y, (1+\Delta)^s]X$ is symmetric in X and Y , we get

$$\nabla_X^{1,s} Y - \nabla_Y^{1,s} X = D_X Y - D_Y X = [X, Y].$$

□

Definition 2.1. $\nabla^{1,s}$ is called the H^s connection.

2.7. Extensions of the Frame Bundle of LM . In this subsection we discuss the choice of structure group for the H^s and Levi-Civita connections on LM .

Let \mathcal{H} be the Hilbert space $H^{s_0}(\gamma^*TM)$ for a fixed s_0 and γ . Let $GL(\mathcal{H})$ be the group of bounded invertible linear operators on \mathcal{H} ; inverses of elements are bounded by the closed graph theorem. $GL(\mathcal{H})$ has the subset topology of the norm topology on $\mathcal{B}(\mathcal{H})$, the bounded linear operators on \mathcal{H} . $GL(\mathcal{H})$ is an infinite dimensional Banach Lie group, as a group which is an open subset of the infinite dimensional Hilbert manifold $\mathcal{B}(\mathcal{H})$ [19, p. 59], and has Lie algebra $\mathcal{B}(\mathcal{H})$. Let $\Psi\text{DO}_{\leq 0}, \Psi\text{DO}_0^*$ denote the algebra of classical ΨDO s of nonpositive order and the group of invertible zeroth order ΨDO s, respectively, where all ΨDO s act on \mathcal{H} . Note that $\Psi\text{DO}_0^* \subset GL(\mathcal{H})$.

Remark 2.5. The inclusions of $\Psi\text{DO}_0^*, \Psi\text{DO}_{\leq 0}$ into $GL(\mathcal{H}), \mathcal{B}(\mathcal{H})$ are trivially continuous in the subset topology. For the Fréchet topology on $\Psi\text{DO}_{\leq 0}$, the inclusion is continuous as in [16].

We recall the relationship between the connection one-form θ on the frame bundle FN of a manifold N and local expressions for the connection on TN . For $U \subset N$, let $\chi : U \rightarrow FN$ be a local section. A metric connection ∇ on TN with local connection one-form ω determines a connection $\theta_{FN} \in \Lambda^1(FN, \mathfrak{o}(n))$ on FN by (i) θ_{FN} is the Maurer-Cartan one-form on each fiber, and (ii) $\theta_{FN}(Y_u) = \omega(X_p)$, for $Y_u = \chi_*X_p$ [23, Ch. 8, Vol. II], or equivalently $\chi^*\theta_{FN} = \omega$.

This applies to $N = LM$. The frame bundle $FLM \rightarrow LM$ is constructed as in the finite dimensional case. The fiber over γ is isomorphic to the gauge group \mathcal{G} of \mathcal{R} and fibers are glued by the transition functions for TLM . Thus the frame bundle is topologically a \mathcal{G} -bundle.

However, by Corollaries 2.5 and 2.6, the H^s connection one-form

$$\omega_X^{1,s}Y = \omega_X^MY + \frac{1}{2}(1 + \Delta)^{-s}[D_X, (1 + \Delta)^s]Y + \frac{1}{2}(1 + \Delta)^{-s}[D_Y, (1 + \Delta)^s]X$$

takes values in $\Psi\text{DO}_{\leq 0}$. The curvature two-form $\Omega^{1,s} = d_{LM}\omega^{1,s} + \omega^{1,s} \wedge \omega^{1,s}$ also takes values in $\Psi\text{DO}_{\leq 0}$. (Here $d_{LM}\omega^{1,s}(X, Y)$ is defined by the Cartan formula for the exterior derivative.) These forms should take values in the Lie algebra of the structure group. Thus we should extend the structure group to the Fréchet Lie group ΨDO_0^* , since its Lie algebra is $\Psi\text{DO}_{\leq 0}$. This leads to an extended frame bundles, also denoted FLM . The transition functions are unchanged, since $\mathcal{G} \subset \Psi\text{DO}_0^*$. Thus (FLM, θ^s) as a geometric bundle (i.e. as a bundle with connection θ^s associated to $\nabla^{1,s}$) is a ΨDO_0^* -bundle. Similar remarks hold for the Levi-Civita connection if $s \in \mathbb{Z}^+$.

In summary, for the H^s connections we have

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & FLM & & \Psi\text{DO}_0^* & \longrightarrow & (FLM, \theta^s) \\ & & \downarrow & & & & \downarrow \\ & & LM & & & & LM \end{array}$$

Remark 2.6. If we extend the structure group of the frame bundle with connection from ΨDO_0^* to $GL(\mathcal{H})$, the frame bundle becomes trivial by Kuiper's theorem. Thus there is a potential loss of information if we pass to the larger frame bundle.

The situation is similar to the following examples. Let $E \rightarrow S^1$ be the $GL(1, \mathbb{R})$ (real line) bundle with gluing functions (multiplication by) 1 at $1 \in S^1$ and 2 at $-1 \in S^1$. E is trivial as a $GL(1, \mathbb{R})$ -bundle, with global section f with $\lim_{\theta \rightarrow -\pi^+} f(e^{i\theta}) = 1$, $f(1) = 1$, $\lim_{\theta \rightarrow \pi^-} f(e^{i\theta}) = 1/2$. However, as a $GL(1, \mathbb{Q})^+$ -bundle, E is nontrivial, as a global section is locally constant. As a second example, let $E \rightarrow M$ be a nontrivial $GL(n, \mathbb{C})$ -bundle. Embed \mathbb{C}^n into a Hilbert space \mathcal{H} , and extend E to an $GL(\mathcal{H})$ -bundle \mathcal{E} with fiber \mathcal{H} and with the same transition functions. Then \mathcal{E} is trivial.

3. Local Symbol Calculations

In this section, we compute the 0 and -1 order symbols of the connection one-form and the curvature two-form of the $s = 1$ Levi-Civita connection. We also compute the 0 and -1 order symbols of the connection one-form for the H^s connection, and the 0 order symbol of the curvature for the H^s connection. These results are used in the calculations of Wodzicki-Chern-Simons classes in §6. The formulas show that the s -dependence of these symbols is linear, which will be used to define regularized Wodzicki-Chern-Simons classes (see Definition 5.2).

3.1. Connection and Curvature Symbols for $s = 1$.

Using Corollary 2.6, we can compute these symbols easily.

Lemma 3.1. (i) At $\gamma(\theta)$, $\sigma_0(\omega_X)_b^a = (\omega_X^M)_b^a = \Gamma_{cb}^a X^c$.

(ii)

$$\frac{1}{i|\xi|^{-2}\xi} \sigma_{-1}(\omega_X) = \frac{1}{2}(-2R(X, \dot{\gamma}) - R(\cdot, \dot{\gamma})X + R(X, \cdot)\dot{\gamma}).$$

Equivalently,

$$\frac{1}{i|\xi|^{-2}\xi} \sigma_{-1}(\omega_X)_b^a = \frac{1}{2}(-2R_{cdb}{}^a - R_{bdc}{}^a + R_{cbd}{}^a)X^c \dot{\gamma}^d.$$

Proof. (i) For $\sigma_0(\omega_X)$, the only term in (2.8) of order zero is the Christoffel term.

(ii) For $\sigma_{-1}(\omega_X)$, label the last six terms on the right hand side of (2.8) by (a), ..., (f). By Leibniz rule for the tensors, the only terms of order -1 come from: in (a), $-\nabla_{\dot{\gamma}}(R(X, \dot{\gamma})Y) = -R(X, \dot{\gamma})\nabla_{\dot{\gamma}}Y + \text{lower order in } Y$; in (b), the term $-R(X, \dot{\gamma})\nabla_{\dot{\gamma}}Y$; in (c), the term $-R(\nabla_{\dot{\gamma}}Y, \dot{\gamma})X$; in (e), the term $R(X, \nabla_{\dot{\gamma}}Y)\dot{\gamma}$.

For any vectors Z, W , the curvature endomorphism $R(Z, W) : TM \rightarrow TM$ has

$$R(Z, W)_b^a = R_{cdb}{}^a Z^c W^d.$$

Also, since $(\nabla_{\dot{\gamma}}Y)^a = \frac{d}{d\theta}Y^a$ plus zeroth order terms, $\sigma_1(\nabla_{\dot{\gamma}}) = i\xi \cdot Id$. Thus in (a) and (b), $\sigma_1(-R(X, \dot{\gamma})\nabla_{\dot{\gamma}})_b^a = -R_{cdb}{}^a X^c \dot{\gamma}^d \xi$.

For (c), we have $-R(\nabla_{\dot{\gamma}}Y, \dot{\gamma})X = -R_{cdb}{}^a(\nabla_{\dot{\gamma}}Y)^c\dot{\gamma}^dX^b\partial_a$, so the top order symbol is $-R_{cdb}{}^a\xi\dot{\gamma}^dX^b = -R_{bdc}{}^a\xi\dot{\gamma}^dX^c$.

For (e), we have $R(X, \nabla_{\dot{\gamma}}Y)\dot{\gamma} = R_{cdb}{}^aX^c(\nabla_{\dot{\gamma}}Y)^d\dot{\gamma}^b\partial_a$, so the top order symbol is $R_{cdb}{}^aX^c\xi\dot{\gamma}^b = R_{cbd}{}^aX^c\xi\dot{\gamma}^d$.

Since the top order symbol of $(1 + \Delta)^{-1}$ is $|\xi|^{-2}$, adding these four terms finishes the proof. \square

We now compute the top symbols of the curvature tensor. $\sigma_{-1}(\Omega)$ involves the covariant derivative of the curvature tensor on M , but fortunately this symbol will not be needed in Part II.

Lemma 3.2. (i) $\sigma_0(\Omega(X, Y))_b^a = R^M(X, Y)_b^a = R_{cdb}{}^aX^cY^d$.

(ii)

$$\begin{aligned} \frac{1}{i|\xi|^{-2}\xi}\sigma_{-1}(\Omega(X, Y)) &= \frac{1}{2}(\nabla_X[-2R(Y, \dot{\gamma}) - R(\cdot, \dot{\gamma})Y + R(Y, \cdot)\dot{\gamma}] \\ &\quad -(X \leftrightarrow Y) \\ &\quad -[-2R([X, Y], \dot{\gamma}) - R(\cdot, \dot{\gamma})[X, Y] + R([X, Y], \cdot)\dot{\gamma}]). \end{aligned}$$

Equivalently, in exponential normal coordinates on M centered at $\gamma(\theta)$,

$$\begin{aligned} \frac{1}{i|\xi|^{-2}\xi}\sigma_{-1}(\Omega(X, Y))_b^a &= \frac{1}{2}X[(-2R_{cdb}{}^a - R_{bdc}{}^a + R_{cbd}{}^a)\dot{\gamma}^d]Y^c - (X \leftrightarrow Y) \\ &= \frac{1}{2}X[-2R_{cdb}{}^a - R_{bdc}{}^a + R_{cbd}{}^a]\dot{\gamma}^dY^c - (X \leftrightarrow Y) \quad (3.1) \\ &\quad + \frac{1}{2}[-2R_{cdb}{}^a - R_{bdc}{}^a + R_{cbd}{}^a]\dot{X}^dY^c - (X \leftrightarrow Y) \end{aligned}$$

Proof. (i)

$$\begin{aligned} \sigma_0(\Omega(X, Y))_b^a &= \sigma_0((d\omega + \omega \wedge \omega)(X, Y))_b^a \\ &= [(d\sigma_0(\omega) + \sigma_0(\omega) \wedge \sigma_0(\omega))(X, Y)]_b^a \\ &= [(d\omega^M + \omega^M \wedge \omega^M)(X, Y)]_b^a \\ &= R^M(X, Y)_b^a = R_{cdb}{}^aX^cY^d. \end{aligned}$$

(ii) Since $\sigma_0(\omega_X)$ is independent of ξ , after dividing by $i|\xi|^{-2}\xi$ we have

$$\begin{aligned} \sigma_{-1}(\Omega(X, Y))_b^a &= (d\sigma_{-1}(\omega)(X, Y))_b^a + \sigma_0(\omega_X)_c^a\sigma_{-1}(\omega_Y)_b^c + \sigma_{-1}(\omega_X)_c^a\sigma_0(\omega_Y)_b^c \\ &\quad - \sigma_0(\omega_Y)_c^a\sigma_{-1}(\omega_X)_b^c + \sigma_{-1}(\omega_Y)_c^a\sigma_0(\omega_X)_b^c. \end{aligned}$$

As an operator on sections of γ^*TM , $\Omega^{LM} - \Omega^M$ has order -1 so $\sigma_{-1}(\Omega^{LM}) = \sigma_{-1}(\Omega^{LM} - \Omega^M)$ is independent of coordinates. In normal coordinates at a fixed

point $\gamma(\theta)$, $\sigma_0(\omega_X) = \sigma_0(\omega_Y) = 0$, so

$$\begin{aligned}\sigma_{-1}(\Omega(X, Y))_b^a &= (d\sigma_{-1}(\omega)(X, Y))_b^a \\ &= X(\sigma_{-1}(\omega_Y))_b^a - Y(\sigma_{-1}(\omega_X))_b^a - \sigma_{-1}(\omega_{[X, Y]})_b^a \\ &= \frac{1}{2}X[(-2R_{cdb}{}^a - R_{bdc}{}^a + R_{cbd}{}^a)Y^c\dot{\gamma}^d] - (X \leftrightarrow Y) \\ &\quad - \frac{1}{2}(-2R_{cdb}{}^a - R_{bdc}{}^a + R_{cbd}{}^a)[X, Y]^c\dot{\gamma}^d.\end{aligned}$$

The terms involving $X(Y^c) - Y(X^c) - [X, Y]^c$ cancel (as they must, since the symbol two-form cannot involve derivatives of X or Y). Thus

$$\sigma_{-1}(\Omega(X, Y))_b^a = \frac{1}{2}X[(-2R_{cdb}{}^a - R_{bdc}{}^a + R_{cbd}{}^a)Y^c\dot{\gamma}^d] - (X \leftrightarrow Y).$$

This gives the first coordinate expression in (3.1). The second expression follows from $X(\dot{\gamma}^d) = \dot{X}^d$ (see (2.7)).

To convert from the coordinate expression to the covariant expression, we follow the usual procedure of changing ordinary derivatives to covariant derivatives and adding bracket terms. For example,

$$\begin{aligned}\nabla_X(R(Y, \dot{\gamma})) &= (\nabla_X R)(Y, \dot{\gamma}) + R(\nabla_X Y, \dot{\gamma}) + R(Y, \nabla_X \dot{\gamma}) \\ &= X^i R_{cdb}{}^a{}_{;i} Y^c \dot{\gamma}^d + R(\nabla_X Y, \dot{\gamma}) + R_{cdb}{}^a Y^c (\nabla_X \dot{\gamma})^d.\end{aligned}$$

In exponential normal coordinates at $\gamma(\theta)$, we have $X^i R_{cdb}{}^a{}_{;i} = X^i \partial_i R_{cdb}{}^a = X(R_{cdb}{}^a)$ and $(\nabla_X \dot{\gamma})^d = X(\dot{\gamma}^d)$. Thus

$$\nabla_X(R(Y, \dot{\gamma})) - (X \leftrightarrow Y) - R([X, Y], \dot{\gamma}) = X(R_{cdb}{}^a \dot{\gamma}^d) Y^c - (X \leftrightarrow Y).$$

The other terms are handled similarly. \square

3.2. Connection and Curvature Symbols for general s .

The noteworthy feature of these computations is the linear dependence of $\sigma_{-1}(\omega^{1,s})$ on s .

Let g be the Riemannian metric on M .

Lemma 3.3. (i) At $\gamma(\theta)$, $\sigma_0(\omega_X^{1,s})_b^a = (\omega_X^M)_b^a = \Gamma_{cb}^a X^c$.

(ii) $\sigma_0(\Omega^{1,s}(X, Y))_b^a = R^M(X, Y)_b^a = R_{cdb}{}^a X^c Y^d$.

(iii) $\frac{1}{i|\xi|^{-2\xi}} \sigma_{-1}(\omega_X^{1,s})_b^a = sT(X, \dot{\gamma}, g)$, where $T(X, \dot{\gamma}, g)$ is tensorial and independent of s .

Proof. (i) By Lemma 2.11, the only term of order zero in Lemma 2.13 is ω_X^M .

(ii) The proof of Lemma 3.2(ii) carries over.

(iii) By Lemma 2.13, we have to compute σ_{-1} for $\frac{1}{2}[D_X, (1 + \Delta)^s]$ and for $\frac{1}{2}[D_\cdot, (1 + \Delta)^s]X$.

Write $D_X = \delta_X + \Gamma \cdot X$ in shorthand. Since $(1 + \Delta)^s$ has scalar leading order symbol, $[\Gamma \cdot X, (1 + \Delta)^s]$ has order $2s - 1$. Thus we can compute $\sigma_{2s-1}([\Gamma \cdot X, (1 + \Delta)^s])$ in

any coordinate system. For fixed θ , at the center point $\gamma(\theta)$ of exponential normal coordinates, the Christoffel symbols vanish. Thus $\sigma_{-1}(\frac{1}{2}[\Gamma \cdot X, (1 + \Delta)^s]) = 0$.

By (2.11), (2.13), $\sigma_{2s-1}(\frac{1}{2}[\delta_X, (1 + \Delta)^s])$ is s times a tensorial expression in $X, \dot{\gamma}, g$, since $\partial_i \Gamma_{\nu j}^\ell = \frac{1}{3}(R_{i\nu j}^\ell + R_{ij\nu}^\ell)$ in normal coordinates. Thus $\sigma_{-1}(\frac{1}{2}(1 + \Delta)^{-s}[\delta_X, (1 + \Delta)^s])$ is s times this tensorial expression.

The argument for $\sigma_{2s-1}(\frac{1}{2}(1 + \Delta)^{-s}[D., (1 + \Delta)^s]X)$ is similar. The term with Γ vanishes, and the term $[\delta., (1 + \Delta)^s]$ is treated by (2.14) and the functional calculus as in (2.13). \square

4. The Loop Group Case

In this section, we relate our work to Freed's work on based loop groups ΩG [10]. We find a particular representation of the loop algebra that controls the order of the curvature of the H^1 metric on ΩG .

$\Omega G \subset LG$ with base point e.g. $e \in G$ has tangent space $T_\gamma \Omega G = \{X \in T_\gamma LG : X(0) = X(2\pi) = 0\}$ in some Sobolev topology. Instead of using $D^2/d\gamma^2$ to define the Sobolev spaces, the usual choice is $\Delta_{S^1} = -d^2/d\theta^2$ coupled to the identity operator on the Lie algebra \mathfrak{g} . Since this operator has no kernel on $T_\gamma \Omega M$, $1 + \Delta$ is replaced by Δ . These changes in the H^s inner product do not alter the spaces of Sobolev sections, but they do change the Levi-Civita connection. In any case, for X, Y, Z left invariant vector fields, the first three terms on the right hand side of (2.3) vanish. Under the standing assumption that G has a left invariant, Ad-invariant inner product, one obtains

$$2\nabla_X^{(s)} Y = [X, Y] + \Delta^{-s}[X, \Delta^s Y] + \Delta^{-s}[Y, \Delta^s X]$$

[10].

It is an interesting question to compute the order of the curvature operator as a function of s . For based loops, Freed proved that this order is at most -1 . In [15], it is shown that the order of Ω^s is at most -2 for all $s \neq 1/2, 1$ on both ΩG and LG , and is exactly -2 for G nonabelian. For the case $s = 1$, we have a much stronger result.

Proposition 4.1. *The curvature of the Levi-Civita connection for the H^1 inner product on ΩG associated to $-\frac{d^2}{d\theta^2} \otimes \text{Id}$ is a ΨDO of order $-\infty$.*

PROOF: We give two proofs.

By [10], the H^1 curvature operator $\Omega = \Omega^1$ satisfies

$$\langle \Omega(X, Y)Z, W \rangle_1 = \left(\int_{S^1} [Y, \dot{Z}], \int_{S^1} [X, \dot{W}] \right)_{\mathfrak{g}} - (X \leftrightarrow Y),$$

where the inner product is the Ad-invariant form on the Lie algebra \mathfrak{g} . We want to write the right hand side of this equation as an H^1 inner product with W , in order to recognize $\Omega(X, Y)$ as a ΨDO .

Let $\{e_i\}$ be an orthonormal basis of \mathfrak{g} , considered as a left-invariant frame of TG and as global sections of γ^*TG . Let $c_{ij}^k = ([e_i, e_j], e_k)_{\mathfrak{g}}$ be the structure constants of \mathfrak{g} . (The Levi-Civita connection on left invariant vector fields for the left invariant metric is given by $\nabla_X Y = \frac{1}{2}[X, Y]$, so the structure constants are twice the Christoffel symbols.) For $X = X^i e_i = X^i(\theta) e_i$, $Y = Y^j e_j$, etc., integration by parts gives

$$\langle \Omega(X, Y)Z, W \rangle_1 = \left(\int_{S^1} \dot{Y}^i Z^j d\theta \right) \left(\int_{S^1} \dot{X}^\ell W^m d\theta \right) c_{ij}^k c_{\ell m}^n \delta_{kn} - (X \leftrightarrow Y).$$

Since

$$\int_{S^1} c_{\ell m}^n \dot{X}^\ell W^m = \int_{S^1} \left(\delta^{mc} c_{\ell c}^n \dot{X}^\ell e_m, W^b e_b \right)_{\mathfrak{g}} = \left\langle \Delta^{-1}(\delta^{mc} c_{\ell c}^n \dot{X}^\ell e_m), W \right\rangle_1,$$

we get

$$\begin{aligned} \langle \Omega(X, Y)Z, W \rangle_1 &= \left\langle \left[\int_{S^1} \dot{Y}^i Z^j \right] c_{ij}^k \delta_{kn} \delta^{ms} c_{\ell s}^n \Delta^{-1}(\dot{X}^\ell e_m), W \right\rangle_1 - (X \leftrightarrow Y) \\ &= \left\langle \left[\int_{S^1} a_j^k(\theta, \theta') Z^j(\theta') d\theta' \right] e_k, W \right\rangle_1, \end{aligned}$$

with

$$a_j^k(\theta, \theta') = \dot{Y}^i(\theta') c_{ij}^r \delta_{rn} \delta^{ms} c_{\ell s}^n \left(\Delta^{-1}(\dot{X}^\ell e_m) \right)^k(\theta) - (X \leftrightarrow Y). \quad (4.1)$$

We now show that $Z \mapsto \left(\int_{S^1} a_j^k(\theta, \theta') Z^j(\theta') d\theta' \right) e_k$ is a smoothing operator. Applying Fourier transform and Fourier inversion to Z^j yields

$$\begin{aligned} \int_{S^1} a_j^k(\theta, \theta') Z^j(\theta') d\theta' &= \int_{S^1 \times \mathbb{R} \times S^1} a_j^k(\theta, \theta') e^{i(\theta' - \theta'') \cdot \xi} Z^j(\theta'') d\theta'' d\xi d\theta' \\ &= \int_{S^1 \times \mathbb{R} \times S^1} \left[a_j^k(\theta, \theta') e^{-i(\theta - \theta') \cdot \xi} \right] e^{i(\theta - \theta'') \cdot \xi} Z^j(\theta'') d\theta'' d\xi d\theta', \end{aligned}$$

so $\Omega(X, Y)$ is a Ψ DO with symbol

$$b_j^k(\theta, \xi) = \int_{S^1} a_j^k(\theta, \theta') e^{i(\theta - \theta') \cdot \xi} d\theta', \quad (4.2)$$

with the usual mixing of local and global notation.

For fixed θ , (4.2) contains the Fourier transform of $\dot{Y}^i(\theta')$ and $\dot{X}^i(\theta')$, as these are the only θ' -dependent terms in (4.1). Since the Fourier transform is taken in a local chart with respect to a partition of unity, and since in each chart \dot{Y}^i and \dot{X}^i times the partition of unity function is compactly supported, the Fourier transform of a_j^k in each chart is rapidly decreasing. Thus $b_j^k(\theta, \xi)$ is the product of a rapidly decreasing function with $e^{i\theta \cdot \xi}$, and hence is of order $-\infty$.

We now give a second proof. For all s ,

$$\nabla_X Y = \frac{1}{2}[X, Y] - \frac{1}{2}\Delta^{-s}[\Delta^s X, Y] + \frac{1}{2}\Delta^{-s}[X, \Delta^s Y].$$

Label the terms on the right hand side (1) – (3). As an operator on Y for fixed X , the symbol of (1) is $\sigma((1))_\mu^a = \frac{1}{2}X^\varepsilon c_{\varepsilon\mu}^a$. Abbreviating $(\xi^2)^{-s}$ by ξ^{-2s} , we have

$$\begin{aligned} \sigma((2))_\mu^a &\sim -\frac{1}{2}c_{\varepsilon\mu}^a \left[\xi^{-2s} \Delta^s X^\varepsilon - \frac{2s}{i} \xi^{-2s-1} \partial_\theta \Delta^s X^\varepsilon \right. \\ &\quad \left. + \sum_{\ell=2}^{\infty} \frac{(-2s)(-2s-1)\dots(-2s-\ell+1)}{i^\ell \ell!} \xi^{-2s-\ell} \partial_\theta^\ell \Delta^s X^\varepsilon \right] \\ \sigma((3))_\mu^a &\sim \frac{1}{2}c_{\varepsilon\mu}^a \left[X^\varepsilon + \sum_{\ell=1}^{\infty} \frac{(-2s)(-2s-1)\dots(-2s-\ell+1)}{i^\ell \ell!} \xi^{-\ell} \partial_\theta^\ell X^\varepsilon \right]. \end{aligned}$$

Thus

$$\begin{aligned} \sigma(\nabla_X)_\mu^a &\sim \frac{1}{2}c_{\varepsilon\mu}^a \left[2X^\varepsilon - \xi^{-2s} \Delta^s X^\varepsilon + \frac{2s}{i} \xi^{-2s-1} \partial_\theta \Delta^s X^\varepsilon \right. \\ &\quad \left. - \sum_{\ell=2}^{\infty} \frac{(-2s)(-2s-1)\dots(-2s-\ell+1)}{i^\ell \ell!} \xi^{-2s-\ell} \partial_\theta^\ell \Delta^s X^\varepsilon \right. \\ &\quad \left. + \sum_{\ell=1}^{\infty} \frac{(-2s)(-2s-1)\dots(-2s-\ell+1)}{i^\ell \ell!} \xi^{-\ell} \partial_\theta^\ell X^\varepsilon \right]. \end{aligned} \quad (4.3)$$

Set $s = 1$ in (4.3), and replace ℓ by $\ell - 2$ in the first infinite sum. Since $\Delta = -\partial_\theta^2$, a little algebra gives

$$\sigma(\nabla_X)_\mu^a \sim c_{\varepsilon\mu}^a \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{i^\ell} \partial_\theta^\ell X^\varepsilon \xi^{-\ell} = \text{ad} \left(\sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{i^\ell} \partial_\theta^\ell X \xi^{-\ell} \right). \quad (4.4)$$

Denote the infinite sum in the last term of (4.4) by $W(X, \theta, \xi)$. The map $X \mapsto W(X, \theta, \xi)$ takes the Lie algebra of left invariant vector fields on LG to the Lie algebra $L\mathfrak{g}[[\xi^{-1}]]$, the space of formal Ψ DOs of nonpositive integer order on the trivial bundle $S^1 \times \mathfrak{g} \rightarrow S^1$, where the Lie bracket on the target involves multiplication of power series and bracketing in \mathfrak{g} . We claim that this map is a Lie algebra homomorphism. Assuming this, we see that

$$\begin{aligned} \sigma(\Omega(X, Y)) &= \sigma([\nabla_X, \nabla_Y] - \nabla_{[X, Y]}) \sim \sigma([\text{ad } W(X), \text{ad } W(Y)] - \text{ad } W([X, Y])) \\ &= \sigma(\text{ad}([W(X), W(Y)]) - \text{ad } W([X, Y])) = 0, \end{aligned}$$

which proves that $\Omega(X, Y)$ is a smoothing operator.

To prove the claim, set $X = x_n^a e^{in\theta} e_a, Y = y_m^b e^{im\theta} e_b$. Then

$$\begin{aligned} W([X, Y]) &= W(x_n^a y_m^b e^{i(n+m)\theta} c_{ab}^k e_k) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{i^\ell} c_{ab}^k \partial_\theta^\ell (x_n^a y_m^b e^{i(n+m)\theta}) \xi^{-\ell} e_k \\ [W(X), W(Y)] &= \sum_{\ell=0}^{\infty} \sum_{p+q=\ell} \frac{(-1)^{p+q}}{i^{p+q}} \partial_\theta^p (x_n^a e^{in\theta}) \partial_\theta^q (y_m^b e^{im\theta}) \xi^{-(p+q)} c_{ab}^k e_k, \end{aligned}$$

and these two sums are clearly equal. \square

It would be interesting to understand how the map W fits into the representation theory of the loop algebra $L\mathfrak{g}$.

Part II. Characteristic Classes on LM

In this part, we construct a general theory of Chern-Simons classes on certain infinite rank bundles including the frame/tangent bundle of loop spaces, following the construction of primary characteristic classes in [20]. The primary classes vanish on the tangent bundles of loop spaces, which forces the consideration of secondary classes. The key ingredient is to replace the ordinary matrix trace in the Chern-Weil theory of invariant polynomials on finite dimensional Lie groups with the Wodzicki residue on invertible bounded Ψ DOs.

As discussed in the Introduction, there are absolute and relative versions of Chern-Simons theory. We use the relative version, which assigns an odd degree form to a pair of connections. In particular, for TLM , we can use the L^2 and $s = 1$ Levi-Civita connections to form Wodzicki-Chern-Simons (WCS) classes associated to a metric on M .

In §5, we develop the general theory of Wodzicki-Chern and WCS classes for bundles with structure group ΨDO_0^* , the group of invertible classical zeroth order pseudodifferential operators. We show the vanishing of the Wodzicki-Chern classes of LM and more general mapping spaces. As in finite dimensions, we show the existence of WCS classes in $H^n(LM, \mathbb{C})$ if $\dim(M) = n$ is odd (Definition 5.1) and give the local expression for the WCS classes associated to the Chern character (Theorem 5.5). In Theorem 5.6, we prove that the Chern character WCS class vanishes if $\dim(M) \equiv 3 \pmod{4}$. In §6, we associate to every circle action $a : S^1 \times M^n \rightarrow M^n$ an n -cycle $[a]$ in LM . For a specific metric on $S^2 \times S^3$ and a specific circle action a , we prove via exact computer calculations that the WCS class is nonzero by integrating it over $[a]$. Since the corresponding integral for the cycle associated to the trivial action is zero, a cannot be homotoped to the trivial action. We use this result to prove that $\pi_1(\text{Diff}(S^2 \times S^3))$ is infinite.

Throughout this part, H^* always refers to de Rham cohomology for complex valued forms. By [2], $H^*(LM) \simeq H_{\text{sing}}^*(LM, \mathbb{C})$.

5. Chern-Simons Classes on Loop Spaces

We begin in §5.1 with a review of Chern-Weil and Chern-Simons theory in finite dimensions, following [6].

In §5.2, we discuss Chern-Weil and Chern-Simons theory on a class of infinite rank bundles including the frame bundles of loop spaces. As in §2.7, the geometric structure group of these bundles is ΨDO_0^* , so we need a trace on the Lie algebra $\Psi\text{DO}_{\leq 0}$ to define invariant polynomials. There are two types of traces, one given by

taking the zeroth order symbol and one given by the Wodzicki residue [17], [21]. Here we only consider the Wodzicki residue trace.

Using this trace, we generalize the usual definitions of Chern and Chern-Simons classes in de Rham cohomology. In particular, given a $U(n)$ -invariant polynomial P of degree k , we define a corresponding WCS class $CS_P^W \in H^{2k-1}(LM)$ if $\dim(M) = 2k - 1$. We are forced to consider these secondary classes, because the Wodzicki-Chern classes of mapping spaces $\text{Maps}(N, M)$ vanish. In Theorem 5.5, we give an exact expression for the WCS classes associated to the Chern character. In Theorem 5.6, we show that these WCS classes in $H^{4k+3}(LM^{4k+3})$ vanish; in contrast, in finite dimensions, the Chern-Simons classes associated to the Chern character vanish in $H^{4k+1}(M^{4k+1})$.

5.1. Chern-Weil and Chern-Simons Theory for Finite Dimensional Bundles. We first review the Chern-Weil construction. Let G be a finite dimensional Lie group with Lie algebra \mathfrak{g} , and let $G \longrightarrow F \longrightarrow M$ be a principal G -bundle over a manifold M . Set $\mathfrak{g}^k = \mathfrak{g}^{\otimes k}$ and let

$$I^k(G) = \{P : \mathfrak{g}^k \longrightarrow \mathbb{C} \mid P \text{ symmetric, multilinear, Ad-invariant}\}$$

be the degree k Ad-invariant polynomials on \mathfrak{g} .

Remark 5.1. For classical Lie groups G , $I^k(G)$ is generated by the polarization of the Newton polynomials $\text{Tr}(A^\ell)$, where Tr is the usual trace on finite dimensional matrices.

For $\phi \in \Lambda^\ell(F, \mathfrak{g}^k)$, $P \in I^k(G)$, set $P(\phi) = P \circ \phi \in \Lambda^\ell(F)$.

Theorem 5.1 (The Chern-Weil Homomorphism [14]). *Let $F \longrightarrow M$ have a connection θ with curvature $\Omega_F \in \Lambda^2(F, \mathfrak{g})$. For $P \in I^k(G)$, $P(\Omega_F)$ is a closed invariant real form on F , and so determines a closed form $P(\Omega_M) \in \Lambda^{2k}(M)$. The Chern-Weil map*

$$\oplus_k I^k(G) \longrightarrow H^*(M), \quad P \mapsto [P(\Omega_M)]$$

is a well-defined algebra homomorphism, and in particular is independent of the choice of connection on F .

The proof depends on:

- (The *commutativity property*) For $\phi \in \Lambda^\ell(F, \mathfrak{g}^k)$,

$$d(P(\phi)) = P(d\phi). \tag{5.1}$$

- (The *infinitesimal invariance property*) For $\psi_i \in \Lambda^{\ell_i}(F, \mathfrak{g})$, $\phi \in \Lambda^1(F, \mathfrak{g})$ and $P \in I^k(G)$,

$$\sum_{i=1}^k (-1)^{\ell_1 + \dots + \ell_i} P(\psi_1 \wedge \dots \wedge [\psi_i, \phi] \wedge \dots \wedge \psi_k) = 0. \tag{5.2}$$

$[P(\Omega_M)]$ is called the *characteristic class* of P . For example, the characteristic class associated to $\text{Tr}(A^k)$ is the k^{th} component of the Chern character of F .

Part of the theorem's content is that for any two connections on F , $P(\Omega_1) - P(\Omega_0) = dCS_P(\theta_1, \theta_0)$ for some odd form $CS_P(\nabla_1, \nabla_0)$. Explicitly,

$$CS_P(\theta_1, \theta_0) = \int_0^1 P(\theta_1 - \theta_0, \overbrace{\Omega_t, \dots, \Omega_t}^{k-1}) dt \quad (5.3)$$

where

$$\theta_t = t\theta_0 + (1-t)\theta_1, \quad \Omega_t = d\theta_t + \theta_t \wedge \theta_t$$

[5, Appendix].

Remark 5.2. For $F \xrightarrow{\pi} M$, $\pi^*F \rightarrow F$ is trivial. Take θ_1 to be the flat connection on π^*F with respect to a fixed trivialization. Let θ_1 also denote the connection $\chi^*\theta_1$ on F , where χ is the global section of π^*F . For any other connection θ_0 on F , $\theta_t = t\theta_0$, $\Omega_t = t\Omega_0 + (t^2 - t)\theta_0 \wedge \theta_0$. Assume an invariant polynomial P takes values in \mathbb{R} . Then we obtain the formulas for the transgression form $TP(\Omega_1)$ on F : for

$$\phi_t = t\Omega_1 + \frac{1}{2}(t^2 - t)[\theta, \theta], \quad TP(\theta) = l \int_0^1 P(\theta \wedge \phi_t^{k-1}) dt, \quad (5.4)$$

$dTP(\theta) = P(\Omega_1) \in \Lambda^{2l}(F)$ [6]. $TP(\Omega_1)$ pushes down to an \mathbb{R}/\mathbb{Z} -class on M , the absolute Chern-Simons class.

As usual, these formulas carry over to connections $\nabla = d + \omega$ on vector bundles $E \rightarrow M$ in the form

$$CS_P(\nabla_1, \nabla_0) = \int_0^1 P(\omega_1 - \omega_0, \Omega_t, \dots, \Omega_t) dt, \quad (5.5)$$

since $\omega_1 - \omega_0$ and Ω_t are globally defined forms.

5.2. Chern-Weil and Chern-Simons Theory for ΨDO_0^* -Bundles. Let $\mathcal{E} \rightarrow \mathcal{M}$ be an infinite rank bundle over a paracompact Banach manifold \mathcal{M} , with the fiber of \mathcal{E} modeled on a fixed Sobolev class of sections of a finite rank hermitian vector bundle $E \rightarrow N$, and with structure group $\Psi\text{DO}_0^*(E)$. For such ΨDO_0^* -bundles, we can produce primary and secondary characteristic classes once we choose a trace on $\Psi\text{DO}_{\leq 0}(E)$. Since the adjoint action of ΨDO_0^* on $\Psi\text{DO}_{\leq 0}$ is by conjugation, a trace on $\Psi\text{DO}_{\leq 0}$ will extend to a polynomial on forms satisfying (5.1), (5.2), so the finite dimensional proofs extend.

These traces were classified in [17], although there are additional traces in our special case $N = S^1$ [21]. Roughly speaking, the traces fall into two classes, the leading order symbol trace [20] and the Wodzicki residue. In this paper, we consider only the Wodzicki residue, and refer to [16] for the leading order symbol trace.

For simplicity, we mainly restrict to the generating invariant polynomials $P_k(A) = A^k$, and only consider $\mathcal{E} = TLM$, which we recall is the complexified tangent bundle. We will work with vector bundles rather than principal bundles.

Definition 5.1. (i) The k^{th} *Wodzicki-Chern (WC) form* of a ΨDO_0^* -connection ∇ on TLM with curvature Ω is

$$c_k^W(\Omega)(\gamma) = \frac{1}{k!} \int_{S^*S^1} \text{tr} \sigma_{-1}(\Omega^k) d\xi dx. \quad (5.6)$$

Here we recall that for each $\gamma \in LM$, $\sigma_{-1}(\Omega^k)$ is a $2k$ -form with values in endomorphisms of a trivial bundle over S^*S^1 .

(ii) The k^{th} *Wodzicki-Chern-Simons (WCS) form* of two ΨDO_0^* -connections ∇_0, ∇_1 on TLM is

$$\begin{aligned} CS_{2k-1}^W(\nabla_1, \nabla_0) &= \frac{1}{k!} \int_0^1 \int_{S^*S^1} \text{tr} \sigma_{-1}((\omega_1 - \omega_0) \wedge (\Omega_t)^{k-1}) dt \\ &= \frac{1}{k!} \int_0^1 \text{res}^w[(\omega_1 - \omega_0) \wedge (\Omega_t)^{k-1}] dt. \end{aligned} \quad (5.7)$$

(iii) The k^{th} *Wodzicki-Chern-Simons form* associated to a Riemannian metric g on M , denoted $CS_{2k-1}^W(g)$, is $CS_{2k-1}^W(\nabla_1, \nabla_0)$, where ∇_0, ∇_1 refer to the L^2 and $s = 1$ Levi-Civita connections on LM , respectively.

(iv) Let $\Sigma = \{\sigma\}$ be the group of permutations of $\{1, \dots, k\}$. Let $I : 1 \leq i_1 < \dots < i_\ell = k$ be a partition of k (i.e. with $i_0 = 0$, $\sum_{j=1}^k (i_j - i_{j-1}) = k$). For the symmetric, $U(n)$ -invariant, multilinear form on $\mathfrak{u}(n)$

$$\begin{aligned} P_I(A_1, A_2, \dots, A_k) &= \frac{1}{k!} \sum_{\sigma} \text{tr}(A_{\sigma(1)} \cdot \dots \cdot A_{\sigma(i_1)}) \text{tr}(A_{\sigma(i_1+1)} \cdot \dots \cdot A_{\sigma(i_2)}) \\ &\quad \dots \cdot \text{tr}(A_{\sigma(i_{\ell-1})} \cdot \dots \cdot A_{\sigma(k)}), \end{aligned}$$

define the symmetric, ΨDO_0^* -invariant, multilinear form on $\Psi\text{DO}_{\leq 0}$ by

$$\begin{aligned} P_I^W(B_1, \dots, B_k) &= \frac{1}{k!} \sum_{\sigma} \left(\int_{S^*S^1} \text{tr} \sigma_{-1}(B_{\sigma(1)} \cdot \dots \cdot B_{\sigma(i_1)}) \right. \\ &\quad \cdot \int_{S^*S^1} \text{tr} \sigma_{-1}(B_{\sigma(i_1+1)} \cdot \dots \cdot B_{\sigma(i_2)}) \\ &\quad \left. \dots \cdot \int_{S^*S^1} \text{tr} \sigma_{-1}(B_{\sigma(i_{\ell-1})} \cdot \dots \cdot B_{\sigma(k)}) \right). \end{aligned}$$

The *Wodzicki-Chern form* associated to P_I for a ΨDO_0^* -connection on TLM with curvature Ω is

$$\begin{aligned} c_{P_I}^W(\Omega) &= P_I^W(\Omega, \Omega, \dots, \Omega) \\ &= \frac{1}{k!} \int_{S^*S^1} \text{tr} \sigma_{-1}(\Omega^{k_1}) \cdot \int_{S^*S^1} \text{tr} \sigma_{-1}(\Omega^{k_2}) \cdot \dots \cdot \int_{S^*S^1} \text{tr} \sigma_{-1}(\Omega^{k_\ell}) \\ &= \frac{k_1! k_2! \cdot \dots \cdot k_\ell!}{k!} c_{k_1}^W(\Omega) c_{k_2}^W(\Omega) \cdot \dots \cdot c_{k_\ell}^W(\Omega), \end{aligned} \quad (5.8)$$

where $k_1 = i_1 - i_0, k_2 = i_2 - i_1, \dots, k_\ell = i_\ell - i_{\ell-1}$.

Setting $K = (k_1, \dots, k_\ell)$, we also denote $c_{P_I}^W(\Omega)$ by $c_K^W(\Omega)$.

(v) Let ∇_0, ∇_1 be ΨDO_0^* -connections on TLM with connection forms ω_0, ω_1 , respectively. The *Wodzicki-Chern-Simon form associated to P_I and ∇_0, ∇_1* is

$$CS_{P_I}^W(\nabla_1, \nabla_0) = \int_0^1 P_I^W(\omega_1 - \omega_0, \Omega_t, \dots, \Omega_t) dt.$$

In (iv) and (v), we do not bother with a normalizing constant, since we do not claim that there is a normalization which gives classes with integral periods. Note that the k^{th} WCS class is associated to $P_k(A_1, \dots, A_k) = \text{tr}(A_1 \cdot \dots \cdot A_k)$, i.e. the partition $K = (k)$, or in other words to the polynomial giving the k^{th} component of the Chern character.

As in finite dimensions, $c_k^W(\nabla)$ is a closed $2k$ -form, with de Rham cohomology class $c_k(LM)$ independent of ∇ , as $c_k^W(\Omega_1) - c_k^W(\Omega_0) = dCS_{2k-1}^W(\nabla_1, \nabla_0)$.

Remark 5.3. It is an interesting question to determine all the ΨDO_0^* -invariant polynomials on $\Psi\text{DO}_{\leq 0}$. As above, $U(n)$ -invariant polynomials combine with the Wodzicki residue (or the other traces on $\Psi\text{DO}_{\leq 0}$) to give ΨDO_0^* -polynomials, but there may be others.

The tangent space TLM , and more generally mapping spaces $\text{Maps}(N, M)$ with N closed have vanishing Wodzicki-Chern classes. Here we take a Sobolev topology on $\text{Maps}(N, M)$ for some large Sobolev parameter, so that $\text{Maps}(N, M)$ is a paracompact Banach manifold. We denote the de Rham class of $c_{P_I}^W(\Omega)$ for a connection on \mathcal{E} by $c_{P_I}(\mathcal{E})$.

Proposition 5.2. *Let N, M be closed manifolds, and let $\text{Maps}_f(N, M)$ denote the component of a fixed $f : N \rightarrow M$. Then the cohomology classes $c_{P_I}^W(\text{Maps}_f(N, M))$ of $T\text{Maps}(M, N)$ vanish.*

Proof. For TLM , the L^2 connection in Lemma 2.1 has curvature Ω which is a multiplication operator. Thus $\sigma_{-1}(\Omega)$ and hence $\sigma_{-1}(\Omega^i)$ are zero, so the WC forms $c_{P_I}(\Omega)$ also vanish.

For $n \in N$ and $h : N \rightarrow M$, let $\text{ev}_n : \text{Maps}_f(N, M)$ be $\text{ev}_n(h) = h(n)$. Then $D_X Y(h)(n) \stackrel{\text{def}}{=} (\text{ev}_h^* \nabla^{LC, M})_X Y(h)(n)$ is the L^2 Levi-Civita connection on $\text{Maps}(N, M)$. As in Lemma 2.1, the curvature of D is a multiplication operator. Details are left to the reader. \square

Remark 5.4. (i) These mapping spaces fit into the framework of the Families Index Theorem in the case of a trivial fibration $Z \rightarrow M \xrightarrow{\pi} B$ of closed manifolds. Given a finite rank bundle $E \rightarrow M$, we get an associated infinite rank bundle $\mathcal{E} = \pi_* E \rightarrow B$. For the fibration $N \rightarrow N \times \text{Maps}(N, M) \rightarrow \text{Maps}(N, M)$ and $E = \text{ev}^* TM$, \mathcal{E} is $T\text{Maps}(N, M)$. A connection ∇ on E induces a connection $\nabla^{\mathcal{E}}$ on \mathcal{E} defined by

$$(\nabla_{Z^s}^{\mathcal{E}})(b)(z) = ((\text{ev}^* \theta^u)_{(Z,0)} u_s)(b, z).$$

Here $u_s(b, z) = s(b)(z)$. The curvature $\Omega^\mathcal{E}$ satisfies

$$\Omega^\mathcal{E}(Z, W)s(b)(z) = (\text{ev}^* \Omega)((Z, 0), (W, 0))u_s(b, z).$$

This follows from

$$\Omega^\mathcal{E}(Z, W)s(b)(z) = [\nabla_Z^\mathcal{E} \nabla_W^\mathcal{E} - \nabla_W^\mathcal{E} \nabla_Z^\mathcal{E} - \nabla_{[Z, W]}^\mathcal{E}]s(b)(z).$$

Thus the connection and curvature forms take values in multiplication operators, and so $c_k^W(\mathcal{E}) = 0$.

If the fibration is nontrivial, the connection on \mathcal{E} depends on the choice of a horizontal complement to TZ in TM , and the corresponding connection and curvature forms take values in first order differential operators.

(ii) In finite dimensions, odd Chern forms of complexified real bundles like $T\text{Maps}(N, M)$ vanish, because the form involves a composition of an odd number of skew-symmetric matrices. In contrast, odd WC forms involve terms like $\sigma_{-1}(\Omega^1) \wedge \Omega^M \wedge \dots \wedge \Omega^M$, where Ω^1 is the curvature of the $s = 1$ Levi-Civita connection. By Lemma 3.2(ii), $\sigma_{-1}(\Omega^1)$ is not skew-symmetric as an endomorphism. Thus it is not obvious that the odd WC forms vanish.

Similarly, in finite dimensions the Chern-Simons form for the odd Chern classes of complexified real bundles vanish, but this need not be the case for WCS forms. In fact, we will produce nonvanishing WCS classes associated to $c_3^W(TLM^5)$ in §6.

In finite dimensions, Chern classes are topological obstructions to the reduction of the structure group and geometric obstructions to the existence of a flat connection. Wodzicki-Chern classes for ΨDO_0^* -bundles are also topological and geometric obstructions, but the geometric information is a little more refined due to the grading on the Lie algebra $\Psi\text{DO}_{\leq 0}$.

Proposition 5.3. *Let $\mathcal{E} \rightarrow \mathcal{B}$ be an infinite rank ΨDO_0^* -bundle, for ΨDO_0^* acting on $E \rightarrow N^n$. If \mathcal{E} admits a reduction to the gauge group $\mathcal{G}(E)$, then $c_k^W(\mathcal{E}) = 0$ for all k , and hence $c_{P_I}^W(\mathcal{E}) = 0$ for all P_I . If \mathcal{E} admits a ΨDO_0^* -connection whose curvature has order $-k$, then $c_\ell(\mathcal{E}) = 0$ for $\ell \geq [n/k]$.*

Proof. If the structure group of \mathcal{E} reduces to the gauge group, there exists a connection one-form with values in $\text{Lie}(\mathcal{G}) = \text{End}(E)$, the Lie algebra of multiplication operators. Thus the Wodzicki residue of powers of the curvature vanishes, so the Wodzicki-Chern classes vanish. For the second statement, the order of the curvature is less than $-n$ for $\ell \geq [n/k]$, so the Wodzicki residue vanishes in this range. \square

However, we do not have examples of nontrivial WC classes; cf. [16], where it is conjectured that these classes always vanish.

The relative WCS form is not difficult to compute.

Proposition 5.4. *Let σ be in the group of permutations of $\{1, \dots, 2k-1\}$. Then*

$$\begin{aligned} CS_{2k-1}^W(g)(X_1, \dots, X_{2k-1}) & \quad (5.9) \\ &= \frac{2}{(2k-1)!} \sum_{\sigma} \operatorname{sgn}(\sigma) \int_{S^1} \operatorname{tr} [(-2R(X_{\sigma(1)}, \dot{\gamma}) - R(\cdot, \dot{\gamma})X_{\sigma(1)} + R(X_{\sigma(1)}, \cdot)\dot{\gamma}) \\ & \quad \cdot (\Omega^M)^k(X_{\sigma(2)}, \dots, X_{\sigma(2k-1)})]. \end{aligned}$$

Proof.

$$\sigma_0((\omega_1 - \omega_0)_X)_b^a = \Gamma_{cb}^a X^c - \Gamma_{cb}^a X^c = 0.$$

Thus

$$CS_{2k-1}^W(g) = \int_0^1 \int_{S^*S^1} \operatorname{tr} \sigma_{-1}(\omega_1 - \omega_0) \wedge (\sigma_0(\Omega_t))^k dt. \quad (5.10)$$

Moreover,

$$\begin{aligned} \sigma_0(\Omega_t) &= td(\sigma_0(\omega_0)) + (1-t)d(\sigma_0(\omega_1)) \\ & \quad + (t\sigma_0(\omega_0) + (1-t)\sigma_0(\omega_1)) \wedge (t\sigma_0(\omega_0) + (1-t)\sigma_0(\omega_1)) \\ &= d\omega^M + \omega^M \wedge \omega^M \\ &= \Omega^M. \end{aligned}$$

Therefore

$$CS_{2k-1}^W(g) = \int_0^1 \int_{S^*S^1} \operatorname{tr} [\sigma_{-1}(\omega_1) \wedge (\Omega^M)^k] dt, \quad (5.11)$$

since $\sigma_{-1}(\omega_0) = 0$. We can drop the integral over t . The integral over the ξ variable contributes a factor of 2: the integrand has a factor of $|\xi|^{-2}\xi$, which equals ± 1 on the two components of S^*S^1 . Since the fiber of S^*S^1 at a fixed θ consists of two points with opposite orientation, the ‘‘integral’’ over each fiber is $1 - (-1) = 2$. Thus

$$\begin{aligned} CS_{2k-1}^W(g)(X_1, \dots, X_{2k-1}) & \quad (5.12) \\ &= \frac{2}{(2k-1)!} \sum_{\sigma} \operatorname{sgn}(\sigma) \int_{S^1} \operatorname{tr} [(-2R(X_{\sigma(1)}, \dot{\gamma}) - R(\cdot, \dot{\gamma})X_{\sigma(1)} + R(X_{\sigma(1)}, \cdot)\dot{\gamma}) \\ & \quad \cdot (\Omega^M)^k(X_{\sigma(2)}, \dots, X_{\sigma(2k-1)})] \end{aligned}$$

by Lemma 3.1. □

This produces odd classes in the de Rham cohomology of the loop space of an odd dimensional manifold.

Theorem 5.5. *(i) Let $\dim(M) = 2k-1$ and let P be a $U(n)$ -invariant polynomial of degree k . Then $c_P^W(\Omega) \equiv 0$ for any ΨDO_0^* -connection ∇ on TLM . Thus $CS_P^W(\nabla_1, \nabla_0)$ is closed and defines a class $[CS_P^W(\nabla_1, \nabla_0)] \in H^{2k-1}(LM)$. In particular, we can define $[CS_P^W(g)] \in H^{2k-1}(LM)$ for a Riemannian metric g on M .*

(ii) For $\dim(M) = 2k - 1$, the k^{th} Wodzicki-Chern-Simons form $CS_{2k-1}^W(g)$ simplifies to

$$\begin{aligned} & CS_{2k-1}^W(g)(X_1, \dots, X_{2k-1}) \\ &= \frac{2}{(2k-1)!} \sum_{\sigma} \text{sgn}(\sigma) \int_{S^1} \text{tr} [(-R(\cdot, \dot{\gamma})X_{\sigma(1)} + R(X_{\sigma(1)}, \cdot)\dot{\gamma}) \\ &\quad \cdot (\Omega^M)^{k-1}(X_{\sigma(2)}, \dots, X_{\sigma(2k-1)})]. \end{aligned} \quad (5.13)$$

Proof. (i) Let Ω be the curvature of ∇ . $c_P^W(\Omega)(X_1, \dots, X_{2k})(\gamma)$ is a sum of monomials of the form (5.8). This is a $2k$ -form on M , and hence vanishes.

(ii) Since

$$R(X_1, \dot{\gamma}) \cdot (\Omega^M)^k(X_2, \dots, X_{2k-1}) = [i_{\dot{\gamma}} \text{tr}(\Omega^k)](X_1, \dots, X_{2k-1}) = \text{tr}(\Omega^k)(\dot{\gamma}, X_1, \dots, X_{2k-1}),$$

the first term on the right hand side of (5.12) vanishes on a $(2k - 1)$ -manifold. \square

Remark 5.5. There are several variants to the construction of relative WCS classes.

(i) If we define the transgression form $T_{c_k}(\nabla)$ with the Wodzicki residue replacing the trace in (5.4), it is easy to check that $T_{c_k}(\nabla)$ involves $\sigma_{-1}(\Omega)$. For ∇ the L^2 connection, this WCS class vanishes. For ∇ the H^s connection, $s > 0$, $\sigma_{-1}(\Omega)$ involves the covariant derivative of the curvature of M (cf. Lemma 3.2 for $s = 1$.) Thus the relative WCS class is easier for computations than the absolute class $[T_{c_k}(\nabla)]$.

(ii) If we define $CS_k^W(g)$ using the H^1 connection instead of the $s = 1$ connection, then we omit the term involving $R(X_{\sigma(1)}, \cdot)\dot{\gamma}$ from (5.9) as well. For the H^1 connection omits the $A_X Y$ term in Proposition 2.3, and hence omits the last two terms in Theorem 2.2 and the last term in Lemma 3.1.

(iii) If we define $CS_k^W(g)$ using the H^s connection instead of the H^1 connection, the WCS class is multiplied by s by Lemma 3.3. Therefore we can remove the dependence of the WCS class on the artificial parameter s by setting $s = 1$:

Definition 5.2. The *regularized k^{th} WCS class* associated to a Riemannian metric g on M is $CS_k^{W,\text{reg}}(g) \stackrel{\text{def}}{=} CS_k^W(\nabla^{1,1}, \nabla_0)$, where $\nabla^{1,1}$ is the H^1 connection and ∇_0 is the L^2 Levi-Civita connection.

By (ii), the H^1 connection one-form is $\sigma_{-1}(\omega_X^{1,1}) = i|\xi|^{-1}\xi[-R(X, \dot{\gamma}) - R(\cdot, \dot{\gamma})]$. By Theorem 5.5, the regularized WCS class is therefore the cohomology class of

$$\begin{aligned} & CS_k^{W,\text{reg}}(g)(X_1, \dots, X_{2k-1}) \\ &= \frac{2}{(2k-1)!} \sum_{\sigma} \text{sgn}(\sigma) \int_{S^1} \text{tr} [-R(\cdot, \dot{\gamma})X_{\sigma(1)} \cdot (\Omega^M)^{k-1}(X_{\sigma(2)}, \dots, X_{\sigma(2k-1)})]. \end{aligned}$$

We conclude this section with a vanishing result that does not have a finite dimensional analogue.

Theorem 5.6. *The k^{th} WCS class $CS_k^W(g)$ vanishes if $\dim(M) \equiv 3 \pmod{4}$.*

Proof. Let $\dim(M) = 2k - 1$. Since Ω^M takes values in skew-symmetric endomorphisms, so does $(\Omega^M)^{k-1}$ if k is even, i.e. if $\dim(M) \equiv 3 \pmod{4}$. The term $-R(\cdot, \dot{\gamma})X_{\sigma(1)} + R(X_{\sigma(1)}, \cdot)\dot{\gamma}$ in (5.13) is a symmetric endomorphism. For in normal coordinates, this term is $(-R_{bdca} + R_{cbda})X^c\dot{\gamma}^d \equiv A_{ab}$, say, so the curvature terms in $A_{ab} - A_{ba}$ are

$$\begin{aligned} -R_{bdca} + R_{cbda} + R_{adcb} - R_{cadb} &= -R_{bdca} + R_{cbda} + R_{cbad} - R_{dbca} \\ &= -R_{bdca} + R_{cbda} - R_{cbda} + R_{bdca} = 0. \end{aligned}$$

Thus the integrand in (5.13) is the trace of a symmetric endomorphism composed with a skew-symmetric endomorphism, and so vanishes. \square

Example 5.7. We contrast Theorem 5.6 with the situation in finite dimensions. Let $\dim(M) = 3$. The only invariant monomials of degree two are $\text{tr}(A_1A_2)$ and $\text{tr}(A_1)\text{tr}(A_2)$ (corresponding to c_2 and c_1^2 , respectively).

For M , $\text{tr}(A_1A_2)$ gives rise to the classical Chern-Simons invariant for M . However, the Chern-Simons class associated to $\text{tr}(A_1)\text{tr}(A_2)$ involves $\text{tr}(\omega_1 - \omega_0)\text{tr}(\Omega_t)$, which vanishes since both forms take values in skew-symmetric endomorphisms.

In contrast, on LM we know that the WCS class CS_3^W associated to $\text{tr}(A_1A_2)$ vanishes. The WCS associated to $\text{tr}(A_1)\text{tr}(A_2)$ involves $\text{tr}\sigma_{-1}(\omega_1 - \omega_0) = \text{tr}\sigma_{-1}(\omega_1)$ and $\text{tr}\sigma_{-1}(\Omega_t)$. Both ω_1 and Ω_t take values in skew-symmetric Ψ DOs, but this does not imply that the terms in their symbol expansions are skew-symmetric. In fact, a calculation using Lemma 3.1 shows that $\sigma_{-1}(\omega_1)$ is not skew-symmetric. Thus the WCS class associated to $\text{tr}(A_1)\text{tr}(A_2)$ may be nonzero.

6. An Application of Wodzicki-Chern-Simons Classes to Circle Actions

In this section we use WCS classes to distinguish different S^1 actions on $M = S^2 \times S^3$. We use this to conclude that $\pi_1(\text{Diff}(M), id)$ is infinite.

Recall that $H^*(LM)$ denotes de Rham cohomology of complex valued forms. In particular, integration of closed forms over homology cycles gives a pairing of $H^*(LM)$ and $H_*(LM, \mathbb{C})$.

For any closed oriented manifold M , let $a_0, a_1 : S^1 \times M \rightarrow M$ be two smooth actions. Thus

$$a_i(0, m) = m, \quad a_i(\theta, a(\psi, m)) = a_i(\theta + \psi, m).$$

Definition 6.1. (i) a_0 and a_1 are *smoothly homotopic* if there exists a smooth map

$$F : [0, 1] \times S^1 \times M \rightarrow M, \quad F(0, \theta, m) = a_0(\theta, m), \quad F(1, \theta, m) = a_1(\theta, m).$$

(ii) a_0 and a_1 are *smoothly homotopic through actions* if $F(t, \cdot, \cdot) : S^1 \times M \rightarrow M$ is an action for all t .

We can rewrite an action in two equivalent ways.

- a determines (and is determined by) $a^D : S^1 \longrightarrow \text{Diff}(M)$ given by $a^D(\theta)(m) = a(\theta, m)$. $a^D(\theta)$ is a diffeomorphism because

$$a^D(-\theta)(a^D(\theta, m)) = a(-\theta, a(\theta, m)) = m.$$

Since $a^D(0) = id$, we get a class $[a^D] \in \pi_1(\text{Diff}(M), id)$, the fundamental group of $\text{Diff}(M)$ based at id . Here $\text{Diff}(M)$ is a Banach manifold as an open subset of the Banach manifold of $\text{Maps}(M) = \text{Maps}(M, M)$ of some fixed Sobolev class.

- a determines (and is determined by) $a^L : M \longrightarrow LM$ given by $a^L(m)(\theta) = a(\theta, m)$. This determines a class $[a^L] \in H_n(LM, \mathbb{Z})$ with $n = \dim(M)$ by setting $[a^L] = a_*^L[M]$. In concrete terms, if we triangulate M as the n -cycle $\sum_i n_i \sigma_i$, with $\sigma_i : \Delta^n \longrightarrow M$, then $[a^L]$ is the homology class of the cycle $\sum_i n_i (a^L \circ \sigma_i)$.

We give a series of elementary lemmas comparing these maps.

Lemma 6.1. a_0 is smoothly homotopic to a_1 through actions iff $[a_0^D] = [a_1^D] \in \pi_1(\text{Diff}(M), id)$.

Proof. (\Rightarrow) Given F as above, set $G : [0, 1] \times S^1 \longrightarrow \text{Diff}(M)$ by $G(t, \theta)(m) = F(t, \theta, m)$. We have $G(0, \theta)(m) = a_0(\theta, m) = a^D(\theta)(m)$, $G(1, \theta)(m) = a_1(\theta, m) = a_1^D(\theta)(m)$. $G(t, \theta) \in \text{Diff}(M)$, because

$$G(t, -\theta)(G(t, \theta)(m)) = F(t, -\theta, F(t, \theta, m)) = F(t, 0, m) = m.$$

(This uses that $F(t, \cdot, \cdot)$ is an action.) Since F is smooth, G is a continuous (in fact, smooth) map of $\text{Diff}(M)$. Thus a_0^D, a_1^D are homotopic as elements of $\text{Maps}(S^1, \text{Diff}(M))$, so $[a_0^D] = [a_1^D]$.

(\Leftarrow) Let $G : [0, 1] \times S^1 \longrightarrow \text{Diff}(M)$ be a continuous homotopy from $a_0^D(\theta) = G(0, \theta)$ to $a_1^D(\theta) = G(1, \theta)$ with $G(t, 0) = id$ for all t . It is possible to approximate G arbitrarily well by a smooth map, since $[0, 1] \times S^1$ is compact. Set $F : [0, 1] \times S^1 \times M \longrightarrow M$ by $F(t, \theta, m) = G(t, \theta)(m)$. F is smooth. Note that $F(0, \theta, m) = G(0, \theta)(m) = a_0^D(\theta)(m) = a_0(\theta, m)$, and $F(1, \theta, m) = a_1(\theta, m)$. Thus a_0 and a_1 are smoothly homotopic. \square

There are similar results for a^L .

Lemma 6.2. a_0 is smoothly homotopic to a_1 iff $a_0^L, a_1^L : M \longrightarrow LM$ are smoothly homotopic.

Proof. Let F be the homotopy from a_0 to a_1 . Set $H : [0, 1] \times M \longrightarrow LM$ by $H(t, m)(\theta) = F(t, \theta, m)$. Then $H(0, m)(\theta) = F(0, \theta, m) = a_0(\theta, m) = a_0^L(m)(\theta)$, $H(1, m)(\theta) = a_1(\theta, m) = a_1^L(m)(\theta)$, so H is a homotopy from a_0^L to a_1^L . It is easy to check that H is smooth.

Conversely, if $H : [0, 1] \times M \longrightarrow LM$ is a smooth homotopy from a_0^L to a_1^L , set $F(t, \theta, m) = H(t, m)(\theta)$. \square

Corollary 6.3. *If a_0 is smoothly homotopic to a_1 , then $[a_0^L] = [a_1^L] \in H_n(LM, \mathbb{Z})$.*

Proof. By the last Lemma, a_0^L and a_1^L are homotopic. Thus $[a_0^L] = a_{0,*}^L[M] = a_{1,*}^L[M] = [a_1^L]$. \square

This yields a technique to use WCS classes to distinguish actions and to investigate $\pi_1(\text{Diff}(M), id)$. From now on, “homotopic” means “smoothly homotopic.”

Proposition 6.4. *Let $\dim(M) = 2k - 1$. Let $a_0, a_1 : S^1 \times M \rightarrow M$ be actions.*

(i) *If $\int_{[a_0^L]} CS_{2k-1}^W \neq \int_{[a_1^L]} CS_{2k-1}^W$, then a_0 and a_1 are not homotopic through actions, and $[a_0^D] \neq [a_1^D] \in \pi_1(\text{Diff}(M), id)$.*

(ii) *If $\int_{[a_1^L]} CS_{2k-1}^W \neq 0$, then $\pi_1(\text{Diff}(M), id)$ is infinite.*

Proof. (i) By Stokes’ Theorem, $[a_0^L] \neq [a_1^L] \in H_n(LM, \mathbb{C})$. By Corollary 6.3, a_0 and a_1 are not homotopic, and hence not homotopic through actions. By Lemma 6.1, $[a_0^D] \neq [a_1^D] \in \pi_1(\text{Diff}(M), id)$.

(ii) Let a_n be the n^{th} iterate of a_1 , i.e. $a_n(\theta, m) = a_1(n\theta, m)$.

We claim that $\int_{[a_n^L]} CS_{2k-1}^W = n \int_{[a_1^L]} CS_{2k-1}^W$. By (5.9), every term in CS_{2k-1}^W is of the form $\int_{S^1} \dot{\gamma}(\theta) f(\theta)$, where f is a periodic function on the circle. Each loop $\gamma \in a_1^L(M)$ corresponds to the loop $\gamma(n \cdot) \in a_n^L(M)$. Therefore the term $\int_{S^1} \dot{\gamma}(\theta) f(\theta)$ is replaced by

$$\int_{S^1} \frac{d}{d\theta} \gamma(n\theta) f(n\theta) d\theta = n \int_0^{2\pi} \dot{\gamma}(\theta) f(\theta) d\theta.$$

Thus $\int_{[a_n^L]} CS_{2k-1}^W = n \int_{[a_1^L]} CS_{2k-1}^W$. By (i), the $[a_n^L] \in \pi_1(\text{Diff}(M), id)$ are all distinct. \square

Remark 6.1. If two actions are homotopic through actions, the S^1 index of an equivariant operator of the two actions is the same. (Here equivariance means for each action $a_t, t \in [0, 1]$.) In contrast to Proposition 6.4(ii), the S^1 index of an equivariant operator cannot distinguish actions on odd dimensional manifolds, as the S^1 index vanishes. This can be seen from the local version of the S^1 index theorem [3, Thm. 6.16]. For the normal bundle to the fixed point set is always even dimensional, so the fixed point set consists of odd dimensional submanifolds. The integrand in the fixed point submanifold contribution to the S^1 -index is the constant term in the short time asymptotics of the appropriate heat kernel. In odd dimensions, this constant term is zero.

In [18], we interpret the S^1 index theorem as the integral of an equivariant characteristic class over $[a^L]$.

We now apply these methods to a Sasaki-Einstein metric on $S^2 \times S^3$ constructed in [12] to prove the following:

Theorem 6.5. (i) *There is an S^1 action on $S^2 \times S^3$ that is not smoothly homotopic to the trivial action.*

(ii) *$\pi_1(\text{Diff}(S^2 \times S^3), id)$ is infinite.*

The content of (i) is that although the S^1 -orbit γ_x through $x \in S^2 \times S^3$ is contractible to x , the contraction cannot be constructed to be smooth in x .

Proof. According to [12], the locally defined metric

$$g = \frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9}[d\psi^2 - \cos\theta d\phi^2] \\ + w(y) \left[d\alpha + \frac{ac - 2y + y^2c}{6(a-y^2)}[d\psi - \cos\theta d\phi] \right]^2, \quad (6.1)$$

with

$$w(y) = \frac{2(a-y^2)}{1-cy}, q(y) = \frac{a-3y^2+2cy^3}{a-y^2},$$

is a family of Sasaki-Einstein metrics on a coordinate ball in the variables $(\phi, \theta, \psi, y, \alpha)$. Here a and c are constants, and we can take $a \in (0, 1], c = 1$. For p, q relatively prime, $q < p$, and satisfying $4p^2 - 3q^2 = n^2$ for some integer n , and for $a = a(p, q) < 1$, the metric extends to a 5-manifold $Y^{p,q}$ which has the coordinate ball as a dense subset. In this case, (ϕ, θ, ψ, y) are spherical coordinates on $S^2 \times S^2$ with a nonstandard metric, and α is the fiber coordinate of an S^1 -fibration $S^1 \rightarrow Y^{p,q} \rightarrow S^2 \times S^2$. $Y^{p,q}$ is diffeomorphic to $S^2 \times S^3$, and has first Chern class which integrates over the two S^2 factors to p and q [12, §2]. The coordinate ranges are $\phi \in (0, 2\pi), \theta \in (0, \pi), \psi \in (0, 2\pi), \alpha \in (0, 2\pi\ell)$, where $\ell = \ell(p, q)$, and $y \in (y_1, y_2)$, with the y_i the two smaller roots of $a - 3y^2 + 2y^3 = 0$. p and q determine a, ℓ, y_1, y_2 explicitly [12, (3.1), (3.4), (3.5), (3.6)].

For these choices of p, q , we get an S^1 -action a_1 on $Y^{p,q}$ by rotation in the α -fiber. We claim that for e.g. $(p, q) = (7, 3)$,

$$\int_{[a_1^L]} CS_5^W(g) \neq 0. \quad (6.2)$$

By Proposition 6.4(iii), this implies $\pi_1(\text{Diff}(S^2 \times S^3), id)$ is infinite. Since the trivial action a_0 has $\int_{[a_0^L]} CS_5^W(g) = 0$ (by the proof of Proposition 6.4(ii) with $n = 0$), a_0 and a_1 are not smoothly homotopic by Proposition 6.4(i). Thus showing (6.2) will prove the theorem.

Set $M = S^2 \times S^3$. Since $a_1^L : M \rightarrow LM$ has degree one on its image,

$$\int_{[a_1^L]} CS_5^W(g) = \int_M a_1^{L,*} CS_5^W(g). \quad (6.3)$$

For $m \in M$,

$$a_1^{L,*} CS_5^W(g)_m = f(m)d\phi \wedge d\theta \wedge dy \wedge d\psi \wedge d\alpha$$

for some $f \in C^\infty(M)$. We determine $f(m)$ by explicitly computing $a_{1,*}^L(\partial_\phi), \dots, a_{1,*}^L(\partial_\alpha)$, (e.g. $a_{1,*}^L(\partial_\phi)(a^L(m))(t) = \partial_\phi|_{a(m,t)}$), and noting

$$\begin{aligned} f(m) &= f(m)d\phi \wedge d\theta \wedge dy \wedge d\psi \wedge d\alpha(\partial_\phi, \partial_\theta, \partial_y, \partial_\psi, \partial_\alpha) \\ &= a_{1,*}^L CS_5^W(g)_m(\partial_\phi, \dots, \partial_\alpha) \\ &= CS_5^W(g)_{a_1^L(m)}(a_{1,*}^L(\partial_\phi), \dots, a_{1,*}^L(\partial_\alpha)). \end{aligned} \tag{6.4}$$

Since $CS_5^W(g)$ is explicitly computable from the formulas in §3, we can compute $f(m)$ from (6.4). Then $\int_{[a_1^L]} CS_5^W(g) = \int_M f(m)d\phi \wedge d\theta \wedge dy \wedge d\psi \wedge d\alpha$ can be computed as an ordinary integral in the dense coordinate space.

Via this method, in the Mathematica file `ComputationsChernSimonsS2xS3.pdf` at <http://math.bu.edu/people/sr/>, $\int_{[a_1^L]} CS_5^W(g)$ is computed as a function of (p, q) . For example, $(p, q) = (7, 3)$,

$$\int_{[a_1^L]} CS_5^W(g) = -\frac{1849\pi^4}{22050}.$$

This formula is exact; the rationality up to π^4 follows from $4p^2 - 3q^2$ being a perfect square, as then the various integrals computed in (6.3) with respect to our coordinates are rational functions evaluated at rational endpoints. In particular, (6.2) holds. \square

Remark 6.2. For $a = 1$, the metric extends to the closure of the coordinate chart, but the total space is S^5 with the standard metric. $\pi_1(\text{Diff}(S^5))$ is torsion [8]. By Proposition 6.4(ii), $\int_{[a_1^L]} CS_5^W = 0$ for any circle action on S^5 . In the formulas in the Mathematica file, $\int_{[a_1^L]} CS_5^W$ is proportional to $(-1 + a)^2$, which vanishes at $a = 1$. This gives a check of the validity of the computation.

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DEPARTMENT OF MATHEMATICS, KEIO UNIVERSITY

E-mail address: maeda@math.keio.ac.jp

DEPARTMENT OF MATHEMATICS AND STATISTICS, BOSTON UNIVERSITY

E-mail address: sr@math.bu.edu

METROPOLITAN COLLEGE, BOSTON UNIVERSITY

E-mail address: fatorres@math.bu.edu