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RIEMANNIAN GEOMETRY ON LOOP SPACES

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Abstract. A Riemannian metric on a manifold M induces a family of Riemannian metrics on the loop space LM depending on a Sobolev space parameter. In Part I, we compute the Levi-Civita connection for these metrics. The connection and curvature forms take values in pseudodifferential operators (ΨDOs), and we compute the top symbols of these forms. In Part II, we develop a theory of Chern-Simons classes $CS^W_{2k-1} \in H^{2k-1}(LM^{2k-1}, \mathbb{R})$, using the Wodzicki residue on ΨDOs. For parallelizable manifolds these "Wodzicki-Chern-Simons" classes are defined for all metrics and are independent of the framing. We use CS^W_3 to distinguish some classical circle actions on S^3.

Contents

1. Introduction

The loop space LM of a manifold M appears frequently in mathematics and mathematical physics. In this paper, using an infinite dimensional version of Chern-Simons theory, we develop a nontrivial, computable theory of secondary characteristic classes on certain infinite rank bundles including the tangent bundles to loop spaces.

The theory of primary characteristic classes on infinite rank bundles was treated via Chern-Weil theory in [24]. While these classes can be nonzero, the Pontrjagin classes vanish for loop spaces (Corollary 5.14).

As in finite dimensions, the suitably interpreted Pontrjagin form $\mathrm{Tr}(\Omega^k)$ vanishes on the loop space of a $(2k-1)$-manifold, which is the precondition for defining Chern-Simons classes. In this paper, we define Chern-Simons classes for loop spaces of parallelizable manifolds (e.g. 3-manifolds). These "Wodzicki-Chern-Simons" classes $CS^W_{2k-1} \in H^{2k-1}(LM^{2k-1}, \mathbb{R})$ are somewhat stronger than their finite dimensional counterparts in that (i) they are frame independent, and hence give real (as opposed to $\mathbb{R}/\mathbb{Z}$) classes (Proposition 5.12) ; (ii) they are potentially nontrivial in all odd dimensions (Remark 5.4). As an application, we use CS^W_3 to homologically distinguish some classical circle actions (rotations, the Hopf action, the trivial action) on S^3. We know no other method that proves these actions cannot be homotoped to each other.

Since Chern-Weil and Chern-Simons theory are geometric, it is necessary to understand connections and curvature on loop spaces. Ideally, one would like to work with smooth loops, but this Frechét manifold is difficult to treat. Instead, we work with Sobolev spaces of highly differentiable loops. A Riemannian metric g on M induces a family of metrics g^s on LM parametrized by a Sobolev space parameter $s \geq 0$, where $s = 0$ gives the usual L^2 metric, and the smooth case should be some sort of limit as $s \longrightarrow \infty$. Thus we think of s as a regularizing parameter, and look for the parts of the theory which are independent of s.

In Part I, we compute the connection and curvature forms for the Levi-Civita connection for g^s. These forms take values in zeroth order pseudodifferential operators (ΨDOs) acting on a trivial bundle over S^1, as first shown by Freed for loop groups [12]. We calculate the principal and subprincipal symbols of the Levi-Civita connection one-form for integer Sobolev parameter; the noninteger case is more technical and will be treated in a subsequent paper.

In Part II, we develop a theory of Chern-Simons classes on loop spaces. The structure group for the Levi-Civita connection for (LM, g^s) is a group of ΨDOs, so we need invariant polynomials on the corresponding Lie algebra. We could use the standard polynomials $\mathrm{Tr}(\Omega^k)$ of the curvature $\Omega = \Omega^s$, where Tr is the operator trace. However, Ω^k is typically zeroth order and hence not trace class, and in any case the operator trace is impossible to compute in general. Instead, as in [24] we use the Wodzicki residue, the only trace on the full algebra of ΨDOs. Following Chern-Simons [7] as much as possible, we build a theory of Wodzicki-Chern-Simons (WCS) classes. The main difference from the finite dimensional theory is the absence of a Narasimhan-Ramanan universal connection theorem. As a result, we do not have a theory of differential characters as in [6], and we only define WCS classes for parallelizable manifolds.[1] We define parameter-free or regularized WCS classes by taking the large s limit of our formulas (Definition 5.3). In contrast to the operator trace, the Wodzicki residue is locally computable, so we can write explicit expressions for the WCS classes. In the last section, using computer calculations we prove the result on circle actions on S^3 mentioned above.

The paper is organized as follows. Part I treats the family of metrics g^s on LM associated to (M,g). §2 discusses the Levi-Civita connection for $s \in \mathbb{Z}^+$. After some preliminary material, we compute the Levi-Civita connection one-form for g^s (Theorem 2.1) and show that the one-form takes values in ΨDOs of order zero (Proposition 2.3).

In §3, we compute the principal and subprincipal symbols of the Levi-Civita connection one-form and the curvature two-form for the g^s metric. The long proofs are in the Appendix. §4 compares of our results with Freed's for loop groups [12].

Part II covers Wodzicki-Chern-Simons classes. In §5.1 we review finite dimensional Chern-Weil and Chern-Simons theory for $O(n)$-bundles. In §5.2 we replace the ordinary matrix trace by the Wodzicki residue to define characteristic and secondary classes on LM. An alternative trace given by the leading order symbol is discussed in §5.3. In §5.4, WCS classes are defined for parallelizable manifolds, and we show these classes are independent of the framing. We also define the regularized WCS classes. In §5.5, we show that the Wodzicki-Pontrjagin classes vanish on LM and more generally on $Maps(N,M)$, the space of maps from one Riemannian manifold to another.

In §6, we define when two circle actions on M are homologically distinct. We use the first WCS class to show that a rotational action, the Hopf action and the trivial action on S^3 are all homologically distinct.

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Part I. The Levi-Civita Connection on the Loop Space LM

In this part of the paper, we compute the Levi-Civita connection on LM associated to a Riemannian metric on M and a Sobolev parameter $s \in \mathbb{Z}^+$. The main result is Theorem 2.1, which computes the Levi-Civita connection explicitly except for one term denoted A_XY. This term is analyzed more concretely in Proposition 2.3.

[1] WCS classes are defined for all manifolds in a subsequent paper.

Part I is organized as follows. In §2, we review background material on LM and pseudodifferential operators on manifolds, and prove Theorem 2.1. In §3, we compute the relevant symbols of the Levi-Civita connection one-form and the curvature two-form. In §4, we compare our results with earlier work of Freed [12] on loop groups.

2. **The Levi-Civita Connection for Integer Sobolev Parameters**

This section covers background material and computes the Levi-Civita connection on LM for integer Sobolev parameter. In §2.1, we review material on LM, and in §2.2 we review pseudodifferential operators and the Wodzicki residue. In §2.3, we give the main computation of the connection one-form for the Levi-Civita connection on LM. In §2.4, we give a more complete calculation of the Levi-Civita connection for integer Sobolev parameter. In §2.5, we prove a technical lemma allowing us to reduce local coordinate computations on LM to local computations on M. In §2.6, we discuss the necessary extension of the structure group of LM from a gauge group to a group of bounded invertible ΨDOs.

2.1. **Preliminaries on LM.** Let $(M, \langle\ ,\ \rangle)$ be a closed oriented Riemannian n-manifold with loop space $LM = C^\infty(S^1, M)$ of smooth loops. LM is a smooth infinite dimensional Fréchet manifold, but it is technically simpler to work with the smooth Hilbert manifold of loops in some Sobolev class $s \gg 0$, as we now recall. For $\gamma \in LM$, the formal tangent space $T_\gamma LM$ is $\Gamma(\gamma^*TM)$, the space of smooth sections of the pullback bundle $\gamma^*TM \longrightarrow S^1$. For $s > 1/2$, we complete $\Gamma(\gamma^*TM \otimes \mathbb{C})$ with respect to the Sobolev inner product

$$\langle X, Y\rangle_s = \frac{1}{2\pi}\int_0^{2\pi} \langle (1+\Delta)^s X(\alpha), Y(\alpha)\rangle_{\gamma(\alpha)} d\alpha,\ X, Y \in \Gamma(\gamma^*TM).$$

Here $\Delta = D^*D$, with $D = D/d\gamma$ the covariant derivative along γ. (We use this notation instead of the classical D/dt to keep track of γ.) We need the complexified pullback bundle, denoted from now on just as γ^*TM, in order to apply the pseudodifferential operator $(1+\Delta)^s$. The construction of $(1+\Delta)^s$ is reviewed in §2.2. We denote this completion by $H^s(\gamma^*TM)$.

A small real neighborhood U_γ of the zero section in $H^s(\gamma^*TM)$ is a coordinate chart near γ in the space of H^s loops via the pointwise exponential map

$$\exp_\gamma : U_\gamma \longrightarrow LM,\ X \mapsto \left(\alpha \mapsto \exp_{\gamma(\alpha)} X(\alpha)\right). \tag{2.1}$$

The differentiability of the transition functions $\exp_{\gamma_1}^{-1} \cdot \exp_{\gamma_2}$ is proved in [9] and [13, Appendix A]. Here γ_1, γ_2 are close loops in the sense that a geodesically convex neighborhood of $\gamma_1(\theta)$ contains $\gamma_2(\theta)$ and vice versa for all θ. Since γ^*TM is (non-canonically) isomorphic to the trivial bundle $\mathcal{R} = S^1 \times \mathbb{R}^n \longrightarrow S^1$, the model space for LM is the set of H^s sections of this trivial bundle.

The tangent bundle TLM has transition functions $d(\exp_{\gamma_1}^{-1}\circ\exp_{\gamma_2})$. Under the isomorphisms $T_{\gamma_1}LM \simeq \mathcal{R} \simeq T_{\gamma_2}LM$, the transition functions lie in the gauge group $\mathcal{G}(\mathcal{R})$, so this is the structure group of TLM.

2.2. **Review of ΨDO Calculus.** We recall the construction of classical pseudodifferential operators (ΨDOs) on a closed manifold M from [14, 27], assuming knowledge of ΨDOs on $\mathbb{R}^n$. We emphasize how to calculate global symbols in local coordinates, since subprincipal terms are coordinate dependent (e.g. (2.2)).

A linear operator $P : C^\infty(M) \longrightarrow C^\infty(M)$ is a ΨDO of order d if for every open chart $U \subset M$ and functions $\phi, \psi \in C_c^\infty(U)$, $\phi P \psi$ is a ΨDO of order d on $\mathbb{R}^n$, where we do not distinguish between U and its diffeomorphic image in $\mathbb{R}^n$. Let $\{U_i\}$ be a finite cover of M with subordinate partition of unity $\{\phi_i\}$. Let $\psi_i \in C_c^\infty(U_i)$ have $\psi_i \equiv 1$ on $\mathrm{supp}(\phi_i)$ and set $P_i = \psi_i P \phi_i$. Then $\sum_i \phi_i P_i \psi_i$ is a ΨDO of M, and P differs from $\sum_i \phi_i P_i \psi_i$ by a smoothing operator, denoted $P \sim \sum_i \phi_i P_i \psi_i$. In particular, this sum is independent of the choices up to smoothing operators. All this carries over to ΨDOs acting on sections of a bundle over M.

An example is the ΨDO $(1+\Delta-\lambda)^{-1}$ for Δ a positive order nonnegative elliptic ΨDO and λ outside the spectrum of $1+\Delta$. In each U_i, we construct a parametrix P_i for $A_i = \psi_i(1+\Delta-\lambda)\phi_i$ by formally inverting $\sigma(A_i)$ and then constructing a ΨDO with the inverted symbol. By [1, App. A], $B = \sum_i \phi_i P_i \psi_i$ is a parametrix for $(1+\Delta-\lambda)^{-1}$. Since $B \sim (1+\Delta-\lambda)^{-1}$, $(1+\Delta-\lambda)^{-1}$ is itself a ΨDO. For $x \in U_i$, by definition

$$\sigma((1+\Delta-\lambda)^{-1})(x,\xi) = \sigma(P)(x,\xi) = \sigma(\phi P \phi)(x,\xi),$$

where ϕ is a bump function with $\phi(x) = 1$ [14, p. 29]; the symbol depends on the choice of (U_i, ϕ_i).

The operator $(1+\Delta)^s$ for $\mathrm{Re}(s) < 0$, which exists as a bounded operator on $L^2(M)$ by the functional calculus, is also a ΨDO. To see this, we construct the putative symbol σ_i of $\psi_i(1+\Delta)^s\phi_i$ in each U_i by a contour integral $\int_\Gamma \lambda^s \sigma[(1+\Delta-\lambda)^{-1}]d\lambda$ around the spectrum of $1+\Delta$. We then construct a ΨDO Q_i on U_i with $\sigma(Q_i) = \sigma_i$, and set $Q = \sum_i \phi_i Q_i \psi_i$. By arguments in [27], $(1+\Delta)^s \sim Q$, so $(1+\Delta)^s$ is a ΨDO.

For $\alpha = (\alpha_1, \ldots, \alpha_n)$, let $\partial_x^\alpha = (\partial^{\alpha_1}/\partial x_1^{\alpha_1})\ldots(\partial^{\alpha_n}/\partial x_n^{\alpha_n})$ in some local coordinates. For any ΨDO $P \sim \sum_i \phi_i P_i \psi_i$ and fixed $x \in U_{i_0}$, the symbol of P in U_{i_0} coordinates is

$$\begin{aligned}
\sigma(P)(x,\xi) &= \sigma(\phi(\sum_i \phi_i P_i \psi_i)\phi) = \sum_i \phi(x)\phi_i(x)\sigma(P\psi_i\phi)\\
&= \sum_i \phi(x)\phi_i(x)\sum_\alpha \frac{1}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha\sigma(P_i)\partial_x^\alpha(\psi_i\phi)\\
&= \sum_i \phi(x)\phi_i(x)\sigma(P_i)(x,\xi)\psi_i(x)\phi(x)\\
&= \sum_i \phi_i(x)\sigma(P_i)(x,\xi),
\end{aligned}$$

where we use $\psi_i \equiv 1$ on $\mathrm{supp}(\phi_i)$, $\partial_x^\alpha \phi(x) = 0$ and $\partial_x^\alpha \psi = 0$ on $\mathrm{supp}(\phi)$ for $\alpha \neq 0$. Thus symbols can be calculated locally.

Recall that the *Wodzicki residue* of a ΨDO P on sections of a bundle $E \longrightarrow M^n$ is

$$\int_{S^*M} \mathrm{tr}\ \sigma_{-n}(P)(x,\xi) d\xi dx,$$

where S^*M is the unit cosphere bundle for some metric. The Wodzicki residue is independent of choice of local coordinates, and up to scaling is the unique trace on the algebra of ΨDOs if $\dim(M) > 1$ (see e.g. [10] in general and [25] for the case $M = S^1$.). It will be used in Part II to define characteristic classes on LM.

If the U_i are diffeomorphic to precompact open balls in $\mathbb{R}^n$, $\sigma(P_i)$ extends smoothly to ∂U_i after possible shrinking of the U_i. Let $V_1 = U_1, V_i = U_i - \cup_{j=1}^{i-1} U_j$. As with any differential form, letting ϕ_1 equal one on "more and more" of U_1 and letting the other ϕ_i equal one on "a little more than" V_i, we get

$$\begin{aligned}\int_{S^*M} \mathrm{tr}\ \sigma_{-n}(P)(x,\xi) d\xi dx &= \sum_i \int_{U_i} \phi_i(x)\, \mathrm{tr}\ \sigma_{-n}(P_i)(x,\xi) d\xi dx \\ &= \sum_i \int_{V_i} \mathrm{tr}\ \sigma_{-n}(P_i)(x,\xi) d\xi dx;\end{aligned}$$

the invariance of the Wodzicki residue makes the right hand side well defined.

Therefore, for Wodzicki residue calculations we can sum up the integrals of the locally defined symbols. In particular, for a bundle E over S^1 with ΨDO P, we can find a closed cover $I_i = [a_i, a_{i+1}]$ with $E|_{I_i}$ trivial, and then

$$\int_{S^*S^1} \mathrm{tr}\ \sigma_{-1}(P)(x,\xi) d\xi d\theta = \sum_i \int_{(a_i, a_{i+1})} \mathrm{tr}\ \sigma_{-1}(P_i)(x,\xi) d\xi dx. \tag{2.2}$$

2.3. **Computing the Levi-Civita Connection.** The H^s metric makes LM a Riemannian manifold. The H^s Levi-Civita connection on LM is determined by the six term formula

$$\begin{aligned}2\langle \nabla_Y^s X, Z\rangle_s &= X\langle Y, Z\rangle_s + Y\langle X, Z\rangle_s - Z\langle X, Y\rangle_s \\ &\quad +\langle [X,Y], Z\rangle_s + \langle [Z,X], Y\rangle_s - \langle [Y,Z], X\rangle_s.\end{aligned} \tag{2.3}$$

Recall that

$$[X,Y]^a = X(Y^a)\partial_a - Y(X^a)\partial_a \equiv \delta_X(Y) - \delta_Y(X) \tag{2.4}$$

in local coordinates on a finite dimensional manifold. Note that $X^i\partial_i(Y^a) = X(Y^a) = (\delta_X Y)^a$ in this notation.

(2.4) continues to hold for vector fields on LM, even though the index a does not refer to coordinates on LM. To see this, one checks that the coordinate-free proof that $L_X Y(f) = [X,Y](f)$ for $f \in C^\infty(M)$ (e.g. [29, p. 70]) carries over to functions on LM. In brief, the usual proof involves a map $H(s,t)$ of a neighborhood of the origin in $\mathbb{R}^2$ into M, where s,t are parameters for the flows of X,Y, resp. For LM, we have a map $H(s,t,\theta)$, where θ is the loop parameter. The usual proof uses only s,t differentiations, so θ is unaffected. The point is that the Y^i are local functions on the (s,t,θ) parameter space, whereas the Y^i are not local functions on M at points where loops cross or self-intersect.

Fix a loop γ and choose a cover $\{(a_i,b_i)\}$ of S^1 such that there is a coordinate cover $\{U_i\}$ of an open neighborhood of $\mathrm{Im}(\gamma)$ in M with

$$\gamma([a_i,b_i]) \subset U_i. \tag{2.5}$$

With these covers fixed, (2.5) holds for all loops near γ. Let $\{\phi_i\}$ be a partition of unity on S^1 subordinate to the cover $\{(a_i,b_i)\}$, and let $g_{ab} = g_{ab}^{(i)}$ be the metric tensor on U_i. The first term on the right hand side of (2.3) is

$$X\langle Y,Z\rangle_s = X\left(\sum_i \int_{(a_i,b_i)} \phi_i \cdot g_{ab}^{(i)}[(1+\Delta)^s Y]^a Z^b\right). \tag{2.6}$$

Since ϕ_j is independent of γ, we have

$$\begin{aligned} X\langle Y,Z\rangle_s \;=\; & \sum_i \int_{(a_i,b_i)} \phi_i \cdot \delta_X g_{ab}^{(i)}[(1+\Delta)^s Y]^a Z^b \\ & +\sum_i \int_{(a_i,b_i)} \phi_i \cdot g_{ab}^{(i)}([\delta_X(1+\Delta)^s]Y)^a Z^b \\ & +\sum_i \int_{(a_i,b_i)} \phi_i \cdot g_{ab}^{(i)}((1+\Delta)^s \delta_X Y)^a \cdot Z^b \\ & +\sum_i \int_{(a_i,b_i)} \phi_i \cdot g_{ab}^{(i)}((1+\Delta)^s Y)^a \cdot \delta_X Z^b \end{aligned} \tag{2.7}$$

We will abbreviate terms like $\sum_i \int_{(a_i,b_i)} \phi_i \cdot \delta_X g_{ab}^{(i)}[(1+\Delta)^s Y]^a Z^b$ by $\int_{S^1} \delta_X g_{ab}[(1+\Delta)^s Y]^a Z^b$, and terms like $[(\delta_X(1+\Delta)^s)(Y)]^a$ by $\delta_X(1+\Delta)^s Y^a$. For example,

$$\langle Y,Z\rangle_s = \sum_i \int_{(a_i,b_i)} \phi_i \cdot g_{ab}^{(i)}((1+\Delta)^s Y)^a Z^b = \int_{S^1} g_{ab}((1+\Delta)^s Y)^a Z^b.$$

Collecting all terms from the six term formula similar to the last two terms in (2.7) gives

$$\begin{aligned} & \int_{S^1} g_{ab}(1+\Delta)^s \left[\delta_X Y^a \cdot Z^b + Y^a \cdot \delta_X Z^b + \delta_Y X^a \cdot Z^b + X^a \cdot \delta_Y Z^b - \delta_Z X^a \cdot Y^b \right. \\ & \qquad \left. -X^a \cdot \delta_Z Y^b + (\delta_X Y - \delta_Y X)^a \cdot Z^b + (\delta_Z X - \delta_X Z)^a \cdot Y^b - (\delta_Y Z - \delta_Z Y)^a \cdot X^b \right] \\ = \;& 2\int_{S^1} g_{ab}(1+\Delta)^s \delta_X Y^a \cdot Z^b. \end{aligned} \tag{2.8}$$

The three terms in the six term formula corresponding to the first term on the right hand side of (2.7) contribute

$$
\begin{aligned}
&\int_{S^1} \delta_X g_{ab}\cdot(1+\Delta)^s Y^a\cdot Z^b + \delta_Y g_{ab}\cdot(1+\Delta)^s X^a\cdot Z^b - \delta_Z g_{ab}\cdot(1+\Delta)^s X^a\cdot Y^b\\
&\quad = \int_{S^1} \delta_X g_{ab}\cdot(1+\Delta)^s Y^a\cdot Z^b + \delta_Y g_{ab}\cdot(1+\Delta)^s X^a\cdot Z^b\\
&\qquad\quad - Z^t\partial_t g_{ab}\cdot(1+\Delta)^s X^a\cdot Y^b \qquad (2.9)\\
&\quad = \int_{S^1} \delta_X g_{ab}\cdot(1+\Delta)^s Y^a\cdot Z^b + \delta_Y g_{ab}\cdot(1+\Delta)^s X^a\cdot Z^b\\
&\qquad\quad - g_{ab}Z^b g^{at}\partial_t g_{ef}\cdot(1+\Delta)^s X^e\cdot Y^f
\end{aligned}
$$

The three terms in the six term formula corresponding to the second term on the right hand side of (2.7) contribute

$$\int_{S^1} g_{ab}\left[\delta_X(1+\Delta)^s Y^a\cdot Z^b + \delta_Y(1+\Delta)^s X^a\cdot Z^b - \delta_Z(1+\Delta)^s X^a\cdot Y^b\right]. \qquad (2.10)$$

The last term in (2.10) is linear in Z. If it is continuous in $Z\in H^s(\gamma^*TM)$, then it is of the form

$$\langle \delta_Z(1+\Delta)^s X, Y\rangle_0 = \langle A_X(Y), Z\rangle_0 \qquad (2.11)$$

for some $A_X(Y)$ which is *a priori* in $H^{-s}(\gamma^*TM)$. In §2.4 we will find A_XY for $s\in\mathbb{Z}^+$.

By (2.7) – (2.11),

$$
\begin{aligned}
&2\langle\nabla_X Y, Z\rangle_s\\
&\quad = \int_{S^1}\big(2g_{ab}(1+\Delta)^s\delta_X Y^a\cdot Z^b + \delta_X g_{ab}\cdot(1+\Delta)^s Y^a\cdot Z^b + g_{ab}\delta_X(1+\Delta)^s Y^a\cdot Z^b\\
&\qquad\quad +\delta_Y g_{ab}(1+\Delta)^s X^a\cdot Z^b + g_{ab}\delta_Y(1+\Delta)^s X^a\cdot Z^b \qquad (2.12)\\
&\qquad\quad -g_{ab}[g^{at}\partial_t g_{ef}\cdot(1+\Delta)^s X^e\cdot Y^f + A_X(Y)^a]Z^b\big).
\end{aligned}
$$

The second term on the right hand side of (2.12) is

$$\delta_X g_{ab}\cdot(1+\Delta)^s Y^a\cdot Z^b = g_{ab}Z^b g^{af}\delta_X g_{ef}\cdot(1+\Delta)^s Y^e,$$

so for $g_{ab}=g_{ab}^{(i)}$, we get

$$
\begin{aligned}
&\int_{S^1}\delta_X g_{ab}((1+\Delta)^s Y^a)Z^b\\
&\quad = \sum_i\int_{(a_i,b_i)}\phi_i\delta_X g_{ab}((1+\Delta)^s Y)^a\cdot Z^b \qquad (2.13)\\
&\quad = \sum_i\int_{(a_i,b_i)}\phi_i g_{ab}\left[(1+\Delta)^s(1+\Delta)^{-s}(g^{tf}\delta_X g_{ef}((1+\Delta)^s Y)^e\partial_t)\right]^a Z^b.
\end{aligned}
$$

In the last line, we take a diffeomorphism of (a_i, b_i) to $\mathbb{R}$, and compute the various ΨDOs in these local charts as in §2.2.

The last expression in (2.13) is not of the form $\langle W, Z\rangle_s$, since $(1+\Delta)^{-s}(g^{tf,(i)}\delta_X g^{(i)}_{ef}((1+\Delta)^s Y)^e\partial_t)$ depends on i. In fact, all the other terms on the right hand side of (2.12) are of the form $\int_{S^1}\phi_i g_{ab}(f^i)^a Z^b$ for some locally defined vector fields f^i, since e.g. $\delta_X Y^a$ also depends on coordinate choices along γ. Since the left hand side of (2.12) is global, the apparently local terms on the right hand side must sum to a global expression. With this understanding, we have

Theorem 2.1. *Assuming (2.11) defines A_XY, the Levi-Civita connection $\nabla = \nabla^{(s)}$ for the H^s-metric on LM is given by*

$$\begin{aligned}(\nabla_X Y)^a &= \delta_X(Y^a) + \frac{1}{2}(1+\Delta)^{-s}\left[g^{af}\delta_X g_{ef}\cdot(1+\Delta)^s Y^e + \delta_X(1+\Delta)^s Y^a\right]\\ &\quad + \frac{1}{2}(1+\Delta)^{-s}\left[g^{af}\delta_Y g_{ef}\cdot(1+\Delta)^s X^e + \delta_Y(1+\Delta)^s X^a\right] \qquad (2.14)\\ &\quad - \frac{1}{2}(1+\Delta)^{-s}\left[g^{at}\partial_t g_{ef}\cdot(1+\Delta)^s X^e\cdot Y^f + A_X(Y)^a\right].\end{aligned}$$

In the theorem, $\delta_X(1+\Delta)^s Y^a$ is shorthand for $[(\delta_X(1+\Delta)^s)(Y)]^a$, and similarly for the other terms.

2.4. **Integer Sobolev Parameters.** If s is a positive integer, it is easy to understand the terms on the right hand side of Theorem 2.1, and in particular to calculate A_XY.

Of the six terms on the right hand side of Theorem 2.1 involving the ΨDO $(1+\Delta)^{-s}$, the first, third and fifth are standard ΨDOs acting on Y.

For the second and fourth terms, we have to analyze the variation of $(1+\Delta)^s$. Let $Z \in T_\gamma LM = H^s(\gamma^*TM)$. If $f : M \longrightarrow \mathbb{R}$ is a (locally defined) function on M, then $\delta_Z f(x) = Z^i\partial_i f(x)$ in local coordinates near $x = \gamma(\theta)$, since $\delta_Z f(x) = (d/dt)|_{t=0} f(\alpha(t))$, where $\alpha(0) = x$, $\dot\alpha(0) = Z_{\gamma(\theta)}$.

The situation is different for (locally defined) functions on $S^1 \times LM$, such as $\dot\gamma^\nu$. Let $\tilde\gamma : [0, 2\pi]\times(-\varepsilon,\varepsilon) \longrightarrow M$ be a smooth map with $\tilde\gamma(\theta, 0) = \gamma(\theta)$, and $\frac{d}{d\tau}|_{\tau=0}\tilde\gamma(\theta,\tau) = Z(\theta)$. Since (θ,τ) are coordinate functions on $S^1\times(-\varepsilon,\varepsilon)$, we have

$$Z(\dot\gamma^\nu) = \partial^Z_\tau(\dot\gamma^\nu) = \frac{\partial}{\partial\tau}\Big|_{\tau=0}\left(\frac{\partial}{\partial\theta}(\tilde\gamma(\theta,\tau)^\nu\right) = \frac{\partial}{\partial\theta}\frac{\partial}{\partial\tau}\Big|_{\tau=0}\tilde\gamma(\theta,\tau)^\nu = \partial_\theta Z^\nu \equiv \dot Z^\nu. \tag{2.15}$$

The covariant derivative along γ is the operator on $W \in \Gamma(\gamma^*TM)$ given by

$$\begin{aligned}\frac{DW}{d\gamma} &= (\gamma^*\nabla^M)_{\partial_\theta}(W) = \partial_\theta W + (\gamma^*\omega^M)(\partial_\theta)(W)\\ &= \partial_\theta(W^i)\partial_i + \dot\gamma^t W^r\Gamma^j_{tr}\partial_j,\end{aligned}$$

where ∇^M is the Levi-Civita connection on M, ω^M is the connection one-form in local coordinates $\{\partial_i\}$ on M, and Γ^j_{tr} are the Christoffel symbols. For $\Delta = (\frac{D}{d\gamma})^*\frac{D}{d\gamma}$, an integration by parts gives

$$(\Delta Y)^k = -\partial_\theta^2 Y^k - 2\Gamma^k_{\nu\mu}\dot\gamma^\nu\partial_\theta Y^\mu - \left(\partial_\theta\Gamma^k_{\nu\delta}\dot\gamma^\nu + \Gamma^k_{\nu\delta}\ddot\gamma^\nu + \Gamma^k_{\nu\mu}\Gamma^\mu_{\varepsilon\delta}\dot\gamma^\varepsilon\dot\gamma^\nu\right) Y^\delta.$$

Therefore, by (2.15)

$$\begin{aligned}((\delta_Z\Delta)Y)^k &= (-2Z^i\partial_i\Gamma^k_{\nu\mu}\dot\gamma^\nu - 2\Gamma^k_{\nu\mu}\dot Z^\nu)\dot Y^\mu - \Big(\dot Z^i\partial_i\Gamma^k_{\nu\delta}\dot\gamma^\nu + Z^j\partial_{ij}\Gamma^k_{\nu\delta}\dot\gamma^\nu + \dot\gamma^i\partial_i\Gamma^k_{\nu\delta}\dot Z^\nu \\ &\quad + Z^i\partial_i\Gamma^k_{\nu\delta}\ddot\gamma^\nu + \Gamma^k_{\nu\delta}\ddot Z^\nu + Z^i\partial_i\Gamma^k_{\nu\mu}\Gamma^\mu_{\varepsilon\delta}\dot\gamma^\varepsilon\dot\gamma^\nu + \Gamma^k_{\nu\mu}Z^i\partial_i\Gamma^\mu_{\varepsilon\delta}\dot\gamma^\varepsilon\dot\gamma^\nu \\ &\quad + \Gamma^k_{\nu\mu}\Gamma^\mu_{\varepsilon\delta}\dot Z^\varepsilon\dot\gamma^\nu + \Gamma^k_{\nu\mu}\Gamma^\mu_{\varepsilon\delta}\dot\gamma^\varepsilon\dot Z^\nu\Big) Y^\delta. \end{aligned} \tag{2.16}$$

Thus $(\delta_Z\Delta)Y$, resp. $\delta_Z(1+\Delta)^sY$, are second, resp. $2s$, order differential operators in Z and first, resp. $2s-1$, order differential operators in Y.

In summary, the second term in (2.14) is order -1 in Y.

We now consider the fourth and sixth terms $(1+\Delta)^{-s}[\delta_Y(1+\Delta)^sX]$, $(1+\Delta)^{-s}[A_XY]$ in (2.14).

Lemma 2.2. *(i) $\delta_Y(1+\Delta)^sX$ is order $2s$ in Y.*

(ii) A_XY is order $2s$ in Y.

Proof. (i)

$$\delta_Y(1+\Delta)^sX = \sum_{k=1}^s (1+\Delta)^{k-1}\cdot\delta_Y(1+\Delta)\cdot(1+\Delta)^{s-k}X.$$

The term with $k=s$ contains the term

$$(-\partial_\theta^2)^{s-1}(-\Gamma^k_{\nu\delta}\ddot Y^\nu X^\delta)\partial_k$$

which contains the term

$$\Gamma^k_{\nu\delta}X^\delta(-\partial_\theta^2)^sY^\nu\partial_k.$$

This is the only term of order $2s$ in Y.

(ii) Recall $\langle A_XY, Z\rangle_0 = \langle\delta_Z(1+\Delta)^sX, Y\rangle_0$. The highest order term in Y in the right hand side term is

$$\begin{aligned}\int_{S^1} g_{ab}(-\partial_\theta^2)^{s-1}(-\Gamma^a_{\nu\delta}\ddot Z^\nu X^\delta)Y^b &\sim -\int_{S^1} g_{ab}\Gamma^a_{\nu\delta}\ddot Z^\nu X^\delta(-\partial_\theta^2)^{s-1}Y^b \\ &\sim \int_{S^1} g_{ab}\Gamma^a_{\nu\delta}Z^\nu X^\delta(-\partial_\theta^2)^sY^b \\ &= (-1)^s\int_{S^1} g_{\nu a}g^{ra}g_{sb}\Gamma^s_{ri}X^iY^{b,(2s)}Z^\nu \\ &= (-1)^s\langle g^{af}g_{en}\Gamma^n_{fi}X^iY^{e,(2s)}\partial_a, Z\rangle_0,\end{aligned}$$

where $\sim$ denotes highest order term in Y. $\square$

We summarize this section:

Proposition 2.3. *For $s \in \mathbb{Z}^+$, the term $A_X Y$ in Theorem 2.1 is a differential operator of order $2s$ acting on Y. The last six terms on the right hand side of Theorem 2.1 are* ΨDO*s in Y of order $0, -1, -2s, 0, -2s, 0$, respectively.*

2.5. **The Levi-Civita Connection One-Form.** In local coordinates on a manifold N, the Levi-Civita connection can be written as $\nabla = d + \omega$. For a vector field Y and a tangent vector X, $d_X(Y) = \delta_X Y$ in the notation of §2.3, and $\omega(X)(Y)$ is the Levi-Civita connection one-form. In Theorem 2.1, the right hand side is not in the form $d + \omega$, since $\delta_X(Y^a)$ is computed in local coordinates on M, not on $N = LM$.

Nevertheless, we make the following definition:

Definition 2.1. *The* Levi-Civita connection one-form *for the H^s metric on LM is the sum of the last six terms on the right hand side of (2.14) and is denoted $\omega = \omega^{(s)}$.*

In this section, we prove a technical local lemma that justifies using this Levi-Civita connection one-form to compute symbols in §3 below.

As just mentioned, for a vector field Y on LM and a tangent vector $X \in T_\gamma LM$, $(\delta_X^{LM} Y)^a \neq \delta_X(Y^a)$ in local coordinates along a portion of γ. Indeed, in local coordinates $(\psi_\alpha)_{\alpha=1}^\infty$ of LM near γ, $\delta_X^{LM} Y$ can only mean $X(Y^\alpha)\psi_\alpha$, while Y^a are components with respect to a finite set of coordinates of M near some $\gamma(\theta_0)$. Of course, we can still substitute $\nabla = \delta + \omega$ from Theorem 2.1 into $\Omega(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ to compute the curvature on LM; as usual, we obtain $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$, where $d\omega$ is defined by the Cartan formula in the a-coordinates. This is very useful for computing symbols of Ω, whereas a decomposition $\nabla = d_{LM} + \omega_{LM}$ in coordinates on LM would be useless for symbol computations.

For Chern-Simons theory in Part II, we have to compute terms of the form $\chi^*\theta$, where χ is a local section of the frame bundle FLM and θ is a Lie algebra valued connection one-form on FLM. We have to compare $\chi^*\theta$ for LM with the M-coordinate expression $\omega^{(s)}$, as we can only compute symbols for ω-like expressions.

The following lemma relates $\chi^*\theta$ to ω.

Lemma 2.4. *Given $\gamma_0 \in LM$, there exists an open neighborhood $V \subset LM$ of γ_0, a local frame $\chi : V \longrightarrow FLM$, and local coordinates $\{U_i\}$ on M covering $Im(\gamma_0)$ such that $\chi^*\theta_\gamma(X)(Y) = \omega_\gamma(X)(Y)$ on each U_i, for all $\gamma \in V$.*

PROOF: For any local frame χ, $\chi^*\theta$ is the connection one-form in the χ trivialization of TLM, so $\nabla_X Y = X(Y^\alpha)\psi_\alpha + \chi^*\theta(X)(Y)$, where $\chi(\gamma) = (\psi_\alpha)$ for $\gamma \in V$. Let $\{\rho_i\}$ be a partition of unity for a cover $\{U_i\}$ as in (2.5), and let $\rho_i Y = Y_i^\alpha \psi_\alpha = Y_i^\alpha \psi_\alpha^j \partial_j$.

Then

$$\begin{aligned}
\nabla_X Y &= \nabla_X\left(\sum_i \rho_i Y\right) = \sum_i \nabla_X(Y_i^\alpha \psi_\alpha) = \sum_i X(Y_i^\alpha)\psi_\alpha + \chi^*\theta(X)(Y)\\
&= \sum_i X(Y_i^\alpha)\psi_\alpha^j \partial_j + \chi^*\theta(X)(Y)\\
&= \sum_i X(Y_i^\alpha \psi_\alpha^j)\partial_j + \chi^*\theta(X)(Y) - \sum_i Y_i^\alpha X(\psi_\alpha^j)\partial_j\\
&= X\left(\sum_i (\rho_i Y)^j\right)\partial_j + \chi^*\theta(X)(Y) - \sum_i Y_i^\alpha X(\psi_\alpha^j)\partial_j\\
&= X(Y^j)\partial_j + \chi^*\theta(X)(Y) - \sum_i Y_i^\alpha X(\psi_\alpha^j)\partial_j\\
&= \delta_X(Y^j)\partial_j + \chi^*\theta(X)(Y) - \sum_i Y_i^\alpha X(\psi_\alpha^j)\partial_j.
\end{aligned}$$

Since $\nabla_X Y = \delta_X(Y^j)\partial_j + \omega(X)(Y)$, we get

$$\omega(X)(Y) = \chi^*\theta(X)(Y) - \sum_i Y_i^\alpha X(\psi_\alpha^j)\partial_j.$$

The lemma follows if we can find χ and U_i (with coordinates) such that $X(\psi_\alpha^j)(\gamma)(\theta) = 0$ whenever $\gamma(\theta) \in U_i$.

Let (ψ_α^0) be a basis of $H^s(\gamma_0^* TM)$, and let $\{e_i\}$ be a global frame of $\gamma_0^* TM$. Fix $p = \gamma_0(\theta_0)$. For short vectors $V_p = \{v \in T_pM : |v| < \epsilon\}$, $\exp_p V$ is a coordinate neighborhood U, with coordinate vectors $\partial_i = d\exp_v e_i$ (thinking of $e_i \in T_vT_pM$). At p, $\partial_i = e_i$. $\{\tilde{e}_i = d\exp_v e_i\}$ is a global frame of γ^*TM for γ close to γ_0. We can trivialize TLM on some neighborhood V of γ_0 by writing $\psi_\alpha^0 = \psi_\alpha^k e_k$ and setting

$$H^s(\gamma_0^*TM) \times V \stackrel{\approx}{\longrightarrow} TLM|_V, \quad (\psi_\alpha, \gamma) \mapsto \psi_k^\alpha \tilde{e}_k.$$

For the section $\chi : \gamma \mapsto (\psi_\alpha) \equiv (\psi_k^\alpha \tilde{e}_k)$, we have $\psi_\alpha = \psi_k^\alpha \tilde{e}_k = \psi_k^\alpha \partial_k$ near p, and so $X(\psi_\alpha) = X(\psi_\alpha^k)\partial_k = 0$. □

Remark 2.1. (i) The metric on M used to define the exponential coordinates and the local frame χ in the proof need not be the fixed metric on M.

(ii) If M is parallelizable with global frame $\{e_i\}$ as in Part II, this frame also trivializes γ^*TM for all $\gamma \in LM$.

To end this section, we check that the local section in Lemma 2.4 can be extended to a global section if the frame bundle FLM is trivial. This justifies using the localized symbol calculations of §3 in the global setting of Part II.

Lemma 2.5. *Assume FLM is trivial. Let $V \subset LM$ be an open set with a local section $\chi : V \longrightarrow FLM$. There exists an open subset $V' \subset V$, and a global section $\tilde{\chi} : LM \longrightarrow FLM$ with $\tilde{\chi}|_{V'} = \chi$.*

PROOF: Let χ_1 be a global section of FLM. There exists a gauge transformation $g : V \longrightarrow Aut(\mathcal{R})$ with $g \circ \chi_1 = \chi$. It suffices to extend g to $g_1 : LM \longrightarrow \mathcal{G} = \mathcal{G}(\mathcal{R})$. Let V' be an open subset of V with $C = \overline{V'} \subset V$. By [8], $g|_C$ has a continuous extension g_2 from the metric space LM to the locally convex vector space $Hom(\mathcal{R})$. Identifying $\mathcal{R}$ with γ^*TM at a loop γ and composing g_2 with the pointwise exponential map on $\mathfrak{gl}(n, \mathbb{C})$ gives a continuous extension $g_3 : LM \longrightarrow \mathcal{G}$.

For a cover $\{U_\alpha\}$ of $\mathcal{G}$, set $V_\alpha = g_3^{-1}(U_\alpha)$. Since LM is a Hilbert manifold, each component function of $g_3|_{V_\alpha - C}$ can be uniformly approximated by a smooth function $g_{4,\alpha}$ [3]. Since LM admits a smooth partition of unity, these local approximations can be glued to a smooth function g_1 which agrees with g on V. □

2.6. **Extensions of the Frame Bundle of LM.** In this subsection we discuss the choice of structure group for the Levi-Civita connections on LM.

Let $\mathcal{H}$ be the Hilbert space $H^s(\gamma^*TM)$ for a fixed s and γ. Let $GL(\mathcal{H})$ be the group of bounded invertible linear operators on $\mathcal{H}$; inverses of elements are bounded by the closed graph theorem. $GL(\mathcal{H})$ has the subset topology of the norm topology on $\mathcal{B}(\mathcal{H})$, the bounded linear operators on $\mathcal{H}$. $GL(\mathcal{H})$ is an infinite dimensional Banach Lie group, as a group which is an open subset of the infinite dimensional Hilbert manifold $\mathcal{B}(\mathcal{H})$ [22, p. 59], and has Lie algebra $\mathcal{B}(\mathcal{H})$. Let $\Psi\mathrm{DO}_{\leq 0}, \Psi\mathrm{DO}_0^*$ denote the algebra of classical ΨDOs of nonpositive order and the group of invertible zeroth order ΨDOs, respectively, where all ΨDOs act on $\mathcal{H}$. Note that $\Psi\mathrm{DO}_0^* \subset GL(\mathcal{H})$.

Remark 2.2. The inclusions of $\Psi\mathrm{DO}_0^*, \Psi\mathrm{DO}_{\leq 0}$ into $GL(\mathcal{H}), \mathcal{B}(\mathcal{H})$ are trivially continuous in the subset topology. For the Fréchet topology on $\Psi\mathrm{DO}_{\leq 0}$, the inclusion is not continuous.

We recall the relationship between the Levi-Civita connection one-form θ on the frame bundle FN of a manifold N and local expressions for the Levi-Civita connection on TN. For $U \subset N$, let $\chi : U \longrightarrow FN$ be a local section. A metric connection ∇ on TN with local connection one-form ω determines a connection $\theta_{FN} \in \Lambda^1(FN, \mathfrak{o}(n))$ on FN by *(i)* θ_{FN} is the Maurer-Cartan one-form on each fiber, and *(ii)* $\theta_{FN}(Y_u) = \omega(X_p)$, for $Y_u = \chi_* X_p$ [28, Ch. 8, Vol. II], or equivalently

$$\chi^*\theta_{FN} = \omega. \tag{2.17}$$

This applies to $N = LM$. The frame bundle $FLM \longrightarrow LM$ is constructed as in the finite dimensional case. The fiber over γ is isomorphic to the gauge group $\mathcal{G}$ of $\mathcal{R}$ and fibers are glued by the transition functions for TLM. Thus the frame bundle is topologically a $\mathcal{G}$-bundle.

However, for $s \in \mathbb{Z}^+$, the connection form and hence the curvature form for the H^s Levi-Civita connection take values in $\Psi\mathrm{DO}_{\leq 0}$. These forms should take values in the Lie algebra of the structure group. Thus we should extend the structure group to the ILH Lie group $\Psi\mathrm{DO}_0^*$, since the Lie algebra is $\Psi\mathrm{DO}_{\leq 0}$. This leads to an extended frame bundles, also denoted FLM. The transition functions are unchanged, since

$\mathcal{G} \subset \Psi\mathrm{DO}_0^*$. Thus (FLM, θ^s) as a geometric bundle (i.e. as a bundle with connection θ^s associated to ∇^s) is a $\Psi\mathrm{DO}_0^*$-bundle.

In summary, for the Levi-Civita connections we have

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & FLM \\ & & \downarrow \\ & & LM \end{array} \qquad\qquad \begin{array}{ccc} \Psi\mathrm{DO}_0^* & \longrightarrow & (FLM, \theta^s) \\ & & \downarrow \\ & & LM \end{array}$$

Remark 2.3. For $s \in \mathbb{Z}^+$, if we extend the structure group of the frame bundle with connection from $\Psi\mathrm{DO}_0^*$ to $GL(\mathcal{H})$, the frame bundle becomes trivial by Kuiper's theorem. This would allow us to define Chern-Simons forms for the trivial connection on LM by the procedures of §5.4. However, there is a potential loss of information if we pass to the larger frame bundle.

The situation is similar to the following examples. Let $E \longrightarrow S^1$ be the $GL(1, \mathbb{R})$ (real line) bundle with gluing functions (multiplication by) 1 at $1 \in S^1$ and 2 at $-1 \in S^1$. E is trivial as a $GL(1, \mathbb{R})$-bundle, with global section f with $\lim_{\theta \longrightarrow -\pi^+} f(e^{i\theta}) = 1, f(1) = 0, \lim_{\theta \longrightarrow \pi^-} f(e^{i\theta}) = 2$. However, as a $GL(1, \mathbb{Q})^+$-bundle, E is nontrivial, as a global section is locally constant. As a second example, let $E \longrightarrow M$ be a nontrivial $GL(n, \mathbb{C})$-bundle. Embed $\mathbb{C}^n$ into a Hilbert space $\mathcal{H}$, and extend E to an $GL(\mathcal{H})$-bundle $\mathcal{E}$ with fiber $\mathcal{H}$ and with the same transition functions. Then $\mathcal{E}$ is trivial.

3. Local Symbol Calculations

In this section, we write down the 0 and -1 order symbols of the connection one-form and the curvature two-form of the H^s Levi-Civita connection for $s \in \mathbb{Z}^+$. Some of the proofs are in Appendix A. These results are used in the calculations of Wodzicki-Chern-Simons classes in §6. The formulas show that the s-dependence of these symbols is linear, which will be used to define regularized Wodzicki-Chern-Simons classes (see Definition 5.3).

Throughout this section, we denote the last six terms on the right hand side of (2.14) by a_X through f_X, so their sum is the Levi-Civita connection form ω_X as an operator on Y.

3.1. The 0 Order Symbols of the Connection and Curvature Forms.

Lemma 3.1. *The Levi-Civita connection form $\omega_X = \omega_X^{(s)}$ is a zeroth order ΨDO with zeroth order symbol*

$$\sigma_0(\omega_X)^a_e = \Gamma^a_{ei} X^i = (\omega^M_X)^a_e. \tag{3.1}$$

Proof: The only terms in (2.14) contributing to the zeroth order symbol of ω_X are a_X, d_X, f_X. We have $\sigma_0(a_X)^a_e = \frac{1}{2} g^{af} X^i \partial_i g_{ef}$. From the identity $\partial_m g_{ab} = \Gamma^n_{am} g_{nb} + \Gamma^n_{bm} g_{an}$, we get $\sigma_0(a_X)^a_e = \frac{1}{2}(\Gamma^a_{ei} + g^{af} g_{en} \Gamma^n_{fi}) X^i$. By Lemma 2.2, $\sigma_0(d_X)^a_e = \frac{1}{2} \Gamma^a_{ei} X^i$, $\sigma_0(f_X)^a_e = -\frac{1}{2} g^{af} g_{en} \Gamma^n_{fi} X^i$. Adding these three terms gives the result. $\square$

Corollary 3.2. $\sigma_0(\Omega^{(s)}(X,Y)) = \Omega_M(X,Y)$.

This follows from $\Omega^{(s)}(X,Y) = [\nabla^{(s)}_X, \nabla^{(s)}_Y] - \nabla^{(s)}_{[X,Y]}$ and the product formula for ΨDOs. (Here it is understood that we only take symbols of the connection forms $\omega^{(s)}$ on the right hand side, as the curvature is tensorial.) For example, $\sigma_0(\nabla^{(s)}_X\nabla^{(s)}_Y) = \sigma_0(\nabla^{(s)}_X)\sigma_0(\nabla^{(s)}_Y) = \nabla^M_X\nabla^M_Y$ with the obvious abuse of notation.

3.2. **The -1 Order Symbols of the Connection and Curvature Forms.** We state the results, with the proofs in Appendix A. These unpleasant formulas are used in computer calculations in §6. For convenience, we write $\frac{1}{i\xi^{-1}}\sigma_{-1}(\omega_X)^a_e = A$ for $\sigma_{-1}(\omega_X)^a_e = i\xi^{-1}A$.

Proposition 3.3.

$$
\begin{aligned}
&\frac{1}{i\xi^{-1}}\sigma_{-1}(\omega_X)^a_e \\
&= s\dot\gamma^\nu\partial_\nu g^{af}X^i\partial_i g_{fe} + sg^{af}\dot X^i\partial_i g_{fe} + sg^{af}X^i\partial_{\nu i}g_{fe}\dot\gamma^\nu \\
&\quad -sX^i\partial_i\Gamma^a_{\nu e}\dot\gamma^\nu - s\Gamma^a_{\nu e}\dot X^\nu - sg^{af}X^i\partial_i g_{mf}\Gamma^m_{\nu e}\dot\gamma^\nu + s\Gamma^a_{k\nu}\dot\gamma^\nu g^{kf}X^i\partial_i g_{ef} \\
&\quad +\partial_\epsilon\Gamma^a_{e\delta}\dot\gamma^\epsilon X^\delta + \Gamma^a_{\epsilon\mu}\Gamma^\mu_{e\delta}\dot\gamma^\epsilon X^\delta \\
&\quad -\Gamma^a_{e\mu}\Gamma^\mu_{\epsilon\delta}\dot\gamma^\epsilon X^\delta - \partial_e\Gamma^a_{i\delta}\dot\gamma^i X^\delta \\
&\quad +(-2s-1)g^{\ell a}g_{tn}\dot\gamma^\nu\Gamma^n_{e\nu}\Gamma^t_{\ell\delta}X^\delta + (4s-3)g^{\ell a}g_{te}\dot\Gamma^t_{\ell\delta}X^\delta \\
&\quad +(4s-1)g^{\ell a}g_{te}\Gamma^t_{\ell\delta}\dot X^\delta + (4s-2)g^{\ell a}g_{te}\Gamma^t_{\nu\mu}\Gamma^\mu_{\ell\delta}\dot\gamma^\nu X^\delta \\
&\quad +g^{\ell a}g_{te}\partial_\ell\Gamma^t_{\nu\delta}\dot\gamma^\nu X^\delta \\
&\quad +g^{\ell a}g_{te}\Gamma^t_{\ell\mu}\Gamma^\mu_{\nu\delta}\dot\gamma^\nu X^\delta + 2s\Gamma^a_{\nu\mu}\dot\gamma^\nu g^{n\mu}g_{te}\Gamma^t_{n\delta}X^\delta \\
&\quad -2s\dot\gamma^j g^{a\ell}\Gamma^\mu_{\ell j}g_{te}\Gamma^t_{\mu\delta}X^\delta - 2s\dot\gamma^j g^{a\ell}\Gamma^w_{\mu j}g_{w\ell}g^{\mu\nu}g_{te}\Gamma^t_{\nu\delta}X^\delta.
\end{aligned}
\tag{3.2}
$$

Theorem 3.4.

$$
\begin{aligned}
&\frac{1}{i\xi^{-1}}\sigma_{-1}(\Omega(X,Y))^a_e \\
&= (DX/d\gamma)^iY^j\left[\frac{4s-1}{3}R_{iej}{}^a - \frac{4s}{3}R_{jei}{}^a\right] - X^i(DY/d\gamma)^j\left[\frac{4s-1}{3}R_{jei}{}^a - \frac{4s}{3}R_{iej}{}^a\right] \\
&\quad +X^iY^j\dot\gamma^\nu\left[(8s-4)R_{ij\nu e;}{}^a + (-40s+28)R_{ije}{}^a{}_{;\nu} + (20s+14)R_{\nu je}{}^a{}_{;i}\right. \\
&\quad -(20s+14)R_{\nu ie}{}^a{}_{;j} + (4s-2)R_{\nu jie;}{}^a - (4s-2)R_{\nu ije;}{}^a \\
&\quad +(16s-12)R_{je\nu}{}^a{}_{;i} - (16s-12)R_{ie\nu}{}^a{}_{;j} + (4s-6)R_{e\nu j}{}^a{}_{;i} \\
&\quad \left.-(4s-6)R_{e\nu i}{}^a{}_{;j} + (-20s+18)R_{iej}{}^a{}_{;\nu} - (-20s+18)R_{jei}{}^a{}_{;\nu}\right].
\end{aligned}
\tag{3.3}
$$

4. The Loop Group Case

In this section, we relate our work to Freed's work on based loop groups ΩG [12]. We find a particular representation of the loop algebra that controls the order of the curvature of the H^1 metric on ΩG.

$\Omega G \subset LG$ with base point e.g. $e \in G$ has tangent space $T_\gamma \Omega G = \{X \in T_\gamma LG : X(0) = X(2\pi) = 0\}$ in some Sobolev topology. Instead of using $D^2/d\gamma^2$ to define the Sobolev spaces, the usual choice is $\Delta_{S^1} = -d^2/d\theta^2$ coupled to the identity operator on the Lie algebra $\mathfrak{g}$. Since this operator has no kernel on $T_\gamma \Omega M$, $1+\Delta$ is replaced by Δ. These changes in the H^s inner product do not alter the spaces of Sobolev sections, but they do change the Levi-Civita connection. In any case, for X, Y, Z left invariant vector fields, the first three terms on the right hand side of (2.3) vanish. Under the standing assumption that G has a left invariant, Ad-invariant inner product, one obtains

$$2\nabla_X^{(s)} Y = [X, Y] + \Delta^{-s}[X, \Delta^s Y] + \Delta^{-s}[Y, \Delta^s X]$$

[12].

It is an interesting question to compute the order of the curvature operator as a function of s. For based loops, Freed proved that this order is at most -1. For the case $s = 1$, we have a much stronger result.

Proposition 4.1. *The curvature of the Levi-Civita connection for the H^1 inner product on ΩG associated to $-\frac{d^2}{d\theta^2} \otimes \mathrm{Id}$ is a* ΨDO *of order $-\infty$.*

PROOF: We give two proofs.

By [12], the H^1 curvature operator $\Omega = \Omega^{(1)}$ satisfies

$$\langle \Omega(X,Y)Z, W\rangle_1 = \left(\int_{S^1} [Y, \dot{Z}], \int_{S^1} [X, \dot{W}]\right)_{\mathfrak{g}} - (X \leftrightarrow Y),$$

where $\dot{Z} = \partial_\theta Z$ as usual, and the inner product is with respect to the Ad-invariant form on the Lie algebra $\mathfrak{g}$. We want to write the right hand side of this equation as an H^1 inner product with W, in order to recognize $\Omega(X,Y)$ as a ΨDO.

Let $\{e_i\}$ be an orthonormal basis of $\mathfrak{g}$, considered as a left-invariant frame of TG and as global sections of γ^*TG. Let $c_{ij}^k = ([e_i, e_j], e_k)_{\mathfrak{g}}$ be the structure constants of $\mathfrak{g}$. (The Levi-Civita connection on left invariant vector fields for the left-invariant metric is given by $\nabla_X Y = \frac{1}{2}[X, Y]$, so the structure constants are twice the Christoffel symbols.) For $X = X^i e_i = X^i(\theta) e_i, Y = Y^j e_j$, etc., integration by parts gives

$$\langle \Omega(X,Y)Z, W\rangle_1 = \left(\int_{S^1} \dot{Y}^i Z^j d\theta\right)\left(\int_{S^1} \dot{X}^\ell W^m d\theta\right) c_{ij}^k c_{\ell m}^n \delta_{kn} - (X \leftrightarrow Y).$$

Since

$$\int_{S^1} c_{\ell m}^n \dot{X}^\ell W^m = \int_{S^1} \left(\delta^{mc} c_{\ell c}^n \dot{X}^\ell e_m, W^b e_b\right)_{\mathfrak{g}} = \left\langle \Delta^{-1}(\delta^{mc} c_{\ell c}^n \dot{X}^\ell e_m), W\right\rangle_1,$$

we get

$$\begin{aligned}\langle\Omega(X,Y)Z,W\rangle_1 &= \left\langle\left[\int_{S^1}\dot{Y}^iZ^j\right]c_{ij}^k\delta_{kn}\delta^{ms}c_{\ell s}^n\Delta^{-1}(\dot{X}^\ell e_m),W\right\rangle_1-(X\leftrightarrow Y)\\ &= \left\langle\left[\int_{S^1}a_j^k(\theta,\theta')Z^j(\theta')d\theta'\right]e_k,W\right\rangle_1,\end{aligned}$$

with

$$a_j^k(\theta,\theta')=\dot{Y}^i(\theta')c_{ij}^r\delta_{rn}\delta^{ms}c_{\ell s}^n\left(\Delta^{-1}(\dot{X}^\ell e_m)\right)^k(\theta)-(X\leftrightarrow Y).$$

We now show that $Z\mapsto\left(\int_{S^1}a_j^k(\theta,\theta')Z^j(\theta')d\theta'\right)e_k$ is a smoothing operator. Applying Fourier transform and Fourier inversion to Z^j yields

$$\begin{aligned}\int_{S^1}a_j^k(\theta,\theta')Z^j(\theta')d\theta' &= \int_{S^1\times\mathbb{R}\times S^1}a_j^k(\theta,\theta')e^{i(\theta'-\theta'')\cdot\xi}Z^j(\theta'')d\theta''d\xi d\theta'\\ &= \int_{S^1\times\mathbb{R}\times S^1}\left[a_j^k(\theta,\theta')e^{-i(\theta-\theta')\cdot\xi}\right]e^{i(\theta-\theta'')\cdot\xi}Z^j(\theta'')d\theta''d\xi d\theta',\end{aligned}$$

so $\Omega(X,Y)Z$ is a ΨDO with symbol

$$b_j^k(\theta,\xi)=\int_{S^1}a_j^k(\theta,\theta')e^{i(\theta-\theta')\cdot\xi}d\theta'$$

with the usual mixing of local and global notation.

For fixed θ, the integral is the Fourier transform of $\dot{Y}^i(\theta')$ (resp. $\dot{X}^i(\theta')$), the only piece of the first (resp. second) term in $a_j^k(\theta,\theta')$ depending on θ'. Since the Fourier transform is taken in a local chart with respect to a partition of unity, and since in each chart $\dot{Y}^i$ and $\dot{X}^i$ times the partition of unity function is compactly supported, the Fourier transform of a_j^k in each chart is rapidly decreasing. Thus $b_j^k(\theta,\xi)$ is the product of a rapidly decreasing function with $e^{i\theta\cdot\xi}$, and hence is of order $-\infty$.

We now give a second proof. For all s

$$\nabla_XY=\frac{1}{2}[X,Y]-\frac{1}{2}\Delta^{-s}[\Delta^sX,Y]+\frac{1}{2}\Delta^{-s}[X,\Delta^sY].$$

Label the terms on the right hand side (1) – (3). As an operator on Y for fixed X, the symbol of (1) is $\sigma((1))_\mu^a=\frac{1}{2}X^ec_{\varepsilon\mu}^a$. Abbreviating $(\xi^2)^{-s}$ by ξ^{-2s}, we have

$$\begin{aligned}\sigma((2))_\mu^a &\sim -\frac{1}{2}c_{\varepsilon\mu}^a\left[\xi^{-2s}\Delta^sX^\varepsilon-\frac{2s}{i}\xi^{-2s-1}\partial_\theta\Delta^sX^\varepsilon\right.\\ &\qquad\left.+\sum_{\ell=2}^\infty\frac{(-2s)(-2s-1)\dots(-2s-\ell+1)}{i^\ell\ell!}\xi^{-2s-\ell}\partial_\theta^\ell\Delta^sX^\varepsilon\right]\\ \sigma((3))_\mu^a &\sim \frac{1}{2}c_{\varepsilon\mu}^a\left[X^\varepsilon+\sum_{\ell=1}^\infty\frac{(-2s)(-2s-1)\dots(-2s-\ell+1)}{i^\ell\ell!}\xi^{-\ell}\partial_\theta^\ell X^\varepsilon\right].\end{aligned}$$

Thus

$$\begin{aligned}\sigma(\nabla_X)^a_\mu \ \sim\ & \frac{1}{2}c^a_{\varepsilon\mu}\Bigg[2X^\varepsilon - \xi^{-2s}\Delta^s X^\varepsilon + \frac{2s}{i}\xi^{-2s-1}\partial_\theta\Delta^s X^\varepsilon \\ & - \sum_{\ell=2}^{\infty}\frac{(-2s)(-2s-1)\dots(-2s-\ell+1)}{i^\ell \ell!}\xi^{-2s-\ell}\partial_\theta^\ell\Delta^s X^\varepsilon \\ & + \sum_{\ell=1}^{\infty}\frac{(-2s)(-2s-1)\dots(-2s-\ell+1)}{i^\ell \ell!}\xi^{-\ell}\partial_\theta^\ell X^\varepsilon.\Bigg].\end{aligned} \tag{4.1}$$

Set $s = 1$ in (4.1), and replace ℓ by $\ell - 2$ in the first infinite sum. Since $\Delta = -\partial_\theta^2$, a little algebra gives

$$\sigma(\nabla_X)^a_\mu \sim c^a_{\varepsilon\mu}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{i^\ell}\partial_\theta^\ell X^\varepsilon\xi^{-\ell} = \mathrm{ad}\left(\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{i^\ell}\partial_\theta^\ell X\xi^{-\ell}\right). \tag{4.2}$$

Denote the infinite sum in the last term of (4.2) by $W(X, \theta, \xi)$. The map $X \mapsto W(X, \theta, \xi)$ takes the Lie algebra of left invariant vector fields on LG to the Lie algebra $L\mathfrak{g}[[\xi^{-1}]]$, the space of formal ΨDOs of nonpositive integer order on the trivial bundle $S^1 \times \mathfrak{g} \longrightarrow S^1$, where the Lie bracket on the target involves multiplication of power series and bracketing in $\mathfrak{g}$. We claim that this map is a Lie algebra homomorphism. Assuming this, we see that

$$\begin{aligned}\sigma\left(\Omega(X,Y)\right) &= \sigma\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) \sim \sigma\left([\mathrm{ad}\, W(X), \mathrm{ad}\, W(Y)] - \mathrm{ad}\, W([X,Y])\right) \\ &= \sigma\left(\mathrm{ad}\left([W(X), W(Y)]\right) - \mathrm{ad}\, W([X,Y])\right) = 0,\end{aligned}$$

which proves that $\Omega(X, Y)$ is a smoothing operator.

To prove the claim, set $X = x^a_n e^{in\theta}e_a, Y = y^b_m e^{im\theta}e_b$. Then

$$\begin{aligned}W([X,Y]) &= W(x^n y^m e^{i(n+m)\theta}c^k_{ab}e_k) = \sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{i^\ell}c^k_{ab}\partial_\theta^\ell\left(x^a_n y^b_m e^{i(n+m)\theta}\right)\xi^{-\ell}e_k \\ [W(X), W(Y)] &= \sum_{\ell=0}^{\infty}\sum_{p+q=\ell}\frac{(-1)^{p+q}}{i^{p+q}}\partial_\theta^p\left(x^a_n e^{in\theta}\right)\partial_\theta^q\left(y^b_m e^{im\theta}\right)\xi^{-(p+q)}c^k_{ab}e_k,\end{aligned}$$

and these two sums are clearly equal. □

It would be interesting to understand how the map W fits into the representation theory of the loop algebra $L\mathfrak{g}$. In [17], it is shown that the order of Ω^s is exactly -2 for all $s \neq 1/2, 1$ on both ΩG and LG.

Part II. Characteristic Classes on LM

In this part of the paper, we construct a general theory of Chern-Simons classes on infinite rank bundles including the frame/tangent bundle of loop spaces, following the construction of primary characteristic classes in [24]. The primary classes vanish on the tangent bundles of loop spaces, which forces the consideration of secondary classes. We discuss the frame dependence of these Wodzicki-Chern-Simons (WCS) classes, and give examples of nontrivial WCS classes on LS^3. The key ingredient is to replace the ordinary matrix trace in the Chern-Weil theory of invariant polynomials on finite dimensional Lie groups with the Wodzicki residue on invertible bounded ΨDOs.

In §5, the general theory is developed for parallelizable manifolds, and we prove a vanishing result for the Pontrjagin classes of LM and more general spaces of maps. In §6, we associate to every circle action on M^n an n-cycle in LM. Using computer calculations, we compute that the three dimensional WCS class in LS^3 integrates to different values on cycles associated to different actions on S^3. Thus these cycles are nonhomologus, and the WCS class distinguishes these actions.

5. Chern-Simons Classes on Loop Spaces

We begin this section with a review of Chern-Weil and Chern-Simons theory in finite dimensions (§5.1), following the seminal paper [7]. In §5.2, we discuss Chern-Simons theory on a class of infinite rank bundles including the frame bundles of loop spaces. Since the geometric structure group of these bundles is $\Psi\mathrm{DO}_0^*$, we need traces on the Lie algebra $\Psi\mathrm{DO}_{\leq 0}$ to define invariant polynomials. There are two types of traces, one given by taking the zeroth order symbol and one given by the Wodzicki residue [18].

In §5.3, we discuss the zeroth order symbol theory. Chern classes are pullbacks of finite dimensional Chern classes, and the same holds for Chern-Simons classes. Thus these primary and secondary classes are not really new. In §5.4, we consider the richer Wodzicki-Chern-Simons theory. On a $(2\ell-1)$-manifold M with trivial tangent bundle, the WCS class $CS^W_{2\ell-1}(LM,\mathbb{R})$ is defined, and is independent of the frame/trivialization of the tangent bundle. As a result, the WCS classes are real, not just $\mathbb{R}/\mathbb{Z}$ classes. We can identify the dependence of the WCS class on the Sobolev parameter s, and this enables us to define regularized WCS classes. In §5.5, we prove that the corresponding Wodzicki-Pontrjagin classes vanish for the tangent bundle to $Maps(N,M)$ for Riemannian manifolds N, M.

5.1. Chern-Weil and Chern-Simons Theory for Finite Dimensional Bundles.

We first review the Chern-Weil construction. Let G be a finite dimensional Lie group with Lie algebra $\mathfrak{g}$, and let $G \longrightarrow E \longrightarrow M$ be a principal G-bundle over a manifold M Set $\mathfrak{g}^l = \mathfrak{g}^{\otimes l}$ and let

$$I^l(G) = \{P : \mathfrak{g}^l \longrightarrow \mathbb{R} \mid P \text{ symmetric, multilinear, Ad-invariant}\}$$

be the Ad-invariant polynomials on $\mathfrak{g}$.

Remark 5.1. For classical Lie groups G, $I^l(G)$ is generated by the polarization of the Newton polynomials $\mathrm{Tr}(A^l)$, where Tr is the usual trace on finite dimensional matrices.

For $\phi \in \Lambda^k(E, \mathfrak{g}^l)$, $P \in I^l(G)$, set $P(\phi) = P \circ \phi \in \Lambda^k(E)$. Two key properties are:

- (The *commutativity property*) For $\phi \in \Lambda^k(E, \mathfrak{g}^l)$,
$$d(P(\phi)) = P(d\phi). \tag{5.1}$$
- (The *infinitesimal invariance property*) For $\psi_i \in \Lambda^{k_i}(E, \mathfrak{g})$, $\phi \in \Lambda^1(E, \mathfrak{g})$ and $P \in I^l(G)$,
$$\sum_{i=1}^{l} (-1)^{k_1+\cdots+k_i} P(\psi_1 \wedge \cdots \wedge [\psi_i, \phi] \wedge \ldots \psi_l) = 0. \tag{5.2}$$

Theorem 5.1 (The Chern-Weil Homomorphism [16])**.** *Let $E \longrightarrow M$ have a connection θ with curvature $\Omega_E \in \Lambda^2(E, \mathfrak{g})$. For $P \in I^l(G)$, $P(\Omega_E)$ is a closed invariant real form on E, and so determines a closed form $P(\Omega_M) \in \Lambda^{2l}(M, \mathbb{R})$. The Chern-Weil map*
$$\oplus_{l=1} I^l(G) \longrightarrow H^*(M, \mathbb{R}), \;\; P \mapsto [P(\Omega_M)]$$
is a well-defined algebra homomorphism.

$[P(\Omega_M)]$ is called the *characteristic class* of P.

We now review Chern-Simons theory. A crucial observation is that $P(\Omega_E)$ is exact, although in general $P(\Omega_M)$ is not.

Proposition 5.2. [7, Prop. 3.2] *Let G be a finite dimensional Lie group. For a G-bundle $E \longrightarrow M$ with connection θ and curvature $\Omega = \Omega_E$, and for $P \in I^l(G)$, set*
$$\phi_t = t\Omega + \frac{1}{2}(t^2 - t)[\theta, \theta], \quad TP(\theta) = l \int_0^1 P(\theta \wedge \phi_t^{l-1})dt.$$
Then $dTP(\theta) = P(\Omega) \in \Lambda^{2l}(E)$.

Proof. We recall the proof for later purposes. Set $f(t) = P(\phi_t^l)$, so $P(\Omega) = \int_0^1 f'(t)dt$. We show $f'(t) = l \cdot dP(\theta \wedge \phi_t^{l-1})$ by computing each side. First, we have
$$\begin{aligned} f'(t) &= \frac{d}{dt}\left(P(\phi_t^l)\right) = P\left(\frac{d}{dt}\phi_t^l\right) = lP\left(\frac{d}{dt}\phi_t \wedge \phi^{l-1}\right) \\ &= lP(\Omega \wedge \phi^{l-1}) + l\left(t - \frac{1}{2}\right) P\left([\theta, \theta] \wedge \phi_t^{l-1}\right), \end{aligned} \tag{5.3}$$
where we have used the commutativity property (5.1). On the other hand, we have
$$\begin{aligned} l \cdot dP(\theta \wedge \phi_t^{l-1}) &= lP(d\theta \wedge \phi_t^{l-1}) - l(l-1)P(\theta \wedge d\phi_t \wedge \phi_t^{l-2}) \quad (5.4) \\ &= lP(\Omega \wedge \phi_t^{l-1}) - \frac{1}{2} lP([\theta, \theta] \wedge \phi_t^{l-1}) - l(l-1)P(\theta \wedge d\phi_t \wedge \phi_t^{l-2}), \end{aligned}$$
by (5.1) and the structure equation $\Omega = d\theta + \frac{1}{2}[\theta, \theta]$. Since $d\phi_t = t[\phi_t, \theta]$, the last term in (5.4) equals
$$l(l-1)P(\theta \wedge d\phi_t \wedge \phi_t^{l-2}) = l(l-1)P(\theta \wedge t[\phi_t, \theta] \wedge \phi_t^{l-2}).$$

Using the invariance property (5.2) with $\phi = \theta$, $\psi_1 = \theta$ and $\psi_k = \psi_t, k = 2, \dots, l-1$, we obtain

$$l(l-1)P(\theta \wedge t[\phi_t, \theta] \wedge \phi_t^{l-2}) = -ltP([\theta, \theta] \wedge \phi_t^{l-1}).$$

This implies (5.4) equals (5.3). □

Setting $E = EG, M = BG$ in Theorem 5.1 gives the universal Chern-Weil homomorphism

$$W : I^l(G) \longrightarrow H^{2l}(BG, \mathbb{R}).$$

We write $P \in I_0^l(G)$ if $W(P) \in H^{2l}(BG, \mathbb{Z})$. For this subalgebra of polynomials, we obtain more information on $TP(\theta)$.

Theorem 5.3. [7, Prop. 3.15]. *Let $E \xrightarrow{\pi} B$ be a G-bundle with connection θ. For $P \in I_0^l(G)$, let $\widetilde{TP(\theta)}$ be the mod $\mathbb{Z}$ reduction of the real cochain $TP(\theta)$. Then there exists a cochain $U \in C^{2l-1}(B, \mathbb{R}/\mathbb{Z})$ such that*

$$\widetilde{TP(\theta)} = \pi^* U + \text{coboundary}.$$

In Theorem 5.10 below, we rework the proof of this theorem in our context.

Corollary 5.4. [7, Thm. 3.16] *Assume $P \in I_0^l(G)$ and $P(\Omega_E) = 0$. Then there exists $CS_P(\theta) \in H^{2l-1}(B, \mathbb{R}/\mathbb{Z})$ such that*

$$\left[\widetilde{TP(\theta)}\right] = \pi^*(CS_P(\theta)).$$

Proof. Choose $U \in C^{2l-1}(B, \mathbb{R}/\mathbb{Z})$ as in Theorem 5.3. Since $P(\Omega_E^l) = 0$, Proposition 5.2 implies $\delta\widetilde{TP(\theta)} = d\widetilde{TP(\theta)} = 0$, where δ is the coboundary map. By Theorem 5.3, $\pi^* U$ and $\widetilde{TP(\theta)}$ are cohomologous. Set $CS_P(\theta) = [U]$. □

Notice that the secondary class or *Chern-Simons class* $CS_P(\theta)$, is defined only when the characteristic form $P(\Omega_E)$ vanishes. This definition of $CS_P(\theta)$ depends on the choice of U, and this dependence is treated systematically in the theory of differential characters [6, viz. Prop. 2.8].

This ambiguity is not an issue for trivial bundles, and the following corollary will be taken as the definition of Chern-Simons classes for trivial ΨDO_0^*-bundles (see Definition 5.2).

Corollary 5.5. *Let $(E, \theta) \xrightarrow{\pi} B$ be a trivial G-bundle with connection, and let χ be a global section. For $P \in I_0^l(G)$,*

$$CS_P(\theta) = \chi^*[\widetilde{TP(\theta)}].$$

Proof. This follows from Corollary 5.4 and $\pi\chi = \text{Id}$. □

If we do not reduce coefficients to $\mathbb{R}/\mathbb{Z}$, this corollary fails; cf. Prop. 5.12.

5.2. **Chern-Simons Theory on Loop Spaces.** In [24], Chern forms are defined on complex vector/principal bundles with structure group $\Psi\mathrm{DO}_0^*$ and with $\Psi\mathrm{DO}_0^*$-connections, where the ΨDOs act on sections of a finite rank hermitian bundle $E \longrightarrow N$ over a closed manifold (e.g. $\gamma^*TM \otimes \mathbb{C} \longrightarrow S^1$ for loop spaces). The key point is to find suitable polynomials $P \in I^l(\Psi\mathrm{DO}_0^*)$. We single out two analogs of the Newton polynomials $\mathrm{Tr}(A^l)$: for $A \in \Psi\mathrm{DO}_0^*$, define

$$P_l^{(0)}(A) = k(l) \int_{S^*N} \mathrm{tr}\ \sigma_0(A^l)(x,\xi)\ d\xi dx. \tag{5.5}$$

Here S^*N is the unit cosphere bundle of N and $k(l) = (2\pi i)^{-l}(\mathrm{Vol}\, S^*N)^{-1}$. Note that $d_l = (2\pi i)^{-l}$ is the normalizing constant such that $d_l[\mathrm{tr}((\Omega^u)^l)] \in H^{2l}(BU(n),\mathbb{Z})$ for a connection θ^u on $EU(n) \longrightarrow BU(n)$[2]. In [23], $P_l^{(0)}$ is called a *Leading Order Symbol Trace.*

The second analog is

$$P_l^W(A) = k(l) \int_{S^*N} \mathrm{tr}\ \sigma_{-n}(A^l)(x,\xi)\ d\xi dx. \tag{5.6}$$

$P_l^W(A)$ is a multiple of the Wodzicki residue of A^l. As usual, $P_l^{(0)}, P_l^W$ determine polynomials by polarization.

For both $P_l^{(0)}, P_l^W$, the commutativity and invariance properties hold because (5.5) and (5.6) are tracial [24]: $\mathrm{tr}[\sigma_0([A,B])] = 0$ for $A, B \in \Psi\mathrm{DO}_{\leq 0}$, and the Wodzicki residue vanishes on commutators. Thus $P_l^{(0)}, P_l^W$ are in both $I^l(\mathcal{G}), I^l(\Psi\mathrm{DO}_0^*)$ (although trivially $P_l^W = 0$ on the gauge group $\mathcal{G}$).

The proof of Proposition 5.2 to carries over to $\Psi\mathrm{DO}_0^*$-bundles with connections. In particular, it applies to the H^s-frame bundle of the loop space with the Levi-Civita connection. Thus, we have

Proposition 5.6. *For a $\Psi\mathrm{DO}_0^*$-bundle with connection $(\mathcal{E},\theta) \xrightarrow{\pi} \mathcal{B}$, and for $P \in I^l(\Psi\mathrm{DO}_0^*)$, set*

$$\begin{aligned} \phi_t &= t\Omega + \frac{1}{2}(t^2 - t)[\theta,\theta], \\ TP(\theta) &= l\int_0^1 P(\theta \wedge \phi_t^{l-1})dt \end{aligned} \tag{5.7}$$

Then $dTP(\theta) = P(\Omega^l)$. We can replace $\Psi\mathrm{DO}_0^*$ by $\mathcal{G}$.

Remark 5.2. By Chern-Weil theory, $P_l^{(0)}(\Omega), P_l^W(\Omega)$ are closed forms with cohomology class independent of the connection θ. The cohomology classes for $P_l^{(0)}, P_l^W$ are the components of the so-called *leading order Chern character* and the *Wodzicki-Chern character.* Using Newton's formulas, the Chern characters define Chern classes

[2]We often omit this normalizing constant in the rest of the paper.

$c_l^{(0)}$, c_l^W as well as Pontrjagin classes for real bundles. Examples of nontrivial leading order Chern classes are given in [24], and examples of nontrivial Wodzicki-Chern classes are given in [26].

5.3. **Leading Order Chern-Weil and Chern-Simons Theory.** In this section, we show that Theorem 5.3 extends to the $\mathbb{R}/\mathbb{Z}$ secondary classes associated to the characteristic forms $P = P_l^{(0)}$ built from the leading order symbol of a connection on a $\mathcal{G}$-bundle. We also show that leading order Chern and Chern-Simons classes are essentially pullbacks of finite dimensional Chern classes, and hence contain limited new information.

For FLM, only the $L^2 = H^0$ Levi-Civita connection is a $\mathcal{G}$-connection. One easily checks that the L^2 connection one-form $\omega_\gamma^{(0)}(X)(\theta) = \omega^M_{\gamma(\theta)}(X_{\gamma(\theta)})$ on LM acts pointwise, as does the curvature two-form. Thus $P_l^{(0)}(\Omega^{(0)})_\gamma = \frac{1}{4\pi}\int_\gamma \operatorname{tr} P_l(\Omega^M)\ d\theta$, so the theory of leading order characteristic forms is a straightforward generalization of the finite dimensional case.

As we now explain, the case of gauge bundles over $Maps(N, M)$ which arise from finite rank bundles over N is similar. This class of bundles includes $TMaps(N, M)$, where the finite rank bundle is $E = \mathrm{ev}^*TM \longrightarrow N$, as shown below. For most of this section, we assume that $\mathcal{G}$ is the gauge group of an oriented real bundle E, but the arguments carry over to e.g. hermitian bundles.

By [2], $B\mathcal{G} = C^\infty_{(0)}(M, BSO(n)) = \{f : M \longrightarrow BSO(n) | f^*ESO(n) \simeq E\}$. For N closed and connected, let $\mathrm{ev} : C^\infty(N, M) \times N \longrightarrow M$ be the evaluation map $\mathrm{ev}(f, n) = f(n)$. The bundle $\mathrm{ev}^* E$ determines an infinite rank bundle $\pi_* \mathrm{ev}^* E \longrightarrow C^\infty(N, M)$, where $\pi_* \mathrm{ev}^* E|_f = \Gamma(f^*E \longrightarrow N)$, with Γ denoting some Sobolev space of sections. (Here $\pi : C^\infty(N, M) \times N \longrightarrow C^\infty(N, M)$ is the projection.) For $n \in N$, define $\mathrm{ev}_n : C^\infty(N, M) \longrightarrow M$ by $\mathrm{ev}_n(f) = f(n)$.

It is well known that connections push down under π_*. For the gauge group case, this gives the following:

Lemma 5.7. *The universal bundle $E\mathcal{G} \longrightarrow B\mathcal{G}$ is isomorphic to $\pi_* \mathrm{ev}^* ESO(n)$. $E\mathcal{G}$ has a universal connection $\theta^{E\mathcal{G}}$ defined on $s \in \Gamma(E\mathcal{G})$ by*

$$(\theta_Z^{E\mathcal{G}} s)(\gamma)(\alpha) = \left((\mathrm{ev}^* \theta^u)_{(Z,0)} u_s\right)(\gamma, \alpha).$$

Here θ^u is the universal connection on $ESO(n) \longrightarrow BSO(n)$, and $u_s : C^\infty(N, M) \times N \longrightarrow \mathrm{ev}^* ESO(n)$ is defined by $u_s(f, n) = s(f)(n)$.

Proof. See [23, §4]. □

Corollary 5.8. *The curvature $\Omega^{E\mathcal{G}}$ of $\theta^{E\mathcal{G}}$ satisfies*

$$\Omega^{E\mathcal{G}}(Z, W)s(f)(n) = \mathrm{ev}^* \Omega^u((Z, 0), (W, 0))u_s(f, n).$$

Proof. This follows from

$$\Omega^{E\mathcal{G}}(Z,W)s(f)(n) = [\nabla^{E\mathcal{G}}_Z\nabla^{E\mathcal{G}}_W - \nabla^{E\mathcal{G}}_W\nabla^{E\mathcal{G}}_Z - \nabla^{E\mathcal{G}}_{[Z,W]}]s(f)(n)$$

and the previous lemma. $\square$

Lemma 5.9. *Let $\mathcal{G}$ be the group of gauge transformations acting on sections of a finite rank bundle $E \longrightarrow N$. Then $P_l^{(0)} \in I_0^l(\mathcal{G})$.*

Proof. For all $n_0 \in N$, the maps ev_{n_0} are homotopic, so the cohomology class

$$\left[P_l^{(0)}(\mathrm{ev}^*_{n_0}\Omega^u)\right] \in H^{2k}(B\mathcal{G}\times\{n_0\},\mathbb{R}) \cong H^{2k}(B\mathcal{G},\mathbb{R})$$

is independent of n_0. Thus

$$\begin{aligned}\left[\frac{d_l}{\mathrm{vol}\ S^*N}\int_{S^*N}\mathrm{tr}\ \sigma_0((\Omega^{E\mathcal{G}})^l)d\xi dx\right] &= \frac{d_l}{\mathrm{vol}\ S^*N}\int_{S^*N}\left[\mathrm{tr}\ \sigma_0\left((\mathrm{ev}^*_{n_0}\Omega^u)^l\right)\right]d\xi dx,\\ &= \left[d_l\,\mathrm{ev}^*_{n_0}\,\mathrm{tr}\ \sigma_0\left((\Omega^u)^l\right)\right], \qquad (5.8)\\ &= \mathrm{ev}^*_{n_0}\left[d_l\,\mathrm{tr}\left((\Omega^u)^l\right)\right],\end{aligned}$$

since Ω^u is a multiplication operator. By the choice of d_l, the last term in (5.8) lies in $\mathrm{ev}^*_{n_0}H^{2l}(BSO(n),\mathbb{Z}) \subset H^{2l}(B\mathcal{G}),\mathbb{Z})$. Thus

$$W(P_l^{(0)}) = [P_l^{(0)}(\Omega^{E\mathcal{G}})] \in H^{2l}(B\mathcal{G},\mathbb{Z}).$$

$\square$

Remark 5.3. Let $(\mathcal{E},\theta) \longrightarrow \mathcal{B}$ be a $\mathcal{G}$-bundle with connection, where $\mathcal{G}$ is the gauge group of the rank n hermitian bundle $E \longrightarrow N$, and let $f : \mathcal{B} \longrightarrow B\mathcal{G}$ be a geometric classifying map. The argument above shows that the l^{th} leading order Chern class equals $f^*\,\mathrm{ev}^*_{n_0}\,c_l(EU(n))$. Thus all leading order Chern classes are pullbacks of finite dimensional Chern classes, although the effect of $\mathrm{ev}^*_{n_0}$ may be difficult to compute. (This argument was developed with S. Paycha.)

As in [7], we have

Theorem 5.10. *Let $(\mathcal{E},\theta) \longrightarrow \mathcal{B}$ be a $\mathcal{G}$-bundle with connection θ and assume $P_l(\Omega) \equiv 0$. Let $\widetilde{TP(\theta)}$ be the mod $\mathbb{Z}$ reduction of $TP(\theta)$. Then there exists a cochain $U \in C^{2l-1}(\mathcal{B},\mathbb{R}/\mathbb{Z})$ such that*

$$\widetilde{TP(\theta)} = \pi^*(U) + \text{ coboundary.}$$

Proof. By Lemma 5.7, $E\mathcal{G} \longrightarrow B\mathcal{G}$ has a universal connection $\hat{\theta}$ (with curvature $\hat{\Omega}$). Thus there exists a geometric classifying map $\phi : \mathcal{B} \longrightarrow B\mathcal{G}$: i.e. $(\mathcal{E},\theta) \simeq (\phi^*E\mathcal{G},\phi^*\hat{\theta})$.

By Lemma 5.9, $P \in I_0^l(\mathcal{G})$, so its mod $\mathbb{Z}$ reduction is zero. From the Bockstein sequence

$$\cdots \longrightarrow H^i(B\mathcal{G},\mathbb{Z}) \longrightarrow H^i(B\mathcal{G},\mathbb{R}) \xrightarrow{\text{mod } \mathbb{Z}} H^i(B\mathcal{G},\mathbb{R}/\mathbb{Z}) \longrightarrow H^{i+1}(B\mathcal{G},\mathbb{Z}) \longrightarrow \cdots$$

we deduce that $P(\hat{\Omega})$ represents an integral class in $B\mathcal{G}$. Thus $\widetilde{P(\hat{\Omega})}$ as a cochain vanishes on all cycles in $B\mathcal{G}$, and hence is an $\mathbb{R}/\mathbb{Z}$ coboundary, i.e. there exists $\bar{u} \in C^{2l-1}(B\mathcal{G},\mathbb{R}/\mathbb{Z})$ such that $\delta\bar{u} = \widetilde{P(\hat{\Omega})}$. We have

$$\delta\pi^*(\bar{u}) = \pi^*(\delta u) = \pi^*(\widetilde{P(\hat{\Omega})}) = d\widetilde{TP(\hat{\theta})} = \delta(\widetilde{TP(\hat{\theta})}).$$

The acyclicity of $E\mathcal{G}$ implies $\widetilde{TP(\hat{\theta})} = \pi^*(\bar{u}) + \text{coboundary}$. Now set $U = \phi^*(\bar{u})$. $\square$

Definition 5.1. *Let $(\mathcal{E},\theta) \longrightarrow \mathcal{B}$ be a $\mathcal{G}$-bundle with connection θ and curvature Ω, and assume $P_l^{(0)}(\Omega) \equiv 0$. In the notation of Theorem 5.10, define the* Chern-Simons class $CS_{2l-1}^{(0)}(\theta) \in H^{2l-1}(B,\mathbb{R}/\mathbb{Z})$ *by*

$$CS_{2l-1}^{(0)}(\theta) = [U].$$

As before, there is a dependence on the choice of $\bar{u}$ for nontrivial bundles. For a trivial bundle, Definition 5.2 below removes this dependence. However, as in the previous Remark, the Chern-Simons class will be a pullback of a finite dimensional Chern-Simons class via the evaulation map.

We can also define leading order Chern-Simons classes for $\Psi\mathrm{DO}_0^*$-bundles with connection. If $\Psi\mathrm{DO}_0^*$ acts on $E \longrightarrow N$, the top order symbol is a homomorphism $\sigma_0 : \Psi\mathrm{DO}_0^* \longrightarrow \mathcal{G}$, where $\mathcal{G}$ is the gauge group of $\pi^*E \longrightarrow S^*N$. A $\Psi\mathrm{DO}_0^*$-bundle $\mathcal{E}$ has an associated $\mathcal{G}$-bundle $\mathcal{E}'$ with transition functions $\sigma_0(A)$, for A a transition function of $\mathcal{E}$. A connection θ with curvature Ω on $\mathcal{E}$ gives rise to a connection $\theta' = \sigma_0(\theta)$ on $\mathcal{E}'$ with curvature $\sigma_0(\Omega)$. Since $P_l^{(0)}(\Omega) = P_l^{(0)}(\sigma_0(\Omega))$, we define $CS_{2l-1}^{(0)}(\theta) = CS_{2l-1}^{(0)}(\theta')$.

The homomorphism σ_0 may lose information from the original $\Psi\mathrm{DO}_0^*$-bundle. This indirect definition is forced on us, because we do not know if $E\Psi\mathrm{DO}_0^* \longrightarrow B\Psi\mathrm{DO}_0^*$ admits a universal connection.

This lack of a Narasimhan-Ramanan theorem prevents us from defining Chern-Simons classes on arbitrary $\Psi\mathrm{DO}_0^*$-bundles using the Wodzicki residue. In the next section, we will define a Wodzicki-Chern-Simons class for FLM when M is parallelizable.

5.4. **Wodzicki-Chern-Simons Classes.** In this section, we extend the classical definition of Chern-Simons classes to P_l^W for trivial $\Psi\mathrm{DO}_0^*$-bundles. In particular, we define a Wodzicki-Chern-Simons class for loop spaces of parallelizable manifolds.

We use Corollary 5.5 to define secondary classes.

Definition 5.2. *Let $(\mathcal{E}, \theta) \longrightarrow \mathcal{B}$ be a trivial $\Psi\mathrm{DO}_0^*$-bundles with connection θ, curvature Ω and global section $\chi : \mathcal{B} \longrightarrow \mathcal{E}$. Let P be an Ad-invariant degree l polynomial on $\Psi\mathrm{DO}_{\leq 0}$. Assume that $P(\Omega) \equiv 0$. The* Chern-Simons class $CS^P_{2l-1}(\theta, \chi) \in H^{2l-1}(\mathcal{B}, \mathbb{R})$ *is*

$$CS^P_{2l-1}(\theta, \chi) = \chi^* \left[TP(\theta)\right].$$

For the case of a trivial frame bundle $FLM \longrightarrow LM$ for a Riemannian manifold (M, g) and $P = P^W_l$ in (5.6), the corresponding Chern-Simons class is denoted

$$CS^W_{2l-1}(\theta^s(g), \chi)$$

and is called the $s^{\rm th}$ Wodzicki-Chern-Simons (WCS) class *of LM with respect to g.*

Remark 5.4. (i) In finite dimensions, the form $TP_{4l-3}(\theta)$ vanishes because the trace of the product of an odd number of skew-symmetric matrices is zero. Thus the usual indexing is $CS_l \in H^{4l-1}(M, \mathbb{R}/\mathbb{Z})$. On LM, the connection and curvature forms are skew-symmetric ΨDOs, but their symbols need not be skew-symmetric. Therefore, we have to allow for the existence of WCS classes in all odd dimensions.

(ii) We have not taken the mod $\mathbb{Z}$ reduction as in finite dimensions, but the *a priori* dependence of the WCS class on χ will be removed in Proposition 5.12.

(iii) As in Remark 2.3, we can always extend the structure group to $GL(\mathcal{H})$ so that a global section χ exists whether or not the original $\Psi\mathrm{DO}_0^*$ bundle is trivial. This yields a general definition of a Wodzicki-Chern-Simons form, but with a possible loss of information.

For the rest of this section, we specialize to the frame bundle FLM.

Theorem 5.11. *Let M be a parallelizable manifold of dimension $2\ell - 1$. Then for any metric g on M, the WCS class $CS^W_{2\ell-1}(\theta^s, \chi) \in H^{2l-1}(LM, \mathbb{R})$ is defined. If M is flat and parallelizable, then WCS classes $CS^W_{2k-1}(\theta^s, \chi)$ are defined for all k.*

Proof: We first check that M parallelizable implies LM is parallelizable, as then Definition 5.2 is applicable. Let $\phi : TM \longrightarrow M \times \mathbb{R}^n$ be a trivialization of TM. For $X_\gamma \in T_\gamma LM = \Gamma(\gamma^* TM)$, define

$$\begin{aligned} \Psi : TLM &\longrightarrow LM \times \Gamma(S^1 \times \mathbb{R}^n \longrightarrow S^1) \\ X_\gamma &\longmapsto (\gamma, \alpha \mapsto \pi_2(\phi(X_\gamma(\alpha)))), \end{aligned}$$

where $\pi_2 : M \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the projection. It is easy to check that α is a smooth trivialization of TLM in the H^s norm.

The Wodzicki-Pontrjagin form $P^W_k(\Omega)$ vanishes for dimension reasons. At each loop γ, the integrand in the definition of $P^W_k(\Omega)$ involves a $2k$-form on M^{2k-1} with values in $Hom(TM, TM)$, and hence vanishes.

By (5.6) with $N = S^1$, every term in $P^W_k(\Omega)$ contains $\sigma_{-1}(\Omega)$. Thm. 3.4 then implies that $P^W_k(\Omega) \equiv 0$ for flat manifolds. Thus WCS classes are defined in all odd degrees. $\square$

To investigate the dependence of the WCS class on the frame whenver it is defined, we recall the Cartan homotopy formula [21, 30]. For $A_0, A_1 \in \Lambda^1(M, \mathfrak{g})$ for a manifold M and a Lie algebra $\mathfrak{g}$, set $A_t = A_0 + t(A_1 - A_0), \Omega_t = dA_t + A_t \wedge A_t$. Define l_t from the algebra F generated by the symbols A_t, Ω_t to $\Lambda^*(M \times [0,1], \mathfrak{g})$ by $l_t A_t = 0$, $l_t \Omega_t = (A_1 - A_0)dt$, with l_t extended as a signed derivation to F. For a polynomial $S(A, \Omega)$, the Cartan homotopy formula is

$$S(A_1, \Omega_1) - S(A_0, \Omega_0) = (dk + kd) S(A_t, \Omega_t), \tag{5.9}$$

where

$$kS(A_t, \Omega_t) = \int_0^1 l_t S(A_t, \Omega_t).$$

This formalism implies $kP(\Omega_t) = TP(A_0)$ on the total space of a bundle $E \longrightarrow M$. In fact, the Cartan homotopy formula is just the standard Cartan formula [28, Ch. 8, Vol. I] applied to polynomials of A, Ω.

A frame $\chi : M \longrightarrow FM$ determines a "loopified" frame $L\chi : LM \longrightarrow FLM$ by $L\chi(\gamma)(\theta) = \chi(\gamma(\theta))$. Denote $L\chi$ just by χ.

Proposition 5.12. *On a parallelizable manifold M, the WCS class $CS^W_{2l-1}(\theta, \chi) \in H^{2l-1}(LM, \mathbb{R})$ is, when defined, independent of the choice of loopified frame $\chi : LM \longrightarrow FLM$.*

Proof. Consider loopified frames $\chi_1, \chi_0 : LM \longrightarrow FLM$. The pullbacks of the connection θ on FLM are related by

$$\chi_1^*(\theta) = g^{-1}\chi_0^*(\theta)g + g^{-1}dg, \ \ g(\gamma) : \chi_0(\gamma) \mapsto \chi_1(\gamma),$$

where g is the loopified gauge transformation taking $\chi_0(m)$ to $\chi_1(m)$. For the family $A_t = tg^{-1}\chi_0^*(\theta)g + g^{-1}dg$ and $S(A, \Omega) = TP(A, \Omega) = \chi^* TP_l^W(\theta)$, (5.9) yields

$$TP(A_1) - TP(g^{-1}dg) = (dk + kd)TP(A_t) = d\alpha + kP(\Omega_t) = d\alpha + TP(A_0),$$

with $\alpha = kTP(A_t)$. Hence,

$$CS^W_{2l-1}(\theta, \chi_1) - CS^W_{2l-1}(\theta, \chi_0) = [TP(A_1)] - [TP(A_0)] = [TP(g^{-1}dg)].$$

Since the gauge transformation g is a multiplication operator on TLM, the Wodzicki residues of the connection $g^{-1}dg$ and its curvature vanish. Thus $TP(g^{-1}dg) = 0$. □

Remark 5.5. (i) For a principal bundle with compact structure group, $[TP(g^{-1}dg)]$ is an integer class, called the instanton number in [11]. This shows that the $\mathbb{R}/\mathbb{Z}$ reduction of $\chi^*TP(\theta)$ is frame independent. A more topological proof, valid for compact structure groups only, is in [7, (6.2)].

(ii) Assume the mod $\mathbb{Z}$ reduction of $CS^W_2(\theta) = CS^W_2(\theta, \chi) \in H^3(LM, \mathbb{R})$ vanishes for a loopified frame on LM for M parallelizable. The Bockstein sequence gives a (non-unique) class in $\alpha \in H^3(LM, \mathbb{Z})$ mapping onto $CS^W_2(\theta)$. α has a representative, a gerbe with connection, whose curvature is $\chi^*TP(\theta)$ [15]. Analogously, for finite dimensional parallelizable manifolds, there is a gerbe associated to a vanishing three dimensional Chern-Simons class. This gerbe functions as a tertiary characteristic class associated to a connection and a framing.

(iii) It is not difficult to extend the constructions in this section to stably parallelizable manifolds such as S^n, i.e. manifolds M with $TM \oplus \varepsilon^k = \varepsilon^r$ for trivial bundles $\varepsilon^k, \varepsilon^r$.

For $M^{2\ell-1}$ parallelizable, by Remark 2.1(ii) we can take a global frame so that Lemma 2.4 applies, which allows us to use the symbol calculations of §3. By the results of §3, the WCS class on $LM^{2\ell-1}$ involves a product of $2\ell-1$ connection and curvature forms and hence is a degree $2\ell-1$ polynomial in s. Therefore, we can take a "large s" limit to remove the choice of the Sobolev parameter. This motivates the following definition:

Definition 5.3. *The* regularized WCS class $CS^{\rm reg}_{2l-1}(\theta)$ *is*

$$\lim_{s\longrightarrow\infty}\frac{1}{s^{2\ell-1}}CS^W_{2l-1}(\theta^s,\chi).$$

It is easy to check that $CS^{\rm reg}_{2l-1}(\theta)$ is closed, and so defines a class in $H^{2\ell-1}(LM^{2\ell-1},\mathbb{R})$. The regularized WCS class is discussed in a subsequent paper.

5.5. **Vanishing Results for Wodzicki-Chern Classes.** The tangent space TLM to a loop space fits into the framework of the Families Index Theorem. In this section, we show that the infinite rank bundles appearing in this framework have vanishing Wodzicki-Chern classes, generalizing [20]. This vanishing indicates that WCS classes could be interesting objects in the more general Families Index Theorem setup.

Recall this setup: there is a fibration $Z \longrightarrow M \stackrel{\pi}{\longrightarrow} B$ of closed manifolds and a finite rank bundle $E \longrightarrow M$, inducing an infinite rank bundle $\mathcal{E} = \pi_*E \longrightarrow B$. For the fibration $N \longrightarrow N \times Maps(N,M) \longrightarrow Maps(N,M)$ and $E = \mathrm{ev}^*TM$, $\mathcal{E}$ is $TMaps(N,M)$.

Theorem 5.13. *If $\mathcal{E} \longrightarrow \mathcal{B}$ satisfies $\mathcal{E} = \pi_*E$ as above, then the Wodzicki-Chern classes $c_k^W(\mathcal{E})$ vanish for all k.*

Proof. As in Lemma 5.7, $\mathcal{E}$ admits a connection whose curvature Ω is a multiplication operator. Ω^l is also a multiplication operator, and hence $c_k^W(\Omega) \equiv 0$. □

For a real infinite rank bundle, Wodzicki-Pontrjagin classes are defined as in finite dimensions: $p_k^W(\mathcal{E}) = (-1)^k c_{2k}^W(\mathcal{E}\otimes\mathbb{C})$.

Corollary 5.14. *The Wodzicki-Pontrjagin classes of $TMaps(N,M)$ and of all naturally associated bundles vanish.*

Proof. Pick an element f_0 in a fixed path component A_0 of $Maps(N,M)$. For $f \in A_0$, $T_fMaps(N,M) \simeq \Gamma(f^*TM \longrightarrow N) \simeq \Gamma(f_0^*TM \longrightarrow N)$ with the second isomorphism noncanonical. Thus over each component, $TMaps(N,M)$ is of the form $\pi_*\,\mathrm{ev}^*TM$, and the previous Theorem applies. The vanishing of the Wodzicki-Pontrjagin classes of associated bundles (such as exterior powers of the tangent bundle) follows as in finite dimensions. □

In finite dimensions, Chern classes are topological obstructions to the reduction of the structure group and geometric obstructions to the extistence of a flat connection. Wodzicki-Chern classes for $\Psi\mathrm{DO}_0^*$-bundles are also topological and geometric obstructions, but the geometric information is a little more refined due to the grading on the Lie algebra $\Psi\mathrm{DO}_{\leq 0}$.

Proposition 5.15. *Let $\mathcal{E} \longrightarrow \mathcal{B}$ be an infinite rank $\Psi\mathrm{DO}_0^*$-bundle, for $\Psi\mathrm{DO}_0^*$ acting on $E \longrightarrow N^n$. If $\mathcal{E}$ admits a reduction to the structure group $\mathcal{G}(E)$, then $c_k^W(\mathcal{E}) = 0$ for all k. If $\mathcal{E}$ admits a $\Psi\mathrm{DO}_0^*$-connection whose curvature has order $-k$, then $c_\ell(\mathcal{E}) = 0$ for $\ell \geq [n/k]$.*

PROOF: If the structure group of $\mathcal{E}$ reduces to the gauge group, there exists a connection one-form with values in $Lie(\mathcal{G}) = Hom(E)$, the Lie algebra of multiplication operators. Thus the Wodzicki residue of powers of the curvature vanishes, so the Wodzicki-Chern classes vanish. For the second statement, we have $ord(\Omega^\ell) < -n$ for $\ell \geq [n/k]$, so the Wodzicki residue vanishes in this range. □

6. An Application of Wodzicki-Chern-Simons Classes to Circle Actions

In this section we use WCS classes to distinguish different S^1 actions on S^3. We also show that WCS classes are not conformal invariants, in contrast to Chern-Simons classes in finite dimensions.

Intuitively, two smooth S^1 actions on a manifold M are homotopic if the orbit starting at $x \in M$ for one action can be homotoped to the orbit starting at x for the second action with this homotopy smooth in x. More precisely, actions $a_1, a_2 : S^1 \times M \longrightarrow M$ are homotopic if there exists a smooth map $F : [0,1] \times S^1 \times M \longrightarrow M$ with $F(0,\cdot,\cdot) = a_1, F(1,\cdot,\cdot) = a_2$. We do not require that $F(t,\cdot,\cdot)$ be an action for other t.

We now introduce a weaker homological notion of equivalent actions. Let M^n have a (possibly trivial) S^1 action $a : S^1 \times M \longrightarrow M$. The action determines a class $h_a \in H_n(LM;\mathbb{Z})$ by setting

$$\tilde{a} : M \longrightarrow LM,\ \tilde{a}(x) = (e^{i\theta} \mapsto e^{i\theta}\cdot x),\ h_a = \tilde{a}_*[M].$$

We will say that two actions a_1, a_2 are *homologically distinct* if $h_{a_1} \neq h_{a_2}$.

We will show that the trivial action on S^3, the action of rotation in a plane, and the Hopf action are all homologically distinct. In particular, these actions are homotopically distinct in the sense above, even though each orbit of each action is contractible. Indeed, such a homotopy would give a homotopy of the cycles representing the two action classes in $H_3(LS^3)$.

To show that these actions are homologically distinct, by the de Rham theorem for LM [4] it suffices to show that $\langle CS_3^W, a_1\rangle \neq \langle CS_3^W, a_2\rangle$ for the standard metric on S^3. In contrast, the S^1 index of an equivariant operator cannot distinguish these actions. If two actions are homotopic, it is easy to check that the S^1 index of the two actions is the same. However, the S^1 index of an odd dimensional manifold vanishes, as can be seen from the local version of the S^1 index theorem [5, Thm. 6.16].[3] In [19], we interpret the S^1 index theorem as the integral of an equivariant chracteristic class over h_a.

Recall that on S^3, we have

$$(x, y, z, w) = (\sin(\rho)\sin(\phi)\cos(\theta), \sin(\rho)\sin(\phi)\sin(\theta), \sin(\rho)\cos(\phi), \cos(\rho))$$

in spherical coordinates $(\theta, \phi, \rho) \in [0, 2\pi] \times [0, \pi] \times [0, \pi]$. We can also think of S^3 as $\{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$, where the two uses of z, w should cause no confusion. We set $a_n, b_n : S^1 \times S^3 \longrightarrow S^3$ by

$$\begin{aligned}
&a_n(e^{i\beta}, (\sin(\rho)\sin(\phi)\cos(\theta), \sin(\rho)\sin(\phi)\sin(\theta), \sin(\rho)\cos(\phi), \cos(\rho))) \\
&\quad = (\sin(\rho)\sin(\phi)\cos(\theta + n\beta), \sin(\rho)\sin(\phi)\sin(\theta + n\beta), \sin(\rho)\cos(\phi), \cos(\rho)) \\
&h_n(e^{i\beta}, (z, w)) = (e^{in\beta}z, e^{in\beta}w).
\end{aligned}$$

Thus a_n is the rotation of S^3 through $2\pi n$ times in the xy-planes, and h_1 is the Hopf action. Note that $a_0 = h_0$ is the trivial action.

For a fixed Sobolev parameter and a fixed metric on S^3, we have

$$\int_{h_{a_n}} CS_3^W(\theta) = n \int_{h_{a_1}} CS_3^W(\theta),$$

for $\theta = \theta^s$, since $\tilde{a}_n$ is the composition of the degree n map $e^{in\beta} : S^1 \longrightarrow S^1$ with a_1. Thus the rotational actions are all homologically distinct provided $\int_{h_{a_1}} CS_3^W(\theta) \neq 0$. The Mathematica notebook `ComputationsRotations.nb` at `http://math.bu.edu/people/sr/articles/notebook.html` computes $\int_{h_{a_1}} CS_3^W(\theta)$ for the standard metric on S^3 for Sobolev parameters $s = 1, 2$. With good precision, we get $\int_{h_{a_1}} CS_3^W(\theta) \neq 0$. This gives the following result:

Theorem 6.1. *For all n, the actions a_n of rotating S^3 n times around the xy-plane are homologically distinct for $s = 1, 2$.*

Strictly speaking, this is a theorem "up to machine accuracy." However, the computed value of $\int_{h_{a_1}} CS_3^W(\theta)$ minus the machine error is far enough from zero to trust the result.

[3]The normal bundle to the fixed point set is always even dimensional, so the fixed point set consists of odd dimensional submanifolds. The integrand in the fixed point submanifold contribution to the S^1-index is the constant term in the short time asymptotics of the appropriate heat kernel. In odd dimensions, this constant term is zero.

The calculation for the Hopf action is similar, although computationally more involved. The notebook `ComputationsHopf.nb` at the website above contains the calculations for $s = 1, 2$. The integrals for $n = 1$ are nonzero and distinct from the $n = 1$ rotation integrals. Therefore, we have

Theorem 6.2. *For $s = 1, 2$, the Hopf actions h_n are pairwise homologically distinct. For $|n|, |k|$ in a finite range, the h_n are homologically distinct from all the xy-plane rotations a_k, except for the trivial action $h_0 = a_0$.*

Because of machine error, we cannot rule out the possibility that $\frac{\int_{a_1} CS^W(\theta)}{\int_{h_1} CS^W(\theta)} \in \mathbb{Q}$. This forces the restrictions on $|n|, |k|$.

In finite dimensions, Chern-Simons classes are conformal invariants. In the notebook `ComputationsConformal.nb`, we compute $\int_{h_{a_1}} CS_3^W(\theta, g) \neq \int_{h_{a_1}} CS_3^W(\theta, g_1)$, where g is the standard metric on S^3 and $g_1 = (1 + \sin^2(\rho))g$. Thus we conclude

Proposition 6.3. *CS_3^W is not a conformal invariant.*

Appendix A. -1 Order Symbols of Connection and Curvature Forms

We prove Proposition 3.3 and Theorem 3.4. We use the conventions in [14] for symbol computations.

The zeroth order (i.e. top) symbol of both the connection and curvature forms are multiplication operators, i.e. the zeroth order symbol does not depend on the cotangent variable ξ, as $\sigma_0(\omega_X^{(s)}) = \omega_X^M, \sigma_0(\Omega(X,Y)^{(s)}) = \Omega(X,Y)^M$. By the standard formulas for the change of symbol under a coordinate transformation, $\omega_X^{(s)} - \omega_X^M$ and $\Omega^{(s)}(X,Y) - \Omega^M(X,Y)$ are well defined -1 order ΨDOs. As a result, the -1 order symbol of ω, Ω can be computed invariantly in any coordinates near a particular $\gamma(\theta)$. Note that this argument is special to ΨDOs with leading order symbol a multiplication operator.

A.1. **Proof of Proposition 3.3.** Label the terms on the right hand side of (2.14) by $a_X, ..., f_X$. By Prop. 2.3, we have to compute σ_{-1} of a_X, b_X, d_X, f_X.

- *The contributions of a_X, b_X*

Denote the kth order symbol of an operator P by $[P]^k$. To simplify notation, set

$$\sigma(\Delta)^j_k = \xi^2\delta^j_k + ih^j_k\xi + r^j_k.$$

The contribution from a_X is

$$
\begin{aligned}
([a_X]^{-1})^a_e &= \frac{1}{2}\Big([(1+\Delta)^{-s}]^{-2s-1}[g^{af}\delta_X g_{ef}(1+\Delta)^s]^{2s} \\
&\quad + [(1+\Delta)^{-s}]^{-2s}[g^{af}\delta_X g_{ef}(1+\Delta)^s]^{2s-1} \\
&\quad -i\frac{\partial}{\partial \xi}([(1+\Delta)^{-s}]^{-2s})\frac{\partial}{\partial\theta}[g^{af}\delta_X g_{ef}(1+\Delta)^s]^{2s}\Big) \\
&= is\xi(\xi^2)^{-1}\left(\partial_\theta(g^{af}\delta_X g_{fe}) + \frac{1}{2}(g^{af}\delta_X g_{mf}h^m_e - h^a_k g^{kf}\delta_X g_{ef})\right).
\end{aligned}
$$

The contribution from b_X is

$$
([b_X]^{-1})^a_e = \frac{1}{2}([(1+\Delta)^{-s}\delta_X(1+\Delta)^s]^{-1})^a_e = \frac{1}{2}is\xi(\xi^2)^{-1}\delta_X h^a_e.
$$

Thus

$$
\begin{aligned}
([a_X+b_X]^{-1})^a_e &= is\xi(\xi^2)^{-1}\partial_\theta(g^{af}\delta_X g_{fe}) \\
&\quad +\frac{1}{2}is\xi(\xi^2)^{-1}\left(\delta_X h^a_e + g^{af}\delta_X g_{mf}h^m_e - h^a_k g^{kf}\delta_X g_{ef}\right),
\end{aligned}
\tag{A.1}
$$

where $h^a_k = -2\Gamma^a_{\nu k}\dot\gamma^\nu$.

- *The contribution of d_X*

We have $d_X Y = \sigma_{-1}((1+\Delta)^{-s}\delta_Y(1+\Delta)^s X)$.

Fix $s \in \mathbb{Z}^+$. Set $A = (1+\Delta)^{-s}, B = \delta_Y(1+\Delta)^s X$. Then A has order $-2s$ and B has order $2s$ as an operator on Y by Lemma 2.2. We have

$$
\begin{aligned}
\sigma_{-1}((1+\Delta)^{-s}\delta_Y(1+\Delta)^s X) &= \sigma_{-2s}(A)\sigma_{2s-1}(B) + \sigma_{-2s-1}(A)\sigma_{2s}(B) \\
&\quad + \frac{1}{i}\partial_\xi\sigma_{-2s}(A)\partial_\theta\sigma_{2s}(B).
\end{aligned}
\tag{A.2}
$$

Label the terms on the right hand side of (A.2) with (1),.., (6). Then

$$
(1) = \xi^{2s},\ (4) = \sigma_{2s}(B)^k_\nu = \Gamma^k_{\nu\delta}X^\delta\xi^{2s}
$$

from Lemma 2.2. Differentiating (4) and (1) gives

$$
(6)^k_\nu = (\dot\gamma^i\partial_i\Gamma^k_{\nu\delta}X^\delta + \Gamma^k_{\nu\delta}\dot X^\delta)\xi^{2s},\ (5) = 2si\xi^{-2s-1}
$$

where (5) includes the factor $\frac{1}{i} = -i$ in (A.2).

We now compute (3). From $0 = \sigma_{-1}((1+\Delta)^{-s}(1+\Delta)^s)$ we get

$$
\begin{aligned}
0 &= \sigma_{-2s}((1+\Delta)^{-s})\sigma_{2s-1}((1+\Delta)^s) + \sigma_{-2s-1}((1+\Delta)^{-s})\sigma_{2s}((1+\Delta)^s) \\
&\quad +\frac{1}{i}\partial_\xi\sigma_{-2s}((1+\Delta)^{-s})\partial_\theta\sigma_{2s}((1+\Delta)^s).
\end{aligned}
$$

The last term vanishes, so

$$
(3) = \sigma_{-2s-1}(A) = -\xi^{-4s}\sigma_{2s-1}((1+\Delta)^s). \tag{A.3}
$$

Recall that $\sigma_1(\Delta)^j_k = ih^j_k\xi = -2i\Gamma^j_{\nu k}\dot\gamma^\nu\xi$. Then

$$\begin{aligned}\sigma_{2s-1}((1+\Delta)^s)^j_k &= \sigma_{2s-1}((1+\Delta)(1+\Delta)^{s-1})\\ &= \sigma_2(1+\Delta)\sigma_{2s-3}((1+\Delta)^{s-1}) + \sigma_1(1+\Delta)\sigma_{2s-2}((1+\Delta)^{s-1})\\ &\qquad\qquad (\text{as } \partial_\theta\sigma_{2s-2}((1+\Delta)^{s-1}) = 0)\\ &= ih^j_k\xi^{2s-1} + \xi^2\sigma_{2s-3}((1+\Delta)^{s-1}).\end{aligned}$$

Continuing, we get

$$\sigma_{2s-1}((1+\Delta)^s)^j_k = ish^j_k\xi^{2s-1}$$

and hence (A.3) is

$$(3)^j_k = -ish^j_k\xi^{-2s-1} = 2is\Gamma^j_{\nu k}\dot\gamma^\nu\xi^{-2s-1}. \tag{A.4}$$

Finally, we compute (2). The idea is that to get $2s-1$ differentiations in Y in $\delta_Y(1+\Delta)^s$, we must have $\ddot Y$ "far to the right" in

$$\delta_Y(1+\Delta)^sX = \sum_{k=1}^s (1+\Delta)^{k-1}\cdot\delta_Y(1+\Delta)\cdot(1+\Delta)^{k-s}X,$$

since the last term is zeroth order in Y.

For example, one term in $\sigma_{2s-1}(\delta_Y(1+\Delta)^s)X$ is

$$\sum_{k=1}^s \sigma_{2k-2}((1+\Delta)^{k-1}\sigma_1(\delta_Y\Delta)\sigma_{2s-2k}((1+\Delta)^{s-k}X).$$

The last term is nonzero only if $2s-2k=0$, i.e. $s=k$. So as an operator on $Y=\cdot$, this term only contributes

$$\begin{aligned}&\xi^{2s-2}\sigma_1(\delta.\Delta)^k_\nu X\\ &\quad= i\xi^{2s-1}(-2\Gamma^k_{\nu\delta}\dot X^\delta - \partial_\nu\Gamma^k_{i\delta}\dot\gamma^iX^\delta - \dot\gamma^i\partial_i\Gamma^k_{\nu\delta}X^\delta - \Gamma^k_{\epsilon\mu}\Gamma^\mu_{\nu\delta}\dot\gamma^\epsilon X^\delta - \Gamma^k_{\nu\mu}\Gamma^\mu_{\epsilon\delta}\dot\gamma^\epsilon X^\delta)\end{aligned}$$

using (2.16) and looking for the terms that are first order in Z.

As another example, a potential term in $\sigma_{2s-1}(\delta_Y(1+\Delta)^s)X$ is

$$\sum_{k=1}^s \sigma_{2k-2}((1+\Delta)^{k-1}\sigma_2(\delta_Y\Delta)\sigma_{2s-2k-1}((1+\Delta)^{s-k}X).$$

Again, the last term must have $2s-2k-1=0$, which is impossible. So this term does not appear.

In the end, we get

$$
\begin{aligned}
(2)^k_\nu &= \Big[\sigma_{2s-2}((1+\Delta)^{s-1})\sigma_1((\delta_Y\Delta)X) + \frac{1}{i}\partial_\xi\sigma_{2s-2}((1+\Delta)^{s-1})\partial_\theta\sigma_2((\delta_Y\Delta)X) \\
&\qquad +\sigma_{2s-3}((1+\Delta)^{s-1})\sigma_2((\delta_Y\Delta)X)\Big]^k_\nu \\
&= i\xi^{2s-1}\Big[(-2\Gamma^k_{\nu\delta}\dot{X}^\delta - \partial_\nu\Gamma^k_{i\delta}\dot{\gamma}^i X^\delta - \dot{\gamma}^i\partial_i\Gamma^k_{\nu\delta}X^\delta - \Gamma^k_{\epsilon\mu}\Gamma^\mu_{\nu\delta}\dot{\gamma}^\epsilon X^\delta - \Gamma^k_{\nu\mu}\Gamma^\mu_{\epsilon\delta}\dot{\gamma}^\epsilon X^\delta) \\
&\qquad -(2s-2)(\Gamma^k_{\nu\delta}\dot{X}^\delta + \dot{\gamma}^i\partial_i\Gamma^k_{\nu\delta}X^\delta) \\
&\qquad -(2s-2)\Gamma^k_{\epsilon\mu}\Gamma^\mu_{\nu\delta}\dot{\gamma}^\epsilon X^\delta)\Big] \\
&= i\xi^{2s-1}\Big[-2s\Gamma^k_{\nu\delta}\dot{X}^\delta - (2s-1)\dot{\gamma}^i\partial_i\Gamma^k_{\nu\delta}X^\delta - \Gamma^k_{\nu\mu}\Gamma^\mu_{\epsilon\delta}\dot{\gamma}^\epsilon X^\delta \\
&\qquad +(-2s+1)\Gamma^k_{\epsilon\mu}\Gamma^\mu_{\nu\delta}\dot{\gamma}^\epsilon X^\delta) - \partial_\nu\Gamma^k_{i\delta}\dot{\gamma}^i X^\delta\Big].
\end{aligned}
$$

We now have computed (1) – (6), so as an operator on $Y = \cdot$, d_X is

$$
\begin{aligned}
\sigma_{-1}((1+\Delta)^{-s}\delta.(1+\Delta)^s X)^k_\nu &= i\xi^{-1}\Big[-2s\Gamma^k_{\nu\delta}\dot{X}^\delta - (2s-1)\dot{\gamma}^i\partial_i\Gamma^k_{\nu\delta}X^\delta - \Gamma^k_{\nu\mu}\Gamma^\mu_{\epsilon\delta}\dot{\gamma}^\epsilon X^\delta \\
&\qquad +(-2s+1)\Gamma^k_{\epsilon\mu}\Gamma^\mu_{\nu\delta}\dot{\gamma}^\epsilon X^\delta) - \partial_\nu\Gamma^k_{i\delta}\dot{\gamma}^i X^\delta \\
&\qquad -s(-2\Gamma^k_{\epsilon\mu}\dot{\gamma}^\epsilon)(\Gamma^\mu_{\nu\delta}X^\delta) \\
&\qquad +2s(\dot{\gamma}^i\partial_i\Gamma^k_{\nu\delta}X^\delta + \Gamma^k_{\nu\delta}\dot{X}^\delta)\Big] \\
&= i\xi^{-1}\left[\partial_\epsilon\Gamma^k_{\nu\delta}\dot{\gamma}^\epsilon X^\delta + \Gamma^k_{\epsilon\mu}\Gamma^\mu_{\nu\delta}\dot{\gamma}^\epsilon X^\delta\right. \qquad\qquad \text{(A.5)} \\
&\qquad \left.-\Gamma^k_{\nu\mu}\Gamma^\mu_{\epsilon\delta}\dot{\gamma}^\epsilon X^\delta - \partial_\nu\Gamma^k_{i\delta}\dot{\gamma}^i X^\delta\right].
\end{aligned}
$$

- *The contribution of* f_X

We have $f_X Y = \sigma_{-1}((1+\Delta)^{-s}A_X Y)$
As an operator on Y, we have

$$
\begin{aligned}
\sigma_{-1}((1+\Delta)^{-s}A_X) &= \sigma_{-2s}((1+\Delta)^{-s})\sigma_{2s-1}(A_X) + \sigma_{-2s-1}((1+\Delta)^{-s})\sigma_{2s}(A_X) \\
&\qquad +\frac{1}{i}\partial_\xi\sigma_{-2s}((1+\Delta)^{-s})\partial_\theta\sigma_{2s}(A_X). \qquad\qquad \text{(A.6)}
\end{aligned}
$$

Call these terms (1) – (6) as before. Terms (1), (5) are immediate, (3) is computed in (A.4), and by Lemma 2.2

$$
\begin{aligned}
(4)^a_b &= g^{ra} g_{sb} \Gamma^s_{ri} X^i \xi^{2s} \\
(6)^a_b &= \partial_\theta (g^{ra} g_{sb} \Gamma^s_{ri} X^i \xi^{2s}) \\
&= \xi^{2s} \left[\dot\gamma^j \partial_j g^{ra} g_{sb} \Gamma^s_{ri} X^i + g^{ra} \dot\gamma^j \partial_j g_{sb} \Gamma^s_{ri} X^i \right. \\
&\qquad \left. + g^{ra} g_{sb} \dot\gamma^j \partial_j \Gamma^s_{ri} X^i + g^{ra} g_{sb} \Gamma^s_{ri} \dot X^i \right] \\
&= \xi^{2s} \left[\dot\gamma^j (-g^{at} \partial_j g_{tu} g^{ur}) g_{sb} \Gamma^s_{ri} X^i + g^{ra} \dot\gamma^j \partial_j g_{sb} \Gamma^s_{ri} X^i \right. \\
&\qquad \left. + g^{ra} g_{sb} \dot\gamma^j \partial_j \Gamma^s_{ri} X^i + g^{ra} g_{sb} \Gamma^s_{ri} \dot X^i \right] \\
&= \xi^{2s} \left[-\dot\gamma^j g^{at} [\Gamma^w_{tj} g_{wu} + \Gamma^w_{uj} g_{wt}] g^{ur} g_{sb} \Gamma^s_{ri} X^i \right. \\
&\qquad + g^{ra} \dot\gamma^j [\Gamma^n_{sj} g_{nb} + \Gamma^n_{bj} g_{ns}] \Gamma^s_{ri} X^i \\
&\qquad \left. + g^{ra} g_{sb} \dot\gamma^j \partial_j \Gamma^s_{ri} X^i + g^{ra} g_{sb} \Gamma^s_{ri} \dot X^i \right] \\
&= \xi^{2s} \left[-\dot\gamma^j g^{at} \Gamma^r_{tj} g_{sb} \Gamma^s_{ri} X^i - \dot\gamma^j g^{at} \Gamma^w_{uj} g_{wt} g^{ur} g_{sb} \Gamma^s_{ri} X^i \right. \\
&\qquad + g^{ra} \dot\gamma^j \Gamma^n_{sj} g_{nb} \Gamma^s_{ri} X^i + g^{ra} \dot\gamma^j \Gamma^n_{bj} g_{ns} \Gamma^s_{ri} X^i \\
&\qquad \left. + g^{ra} g_{sb} \dot\gamma^j \partial_j \Gamma^s_{ri} X^i + g^{ra} g_{sb} \Gamma^s_{ri} \dot X^i \right]
\end{aligned} \tag{A.7}
$$

Thus we have to compute $(2) = \sigma_{2s-1}(A_X)$. Since $\langle A_X Y, Z\rangle_0 = \langle \delta_Z (1+\Delta)^s X, Y\rangle_0$, we need all terms in $\delta_Z(1+\Delta)^s X$ with $2s$ or $2s-1$ derivatives in Z, and then we will move $2s-1$ of these derivatives to Y.

(i) The term in $\delta_Z(1+\Delta)^s X$ with $2s$ derivatives in Z

As in Lemma 2.2, in $\langle \delta_Z(1+\Delta)^s X, Y\rangle_0$ this term is

$$
\begin{aligned}
&\int_{S^1} g_{ab} (-\partial_\theta^2)^{s-1} (-\Gamma^a_{\nu\delta} \ddot Z^\nu X^\delta) Y^b \\
&= -\int_{S^1} g_{ab} \Gamma^a_{\nu\delta} \ddot Z^\nu X^\delta (-\partial_\theta^2)^{s-1} Y^b \\
&\sim (-1)^s \int_{S^1} \dot g_{ab} \Gamma^a_{\nu\delta} Z^\nu X^\delta Y^{b,(2s-1)} + (-1)^s \int_{S^1} g_{ab} \dot\Gamma^a_{\nu\delta} Z^\nu X^\delta Y^{b,(2s-1)} \\
&\qquad + (-1)^s \int_{S^1} g_{ab} \Gamma^a_{\nu\delta} Z^\nu \dot X^\delta Y^{b,(2s-1)} \\
&= (-1)^s \int_{S^1} g^{\ell r} g_{r\nu} \dot g_{ab} \Gamma^a_{\ell\delta} Z^\nu X^\delta Y^{b,(2s-1)} + \ldots \\
&= (-1)^s \langle g^{\ell r} \dot g_{ab} \Gamma^a_{\ell\delta} X^\delta Y^{b,(2s-1)} \partial_r, Z\rangle_0 + (-1)^s \langle g^{\ell r} g_{ab} \dot\Gamma^a_{\ell\delta} X^\delta Y^{b,(2s-1)} \partial_r, Z\rangle_0 \\
&\qquad + (-1)^s \langle g^{\ell r} g_{ab} \Gamma^a_{\ell\delta} \dot X^\delta Y^{b,(2s-1)} \partial_r, Z\rangle_0,
\end{aligned}
$$

where $\sim$ indicates the terms of order $2s-1$. Thus the contribution of this term to $(2)^r_b$ is

$$\begin{aligned} & -i(g^{\ell r}\dot{g}_{ab}\Gamma^a_{\ell\delta}X^\delta + g^{\ell r}g_{ab}\dot{\Gamma}^a_{\ell\delta}X^\delta + g^{\ell r}g_{ab}\Gamma^a_{\ell\delta}\dot{X}^\delta)\xi^{2s-1} \\ = & -i(g^{\ell r}\dot{\gamma}^\nu[\Gamma^n_{a\nu}g_{nb} + \Gamma^n_{b\nu}g_{na}]\Gamma^a_{\ell\delta}X^\delta + g^{\ell r}g_{ab}\dot{\Gamma}^a_{\ell\delta}X^\delta + g^{\ell r}g_{ab}\Gamma^a_{\ell\delta}\dot{X}^\delta)\xi^{2s-1}, \end{aligned} \tag{A.8}$$

using $\dot{g}_{ab} = \dot{\gamma}^\nu\partial_\nu g_{ab} = \dot{\gamma}^\nu[\Gamma^n_{a\nu}g_{nb} + \Gamma^n_{b\nu}g_{na}]$.

(ii) The term in $\delta_Z(1+\Delta)^sX$ with $2s-1$ derivatives in Z

Since

$$\delta_Z(1+\Delta)^sX = \sum_{k=1}^{s}(1+\Delta)^{k-1}\cdot\delta_Z(1+\Delta)\cdot(1+\Delta)^{s-k}X,$$

and $\delta_Z(1+\Delta)$ is order two in Z, we can only get $2s-1$ derivatives in Z if $2k-2+2 \geq 2s-1$, i.e. if $k=s$. In this term $(1+\Delta)^{s-1}\delta_Z(1+\Delta)X$, to get $2s-1$ derivatives in Z there are three cases:

(a) The term in $(1+\Delta)^{s-1}$ with ∂_θ^{2s-3} and the term in $\delta_Z(1+\Delta)X$ with $\ddot{Z}$;
(b) The term in $(1+\Delta)^{s-1}$ with ∂_θ^{2s-2} and the term in $\delta_Z(1+\Delta)X$ with $\dot{Z}$;
(c) The term in $(1+\Delta)^{s-1}$ with ∂_θ^{2s-2} and the term in $\delta_Z(1+\Delta)X$ with $\ddot{Z}$, but only $2s-3$ of these derivatives act on $\ddot{Z}$.

Case (a) contributes

$$(a) = (-1)^s(s-1)(2\Gamma^a_{\nu\mu}\dot{\gamma}^\nu\Gamma^\mu_{\epsilon\delta}X^\delta)Z^{\epsilon,(2s-1)}\partial_a.$$

This follows from (A.4) and (2.16).

Case (b) contributes $(-\partial_\theta^2)^{s-1}$ times the terms in (2.16) with $\dot{Z}$, i.e.

$$(b) = (-1)^{s-1}(-1)(2\Gamma^a_{\epsilon\delta}\dot{X}^\delta + \partial_\epsilon\Gamma^a_{\nu\delta}\dot{\gamma}^\nu X^\delta + \dot{\gamma}^\nu\partial_\nu\Gamma^a_{\epsilon\delta}X^\delta + \Gamma^a_{\nu\mu}\Gamma^\mu_{\epsilon\delta}\dot{\gamma}^\nu X^\delta + \Gamma^a_{\epsilon\mu}\Gamma^\mu_{\nu\delta}\dot{\gamma}^\nu X^\delta)Z^{\epsilon,(2s-1)}\partial_a.$$

Case (c) contributes $(-\partial_\theta^2)^{s-1}$ applied to $-\Gamma^a_{\nu\delta}\ddot{Z}^\nu X^\delta$, but with one of the ∂_θ not applied to Z, so

$$(c) = (-1)^s(2s-2)[\dot{\Gamma}^a_{\nu\delta}X^\delta + \Gamma^a_{\nu\delta}\dot{X}^\delta]Z^{\nu,(2s-1)}\partial_a.$$

The final contribution of (a) to $\langle\delta_Z(1+\Delta)^sX, Y\rangle_0$ is

$$\begin{aligned} & (2s-2)(-1)^s\int_{S^1} g_{ab}\Gamma^a_{\nu\mu}\dot{\gamma}^\nu\Gamma^\mu_{\epsilon\delta}Z^{\epsilon,(2s-1)}X^\delta Y^b \\ \sim\ & (2s-2)(-1)^s(-1)^{2s-1}\int_{S^1} g_{ab}\Gamma^a_{\nu\mu}\dot{\gamma}^\nu\Gamma^\mu_{\epsilon\delta}Z^\epsilon X^\delta Y^{b,(2s-1)} \\ =\ & (2s-2)(-1)^{s-1}\int_{S^1} g_{\epsilon\ell}g^{\ell r}g_{ab}\Gamma^a_{\nu\mu}\dot{\gamma}^\nu\Gamma^\mu_{r\delta}Z^\epsilon X^\delta Y^{b,(2s-1)} \\ =\ & \langle(2s-2)(-1)^{s-1}g^{\ell r}g_{ab}\Gamma^a_{\nu\mu}\dot{\gamma}^\nu\Gamma^\mu_{r\delta}X^\delta Y^{b,(2s-1)}\partial_\ell, Z\rangle_0. \end{aligned}$$

Thus (a) contributes

$$(a)^r_b = (2s-2)(-1)^{s-1}i^{2s-1}g^{r\ell}g_{ab}\Gamma^a_{\nu\mu}\dot\gamma^\nu\Gamma^\mu_{\ell\delta}X^\delta\xi^{2s-1} = (2s-2)ig^{r\ell}g_{ab}\Gamma^a_{\nu\mu}\dot\gamma^\nu\Gamma^\mu_{\ell\delta}X^\delta\xi^{2s-1} \tag{A.9}$$

to $\sigma_{2s-1}(A_X)$.

Similarly, (b) and (c) contribute

$$\begin{aligned}(b)^r_b &= ig^{r\ell}g_{ab}(2\Gamma^a_{\ell\delta}\dot X^\delta + \partial_\ell\Gamma^a_{\nu\delta}\dot\gamma^\nu X^\delta + \dot\gamma^\nu\partial_\nu\Gamma^a_{\ell\delta}X^\delta \\ &\quad +\Gamma^a_{\nu\mu}\Gamma^\mu_{\ell\delta}\dot\gamma^\nu X^\delta + \Gamma^a_{\ell\mu}\Gamma^\mu_{\nu\delta}\dot\gamma^\nu X^\delta) \\ (c)^r_b &= (2s-2)ig^{r\ell}g_{ab}[\dot\Gamma^a_{\ell\delta}X^\delta + \Gamma^a_{\ell\delta}\dot X^\delta]\xi^{2s-1}.\end{aligned} \tag{A.10}$$

This finishes the contribution of (ii).

Combining (A.8) – (A.10) gives

$$\begin{aligned}\frac{(2)^r_b}{i\xi^{2s-1}} &= -g^{\ell r}\dot\gamma^\nu[\Gamma^n_{a\nu}g_{nb} + \Gamma^n_{b\nu}g_{na}]\Gamma^a_{\ell\delta}X^\delta - g^{\ell r}g_{ab}\dot\Gamma^a_{\ell\delta}X^\delta - g^{\ell r}g_{ab}\Gamma^a_{\ell\delta}\dot X^\delta \\ &\quad +(2s-2)g^{r\ell}g_{ab}\Gamma^a_{\nu\mu}\dot\gamma^\nu\Gamma^\mu_{\ell\delta}X^\delta \\ &\quad +g^{r\ell}g_{ab}(2\Gamma^a_{\ell\delta}\dot X^\delta + \partial_\ell\Gamma^a_{\nu\delta}\dot\gamma^\nu X^\delta + \dot\gamma^\nu\partial_\nu\Gamma^a_{\ell\delta}X^\delta \\ &\quad +\Gamma^a_{\nu\mu}\Gamma^\mu_{\ell\delta}\dot\gamma^\nu X^\delta + \Gamma^a_{\ell\mu}\Gamma^\mu_{\nu\delta}\dot\gamma^\nu X^\delta) \\ &\quad +(2s-2)g^{r\ell}g_{ab}[\dot\Gamma^a_{\ell\delta}X^\delta + \Gamma^a_{\ell\delta}\dot X^\delta] \\ &= -g^{\ell r}g_{an}\dot\gamma^\nu\Gamma^n_{b\nu}\Gamma^a_{\ell\delta}X^\delta + (2s-3)g^{\ell r}g_{ab}\dot\Gamma^a_{\ell\delta}X^\delta \\ &\quad +(2s-1)g^{\ell r}g_{ab}\Gamma^a_{\ell\delta}\dot X^\delta + (2s-2)g^{\ell r}g_{ab}\Gamma^a_{\nu\mu}\Gamma^\mu_{\ell\delta}\dot\gamma^\nu X^\delta \\ &\quad +g^{\ell r}g_{ab}\partial_\ell\Gamma^a_{\nu\delta}\dot\gamma^\nu X^\delta + g^{\ell r}g_{ab}\Gamma^a_{\ell\mu}\Gamma^\mu_{\nu\delta}\dot\gamma^\nu X^\delta.\end{aligned} \tag{A.11}$$

By (A.6), (A.7), (A.11), we get

$$\begin{aligned}&\frac{\sigma_{-1}((1+\Delta)^{-s}A_X)^r_b}{i\xi^{-1}} \\ &= -g^{\ell r}g_{an}\dot\gamma^\nu\Gamma^n_{b\nu}\Gamma^a_{\ell\delta}X^\delta + (2s-3)g^{\ell r}g_{ab}\dot\Gamma^a_{\ell\delta}X^\delta \\ &\quad +(2s-1)g^{\ell r}g_{ab}\Gamma^a_{\ell\delta}\dot X^\delta + (2s-2)g^{\ell r}g_{ab}\Gamma^a_{\nu\mu}\Gamma^\mu_{\ell\delta}\dot\gamma^\nu X^\delta \\ &\quad +g^{\ell r}g_{ab}\partial_\ell\Gamma^a_{\nu\delta}\dot\gamma^\nu X^\delta + g^{\ell r}g_{ab}\Gamma^a_{\ell\mu}\Gamma^\mu_{\nu\delta}\dot\gamma^\nu X^\delta \\ &\quad +(2s\Gamma^r_{\nu\mu}\dot\gamma^\nu)(g^{n\mu}g_{sb}\Gamma^s_{ni}X^i) \\ &\quad -(-2s)\left[-\dot\gamma^j g^{r\ell}\Gamma^\mu_{\ell j}g_{ab}\Gamma^a_{\mu\delta}X^\delta - \dot\gamma^j g^{r\ell}\Gamma^w_{\mu j}g_{w\ell}g^{\mu\nu}g_{ab}\Gamma^a_{\nu\delta}X^\delta\right. \\ &\quad + g^{r\ell}\dot\gamma^j\Gamma^a_{\mu j}g_{ab}\Gamma^\mu_{\ell\delta}X^\delta + g^{r\ell}\dot\gamma^j\Gamma^a_{bj}g_{as}\Gamma^s_{\ell\delta}X^\delta \\ &\quad \left.+ g^{r\ell}g_{ab}\dot\gamma^j\partial_j\Gamma^a_{\ell\delta}X^\delta + g^{r\ell}g_{ab}\Gamma^a_{\ell\delta}\dot X^\delta\right].\end{aligned} \tag{A.12}$$

Label the terms on the right hand side of (A.12) as (1),...,(13). We can combine (1) + (11), (2) + (12), (3) + (13), (4) + (10). We get f_X equal to

$$\begin{aligned}
&\frac{\sigma_{-1}((1+\Delta)^{-s}A_X)^r_b}{i\xi^{-1}} \\
&= (-2s-1)g^{\ell r}g_{an}\dot\gamma^\nu\Gamma^n_{b\nu}\Gamma^a_{\ell\delta}X^\delta + (4s-3)g^{\ell r}g_{ab}\dot\Gamma^a_{\ell\delta}X^\delta \\
&\quad +(4s-1)g^{\ell r}g_{ab}\Gamma^a_{\ell\delta}\dot X^\delta + (4s-2)g^{\ell r}g_{ab}\Gamma^a_{\nu\mu}\Gamma^\mu_{\ell\delta}\dot\gamma^\nu X^\delta \\
&\quad +g^{\ell r}g_{ab}\partial_\ell\Gamma^a_{\nu\delta}\dot\gamma^\nu X^\delta \\
&\quad +g^{\ell r}g_{ab}\Gamma^a_{\ell\mu}\Gamma^\mu_{\nu\delta}\dot\gamma^\nu X^\delta + 2s\Gamma^r_{\nu\mu}\dot\gamma^\nu g^{n\mu}g_{ab}\Gamma^a_{n\delta}X^\delta \\
&\quad -2s\dot\gamma^j g^{r\ell}\Gamma^\mu_{\ell j}g_{ab}\Gamma^a_{\mu\delta}X^\delta - 2s\dot\gamma^j g^{r\ell}\Gamma^w_{\mu j}g_{w\ell}g^{\mu\nu}g_{ab}\Gamma^a_{\nu\delta}X^\delta.
\end{aligned} \tag{A.13}$$

Combining (A.1), (A.5), (A.13) gives Prop. 3.3.

A.2. **Proof of Theorem 3.4.** As shown in §2.5, we can compute the curvature Ω using the usual formula $\Omega = d_{LM}\omega + \omega\wedge\omega$ is a neighborhood of γ. Then

$$\begin{aligned}
\sigma_{-1}(\Omega(X,Y)) &= d\sigma_{-1}(\omega)(X,Y) + \sigma_{-1}(\omega_X)\wedge\sigma_0(\omega_Y) - (X\leftrightarrow Y) \\
&\quad +\frac{1}{i}\partial_\xi\sigma_0(\omega_X)\wedge\partial_\theta\sigma_0(\omega_Y) - (X\leftrightarrow Y),
\end{aligned} \tag{A.14}$$

where $(X\leftrightarrow Y)$ means the previous term with X, Y exchanged. The last line vanishes, since $\sigma_0(\omega)$ is independent of ξ. We can compute the curvature tensor in any coordinates on M, so in particular we can use normal coordinates centered at a fixed point in M. The last two terms in the first line of (A.14) vanish, since $\sigma_0(\omega)$ vanishes in normal coordinates by Lemma 3.1. Thus in normal coordinates

$$\begin{aligned}
\sigma_{-1}(\Omega(X,Y)) &= d\sigma_{-1}(\omega)(X,Y)) = \sigma_{-1}(d\omega(X,Y)) \\
&= X(\sigma_{-1}(\omega_Y)) - (X\leftrightarrow Y) - \sigma_{-1}(\omega_{[X.Y]}).
\end{aligned}$$

The last term cancels the nontensorial terms containing $\delta_XY, \delta_YX, \delta_X\dot Y, \delta_Y\dot X$ from the first two terms, so we might as well assume $\delta_XY = \delta_YX = 0$. In other words, under this assumption,

$$\sigma_{-1}(\Omega(X,Y)) = X(\sigma_{-1}(\omega_Y)) - (X\leftrightarrow Y).$$

Remark A.1. We can also assume that $\delta_X\dot Y = \delta_Y\dot X = 0$ because these terms cannot appear in the curvature tensor. More precisely, every term α in (3.2) with $\dot X$ has $Y(\alpha) = ... + A\delta_Y\dot X$, where A vanishes in normal coordinates.

As in the last subsection, we label the six terms in ω_X in (2.14) $a_X, ..., f_X$ and calculate the contributions from the terms a_X, b_X, d_X, f_X.

- *The contributions from* a_X, b_X

Call the right hand side of (A.1) $\sigma_{-1}(\omega_X, a, b)$. Then

$$\begin{aligned}
&\frac{\sigma_{-1}(\omega_X, a, b)^a_e}{i\xi^{-1}} \\
&= s\partial_\theta(g^{af}\delta_X g_{fe}) + \frac{s}{2}(\delta_X h^a_e + g^{af}\delta_X g_{mf}h^m_e - h^a_k g^{kf}\delta_X g_{ef}) \\
&= s\dot\gamma^\nu\partial_\nu g^{af}X^i\partial_i g_{fe} + sg^{af}\dot X^i\partial_i g_{fe} + sg^{af}X^i\partial_{\nu i}g_{fe}\dot\gamma^\nu \qquad \text{(A.15)} \\
&\qquad -sX^i\partial_i\Gamma^a_{\nu e}\dot\gamma^\nu - s\Gamma^a_{\nu e}\dot X^\nu - sg^{af}X^i\partial_i g_{mf}\Gamma^m_{\nu e}\dot\gamma^\nu + s\Gamma^a_{k\nu}\dot\gamma^\nu g^{kf}X^i\partial_i g_{ef}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{i\xi}\left(X(\sigma_{-1}(\omega_Y, a, b))^a_e - Y(\sigma_{-1}(\omega_X, a, b))^a_e\right) \\
&= s\dot X^\nu\partial_\nu g^{af}Y^i\partial_i g_{fe} + s\dot\gamma^\nu X^j\partial_{j\nu}g^{af}Y^i\partial_i g_{fe} + s\dot\gamma^\nu\partial_\nu g^{af}Y^iX^j\partial_{ji}g_{fe} \\
&\qquad +sX^j\partial_j g^{af}\dot Y^i\partial_i g_{fe} + sg^{af}\dot Y^i\partial_{ji}g_{fe}X^j + sX^j\partial_j g^{af}Y^i\partial_{\nu i}g_{ef}\dot\gamma^\nu \\
&\qquad +sg^{af}Y^iX^j\partial_{j\nu i}g_{fe}\dot\gamma^\nu + sg^{af}Y^i\partial_{\nu i}g_{fe}\dot X^\nu - sY^iX^j\partial_{ji}\Gamma^a_{\nu e}\dot\gamma^\nu \\
&\qquad -sY^i\partial_i\Gamma^a_{\nu e}\dot X^\nu - sX^i\dot Y^\nu\partial_i\Gamma^a_{\nu e} - sX^j\partial_j g^{af}Y^i\partial_i g_{mf}\Gamma^m_{\nu e}\dot\gamma^\nu - sg^{af}Y^iX^j\partial_{ji}g_{mf}\Gamma^m_{\nu e}\dot\gamma^\nu \\
&\qquad -sg^{af}Y^i\partial_i g_{mf}X^j\partial_j\Gamma^m_{\nu e}\dot\gamma^\nu - sg^{af}Y^i\partial_i g_{mf}\Gamma^m_{\nu e}\dot X^\nu + sX^j\partial_j\Gamma^a_{k\nu}\dot\gamma^\nu g^{kf}Y^i\partial_i g_{ef} \\
&\qquad +s\Gamma^a_{k\nu}\dot X^\nu g^{kf}Y^i\partial_i g_{ef} + s\Gamma^a_{k\nu}\dot\gamma^\nu X^j\partial_j g^{kf}Y^i\partial_i g_{ef} + s\Gamma^a_{k\nu}\dot\gamma^\nu g^{kf}Y^iX^j\partial_{ji}g_{ef} \\
&\qquad -(X \leftrightarrow Y) \\
&= sg^{af}\partial_{ji}g_{fe}\dot Y^iX^j + sg^{af}Y^iX^j\partial_{j\nu i}g_{fe}\dot\gamma^\nu + sg^{af}Y^i\partial_{\nu i}g_{fe}\dot X^\nu \qquad \text{(A.16)} \\
&\qquad -sY^iX^j\partial_{ji}\Gamma^a_{\nu e}\dot\gamma^\nu - sX^i\dot Y^\nu\partial_i\Gamma^a_{\nu e} - sY^i\dot X^\nu\partial_i\Gamma^a_{\nu e} \\
&\qquad -(X \leftrightarrow Y),
\end{aligned}$$

in normal coordinates. Call the terms on the right hand side of the last expression (1),...,(12). Then (1) + (9), (3) + (7), (5) + (12), (6) + (11) cancel. Using

$$\partial_{ji}g_{ef} = \frac{1}{3}(R_{iejf} - R_{ifej}) = \partial_{ij}g_{ef}, \quad \partial_i\Gamma^a_{jk} = \frac{1}{3}(R_{ijk}{}^a + R_{ikj}{}^a) = \frac{1}{3}(R_{ijka} + R_{ikja}), \tag{A.17}$$

we get

$$\begin{aligned}
&\frac{1}{is\xi}\left(X(\sigma_{-1}(\omega_Y, a, b))_{ae} - Y(\sigma_{-1}(\omega_X, a, b))_{ae}\right) \qquad \text{(A.18)} \\
&= Y^iX^j\dot\gamma^\nu(\partial_{j\nu i}g_{ae} - \partial_{i\nu j}g_{ae} + \partial_{ij}\Gamma^a_{\nu e} - \partial_{ji}\Gamma^a_{\nu e}) \\
&= 0,
\end{aligned}$$

since in local coordinates mixed partials are equal.

Thus the contribution from a_X, b_X is zero.

- *The contribution from* d_X

In normal coordinates, we have

$$\begin{aligned}
&\frac{1}{i\xi}\left(X(d_Y)^a_e - Y(d_X)^a_e\right)\\
&= X\left(\partial_\nu \Gamma^a_{e\delta}\dot\gamma^\nu Y^\delta + \Gamma^a_{\nu\mu}\Gamma^\mu_{e\delta}\dot\gamma^\nu Y^\delta\right.\\
&\qquad \left. -\Gamma^a_{e\mu}\Gamma^\mu_{\nu\delta}\dot\gamma^\nu Y^\delta - \partial_e\Gamma^a_{\nu\delta}\dot\gamma^\nu Y^\delta\right)\\
&\qquad -(X\leftrightarrow Y)\\
&= X^i\partial_{i\nu}\Gamma^a_{ej}\dot\gamma^\nu Y^j + \partial_i\Gamma^a_{ej}\dot X^i Y^j - X^i\partial_{ie}\Gamma^a_{\nu j}\dot\gamma^\nu Y^j - \partial_e\Gamma^a_{ij}\dot X^i Y^j\\
&\qquad -Y^j\partial_{j\nu}\Gamma^a_{ei}\dot\gamma^\nu X^i - \partial_j\Gamma^a_{ei}\dot Y^j X^i + Y^j\partial_{je}\Gamma^a_{\nu i}\dot\gamma^\nu X^i + \partial_e\Gamma^a_{ji}\dot Y^j X^i\\
&= X^iY^j\dot\gamma^\nu(\partial_{i\nu}\Gamma^a_{ej} - \partial_{ie}\Gamma^a_{\nu j} - \partial_{j\nu}\Gamma^a_{ei} + \partial_{je}\Gamma^a_{\nu i})\\
&\qquad +\dot X^iY^j(\partial_i\Gamma^a_{ej} - \partial_e\Gamma^a_{ij}) + X^i\dot Y^j(\partial_e\Gamma^a_{ji} - \partial_j\Gamma^a_{ei})\\
&= X^iY^j\dot\gamma^\nu(\partial_{i\nu}\Gamma^a_{ej} - \partial_{ie}\Gamma^a_{\nu j} - \partial_{j\nu}\Gamma^a_{ei} + \partial_{je}\Gamma^a_{\nu i})\\
&\qquad +\dot X^iY^jR_{iej}{}^a - X^i\dot Y^jR_{jei}{}^a,
\end{aligned} \tag{A.19}$$

where we use (A.17) and the Bianchi identity in the last line.

• *The contribution from* f_X

The explicit formula for f_X is (A.13). Call the terms on the right hand side of (A.13) (1),...,(8). In normal coordinates, only terms (2), (3), (5) have nonzero X or Y derivatives, as the other terms have two Christoffel symbols. For (2), using $\partial_i g_{ab} = 0$ in normal coordinates, we get for the symbol with indices $\cdot^a_e$ (omiting $4s-3$)

$$\begin{aligned}
&X(g^{\ell a}g_{re}\dot\Gamma^r_{\ell\delta}Y^\delta) - (X\leftrightarrow Y)\\
&= \delta^{\ell a}\delta_{re}X(\dot\gamma^\nu\partial_\nu\Gamma^r_{\ell j})Y^j - (X\leftrightarrow Y)\\
&= \delta^{\ell a}\delta_{re}\dot X^i\partial_i\Gamma^r_{\ell j}Y^j + \delta^{\ell a}\delta_{re}\dot\gamma^\nu X^i\partial_{i\nu}\Gamma^r_{\ell j}Y^j - (X\leftrightarrow Y)\\
&= \frac{1}{3}\dot X^iY^j(R_i{}^a{}_{je} + R_{ij}{}^a{}_e) + \delta^{\ell a}\delta_{re}X^iY^j\dot\gamma^\nu\partial_{i\nu}\Gamma^r_{\ell j} - (X\leftrightarrow Y)\\
&= \frac{1}{3}\dot X^iY^j(R_{jei}{}^a + R_{jie}{}^a) + \delta^{\ell a}\delta_{re}X^iY^j\dot\gamma^\nu\partial_{i\nu}\Gamma^r_{\ell j} - (X\leftrightarrow Y).
\end{aligned} \tag{A.20}$$

The contribution from (3) is $4s-1$ times

$$\begin{aligned}
&\delta^{\ell a}\delta_{re}X^i\partial_i\Gamma^r_{\ell j}\dot Y^j - (X\leftrightarrow Y)\\
&= \frac{1}{3}X^i\dot Y^j(R_{jei}{}^a + R_{jie}{}^a) - (X\leftrightarrow Y),
\end{aligned} \tag{A.21}$$

and the contribution from (5) is

$$\begin{aligned}
&X(\delta^{\ell a}\delta_{re}\partial_\ell\Gamma^r_{\nu j}\dot\gamma^\nu Y^j) - (X\leftrightarrow Y)\\
&= \frac{1}{3}\dot X^iY^j(R_{eji}{}^a + R_{eij}{}^a) + \delta^{\ell a}\delta_{re}\partial_{i\ell}\Gamma^r_{\nu j}\dot\gamma^\nu X^iY^j - (X\leftrightarrow Y).
\end{aligned} \tag{A.22}$$

Summing the contributions from (A.20), (A.21), (A.22), the contribution from f_X is

$$\begin{aligned}
&(X(f_Y))^a_e - (X \leftrightarrow Y)\\
&= \dot{X}^i Y^j \left[\frac{4s-3}{3}R_{jei}{}^a + \frac{8s-4}{3}R_{ije}{}^a - \frac{4s}{3}R_{iej}{}^a + \frac{1}{3}R_{eji}{}^a\right] - (X \leftrightarrow Y)\\
&\quad +\delta^{\ell a}\delta_{re}\dot{\gamma}^\nu X^i Y^j \left[(4s-3)\partial_{i\nu}\Gamma^r_{\ell j} + \partial_{i\ell}\Gamma^r_{\nu j}\right] - (X \leftrightarrow Y)\\
&= \dot{X}^i Y^j \left[\frac{8s-4}{3}(R_{jei}{}^a + R_{ije}{}^a - \frac{4s-1}{3}R_{jei}{}^a - \frac{4s}{3}R_{iej}{}^a + \frac{1}{3}R_{eji}{}^a\right] - (X \leftrightarrow Y)\\
&\quad +\delta^{\ell a}\delta_{re}\dot{\gamma}^\nu X^i Y^j \left[(4s-3)\partial_{i\nu}\Gamma^r_{\ell j} + \partial_{i\ell}\Gamma^r_{\nu j}\right] - (X \leftrightarrow Y) \qquad (A.23)\\
&= \dot{X}^i Y^j \left[\frac{8s-4}{3}(-R_{eij}{}^a) - \frac{4s}{3}R_{jei}{}^a - \frac{4s}{3}R_{iej}{}^a\right] - (X \leftrightarrow Y)\\
&\quad +\delta^{\ell a}\delta_{re}\dot{\gamma}^\nu X^i Y^j \left[(4s-3)\partial_{i\nu}\Gamma^r_{\ell j} + \partial_{i\ell}\Gamma^r_{\nu j}\right] - (X \leftrightarrow Y)\\
&= \dot{X}^i Y^j \left[\frac{4s-4}{3}R_{iej}{}^a - \frac{4s}{3}R_{jei}{}^a\right] - (X \leftrightarrow Y)\\
&\quad +\delta^{\ell a}\delta_{re}\dot{\gamma}^\nu X^i Y^j \left[(4s-3)\partial_{i\nu}\Gamma^r_{\ell j} + \partial_{i\ell}\Gamma^r_{\nu j}\right] - (X \leftrightarrow Y).
\end{aligned}$$

Putting together the three bullets (A.18) (A.19), (A.23), we get

$$\begin{aligned}
&\frac{1}{i\xi}\sigma_{-1}(\Omega(X,Y))^a_e\\
&= X^i Y^j \dot{\gamma}^\nu(\partial_{i\nu}\Gamma^a_{ej} - \partial_{ie}\Gamma^a_{\nu j} - \partial_{j\nu}\Gamma^a_{ei} + \partial_{je}\Gamma^a_{\nu i})\\
&\quad +\dot{X}^i Y^j R_{iej}{}^a - X^i \dot{Y}^j R_{jei}{}^a \qquad (A.24)\\
&\quad +\dot{X}^i Y^j \left[\frac{4s-4}{3}R_{iej}{}^a - \frac{4s}{3}R_{jei}{}^a\right] - (X \leftrightarrow Y)\\
&\quad +\delta^{\ell a}\delta_{re}\dot{\gamma}^\nu X^i Y^j \left[(4s-3)\partial_{i\nu}\Gamma^r_{\ell j} + \partial_{i\ell}\Gamma^r_{\nu j}\right] - (X \leftrightarrow Y)\\
&= \dot{X}^i Y^j \left[\frac{4s-1}{3}R_{iej}{}^a - \frac{4s}{3}R_{jei}{}^a\right] - X^i \dot{Y}^j \left[\frac{4s-1}{3}R_{jei}{}^a - \frac{4s}{3}R_{iej}{}^a\right]\\
&\quad +X^i Y^j \dot{\gamma}^\nu \left[\partial_{i\nu}\Gamma^a_{ej} - \partial_{ie}\Gamma^a_{\nu j} - \partial_{j\nu}\Gamma^a_{ei} + \partial_{je}\Gamma^a_{\nu i}\right.\\
&\quad \left. +\delta^{\ell a}\delta_{re}[(4s-3)\partial_{i\nu}\Gamma^r_{\ell j} + \partial_{i\ell}\Gamma^r_{\nu j} - (4s-3)\partial_{j\nu}\Gamma^r_{\ell i} + \partial_{j\ell}\Gamma^r_{\nu i}]\right].
\end{aligned}$$

We now use the formula in normal coordinates

$$\partial_{\ell k}\Gamma^p_{ij} = \frac{1}{24}S_{i,j}S_{k.\ell}(5R_{kji}{}^p{}_{;\ell} + R_{\ell ik}{}^p{}_{;j}), \qquad (A.25)$$

where $S_{i,j}$ denotes the symmetric sum over i,j. Plugging (A.25) into (A.24) gives

$$\begin{aligned}
&\frac{1}{i\xi}\sigma_{-1}(\Omega(X,Y))^a_e\\
&= \dot{X}^iY^j\left[\frac{4s-1}{3}R_{iej}{}^a - \frac{4s}{3}R_{jei}{}^a\right] - X^i\dot{Y}^j\left[\frac{4s-1}{3}R_{jei}{}^a - \frac{4s}{3}R_{iej}{}^a\right]\\
&\quad + X^iY^j\dot\gamma^\nu\left[(20s+20)R_{\nu je}{}^a{}_{;i} + (20s-15)R_{je\nu}{}^a{}_{;i} - 5R_{e\nu j}{}^a{}_{;i}\right.\\
&\quad -2R_{i\nu e}{}^a{}_{;j} + (4s-1)R_{\nu ei}{}^a{}_{;j} + (4s-3)R_{ie\nu}{}^a{}_{;j}\\
&\quad + 4R_{iej}{}^a{}_{;\nu} + (-20s+18)R_{ije}{}^a{}_{;\nu} + (-20s+14)R_{eji}{}^a{}_{;\nu}\\
&\quad -4R_{ij\nu}{}^a{}_{;e} - 5R_{i\nu j}{}^a{}_{;e} + R_{\nu ji}{}^a{}_{;e}\\
&\quad \left.+(4s+2)R_{ij\nu e;}{}^a + (4s-3)R_{\nu jie;}{}^a + 5R_{i\nu je;}{}^a - (i\leftrightarrow j)\right].
\end{aligned} \tag{A.26}$$

It is tedious to expand this expression and simplify using the second Bianchi identity. The result is

$$\begin{aligned}
&\frac{1}{i\xi}\sigma_{-1}(\Omega(X,Y))^a_e\\
&= \dot{X}^iY^j\left[\frac{4s-1}{3}R_{iej}{}^a - \frac{4s}{3}R_{jei}{}^a\right] - X^i\dot{Y}^j\left[\frac{4s-1}{3}R_{jei}{}^a - \frac{4s}{3}R_{iej}{}^a\right]\\
&\quad + X^iY^j\dot\gamma^\nu\left[(8s-4)R_{ij\nu e;}{}^a + (-40s+28)R_{ije}{}^a{}_{;\nu} + (20s+14)R_{\nu je}{}^a{}_{;i}\right.\\
&\quad -(20s+14)R_{\nu ie}{}^a{}_{;j} + (4s-2)R_{\nu jie;}{}^a - (4s-2)R_{\nu ije;}{}^a\\
&\quad +(16s-12)R_{je\nu}{}^a{}_{;i} - (16s-12)R_{ie\nu}{}^a{}_{;j} + (4s-6)R_{e\nu j}{}^a{}_{;i}\\
&\quad \left.-(4s-6)R_{e\nu i}{}^a{}_{;j} + (-20s+18)R_{iej}{}^a{}_{;\nu} - (-20s+18)R_{jei}{}^a{}_{;\nu}\right].
\end{aligned} \tag{A.27}$$

The operator $A(X,Y) = \Omega^s(X,Y) - \Omega^M(X,Y)$ on γ^*TM has order -1, with top order symbol $\sigma_{-1}(A) = \sigma_{-1}(\Omega^s)$. Since the top order symbol is invariant, (A.27) must be tensorial. This is accomplished by replacing $\dot{X}^i$ by $(DX/d\gamma)^i$ and similarly for Y^j. This finishes the proof of Theorem 3.4.

Remark A.2. We recall a short proof of (A.25). For any tangent vector $\vec{x}$ to M, the radial line $\gamma(t) = t\vec{x}$ is the normal coordinates expression for a geodesic. Applying $\frac{d^2}{dt^2}$ to the geodesic equation

$$\frac{d^2\gamma^p}{dt^2} + \Gamma^p_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0$$

yields $S_{m,\ell,i,j}\partial_{m\ell}\Gamma^p_{ij} = 0$. Using $\partial_{\ell m} = \partial_{m\ell}$ and $\Gamma^p_{ij} = \Gamma^p_{ji}$, this reduces to

$$\partial_{\ell m}\Gamma^p_{jk} + \partial_{km}\Gamma^p_{j\ell} + \partial_{\ell k}\Gamma^p_{jm} + \partial_{j\ell}\Gamma^p_{km} + \partial_{jm}\Gamma^p_{k\ell} + \partial_{kj}\Gamma^p_{\ell m} = 0. \tag{A.28}$$

Using $R_{ijk}{}^\ell = \partial_i\Gamma^\ell_{jk} - \partial_j\Gamma^\ell_{ik} + \Gamma^\ell_{i\mu}\Gamma^\mu_{jk} - \Gamma^\ell_{j\mu}\Gamma^\mu_{ik}$ and $\Gamma^p_{ij} = 0$ at the origin in normal coordinates, we get

$$R_{ijk}{}^\ell{}_{;m} = \partial_{mi}\Gamma^\ell_{jk} - \partial_{mj}\Gamma^\ell_{ik} \tag{A.29}$$

at the origin. Using (A.29), a straightforward calculation gives

$$\begin{aligned} & S_{i,j}S_{k,\ell}(5R_{kji}{}^{\ell}{}_{;p} + R_{\ell ik}{}^{p}{}_{;j}) \\ = \ & 20\partial_{\ell k}\Gamma^p_{ij} - 4\partial_{ji}\Gamma^p_{\ell k} - 4\partial_{\ell j}\Gamma^p_{ki} - 4\partial_{\ell i}\Gamma^p_{jk} - 4\partial_{ki}\Gamma^p_{\ell j} - 4\partial_{kj}\Gamma^p_{\ell i} \\ = \ & 20\partial_{\ell k}\Gamma^p_{ij} + 4\partial_{\ell k}\Gamma^p_{ij} \end{aligned}$$

by (A.28).

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