Lefschetz theory, Mathai-Quillen forms and the far point set

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Abstract

The far point set of a self-map of a closed Riemannian manifold is defined to be the set of points mapped into their cut locus. We prove a "far point formula" analogous to the Lefschetz fixed point formula, with the contribution from each far point a real number in [-1,1]. Using the far point formula, we show that for most metrics, a diffeomorphism with Lefschetz number different from the Euler characteristic must have infinite far point set. The main technique is the use of Mathai-Quillen forms. These forms also provide a new integral formula for the Lefschetz number, which reduces to the Chern-Gauss-Bonnet formula when the function is the identity.

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1 Introduction

The Lefschetz fixed point formula for a smooth function f on a closed manifold M expresses the Lefschetz number $L(f)$ as a sum over the fixed point set of f: $L(f) = \sum_{p \in \mathrm{Fix}(f)} \operatorname{sgn} \det(\mathrm{Id} - df_p)$. This gives the important estimate $|L(f)| \leq |\mathrm{Fix}(f)|$ which guarantees the existence of fixed points if $L(f) \neq 0$.

In this paper, we show that the set of points mapped far from themselves has a similar topological content. More precisely, if M has a Riemannian metric and $\mathcal{C}_x$ denotes the cut locus of $x \in M$, we set the *far point set* of f to be $\mathrm{Far}(f) = \{x : f(x) \in \mathcal{C}_x\}$. The main result is a "far point formula"

$$L(f) - \chi(M) = \sum_{p \in \mathrm{Far}(f)} \alpha_p \tag{1}$$

for suitably transverse f (Theorem 2.7), where $\alpha_p \in [-1, 1]$ measures the signed ratio $|\exp_p^{-1} f(p)|/|\exp_p^{-1} \mathcal{C}_x|$. This yields a sharp estimate $|L(f) - \chi(M)| \leq |\mathrm{Far}(f)|$ which holds for all diffeomorphisms. (For example, if $\mathrm{Far}(f) = \emptyset$, there is a unique minimal geodesic joining x to $f(x)$, and so $L(f) = \chi(M)$.) Unlike the fixed point case, the transversality condition is far from generic. In particular, for diffeomorphisms of non-positively curved manifolds with $L(f) \neq \chi(M)$, the far point set is infinite (Theorem 2.9).

The main technical tool used to study the far point set is Mathai-Quillen forms [12], which are geometric representatives for the Thom class of a vector bundle over M. Using a one-parameter family of pullbacks of this form on the tangent bundle, Mathai and Quillen gave a new proof of both the Hopf index formula and the Chern-Gauss-Bonnet formula for the Euler characteristic. Adapting their methods, we easily derive in §2.1 a new integral expression for the Lefschetz number (Theorem 2.3). A deformation of the function f along geodesics joining x to $f(x)$ yields a one parameter family of pullbacks of the Lefschetz integrand in §2.2. As the parameter t goes to zero, the integrand becomes singular on $\mathrm{Far}(f)$, and an examination of the singularity leads to (1). In §2.3 the transversality assumption is dropped. The difference $L(f) - \chi(M)$ is realized the singular part of a current supported on $\mathrm{Far}(f)$, just as $L(f)$ can be considered as a current supported on $\mathrm{Fix}(f)$. In contrast, the Mathai-Quillen deformation of the Thom form has no singular behavior at zero. In §3, a new topological proof of the Lefschetz fixed point/submanifold formula is derived by letting $t \to \infty$. The proof has the flavor of the heat equation proof of the Lefschetz formula in [8], but uses less machinery.

In summary, on a Riemannian manifold the fixed point set and the far point set appear in the same setting, the first as $t \to \infty$ and the second as $t \to 0$

(see Figure 2). In [12], only the $t = 0$ and $t \to \infty$ behavior of pullbacks of Mathai-Quillen forms is of interest. In our case, the $t = 1$ pullback is the basic integral formula for the Lefschetz number. The Lefschetz integrand is nontrivial even for flat manifolds, and reduces to the Chern-Gauss-Bonnet theorem when $f = \mathrm{Id}$. In §4.1 the integrand is explicitly calculated for flat manifolds, and in §4.2 the integrand for arbitrary metrics is examined. Here the integrand depends on solutions of Jacobi equations on M, and so is less explicit. However, the integrand is computable for constant curvature metrics. Finally, in the Appendix we give a Hodge theory estimate for the Lefschetz number in terms of the norm of df and the geometry of M.

2 Geometric deformations of the Lefschetz integral

Let $f : M \to M$ be a smooth map of a closed oriented Riemannian manifold M. The Lefschetz number of f is

$$L(f) = \sum_q (-1)^q \ \mathrm{tr}\ f^q,$$

where f^q denotes the induced map on the real cohomology group $H^q(M)$. In this section we give an integral formula (Theorem 2.3) for the Lefschetz number of f, introduce a one parameter family f_t of deformations of f, and study the $t \to 0$ behavior of f_t. Using the fact that that f_0 is singular precisely on $\mathrm{Far}(f)$, we derive the main theorem (1) under a transversality condition, and study the role of the far point set for general maps.

2.1 Mathai-Quillen forms and the Lefschetz integral

The Lefschetz number has a well known Poincaré duality formulation. Recall that the Poincaré dual η_N of an oriented k-submanifold N of a closed oriented manifold X is the real cohomology class defined by (or characterized by, depending on one's definition)

$$\int_N \omega = \int_X \omega \wedge \eta_N, \tag{2}$$

for all closed k-forms ω on X [1, (5.13)]. The following results are in [1, Ch. 5, viz. Ex 11.26].

Theorem 2.1 *(i) Let N, N' be closed oriented submanifolds of X with transverse intersection. Then*

$$\eta_{N\cap N'} = \eta_N \wedge \eta_{N'}.$$

(ii) A closed form $U \in H^k_c(E)$, the compactly supported cohomology of an oriented rank k bundle E over X, represents the Thom class of E iff the integral of U over each fiber of E is one.

(iii) Identify the total space of ν^X_N, the normal bundle of N in X, with a tubular neighborhood of N in X, so that the Thom class of the normal bundle can be considered as a cohomology class on X. Then the Poincaré dual of N is the Thom class of ν^X_N.

(iv) Let $f : M \to M$ be a smooth map of a closed oriented manifold. Then

$$L(f) = \int_\Delta \eta_\Gamma,$$

where η_Γ is the Poincaré dual of the graph Γ of f in $M \times M$ and Δ is the diagonal in $M \times M$.

It is pointed out in [12] that the cohomology with compact support in (ii) may be replaced with the cohomology of forms with exponential decay in the fibers.

Remark: We will need to make the identification in (iii) more explicit. To consider a closed form U on ν^X_N as a closed form on X, we first use a fiberwise diffeomorphism $\alpha : B_\epsilon(0) \to \nu^X_N$ on the ϵ-ball of the zero section of ν^X_N. For ϵ small enough, the exponential map $\exp : B_\epsilon(0) \to X$ is a diffeomorphism onto a tubular neighborhood of N, and it is really $(\exp^{-1})^*\alpha^* U$ which is a form supported on the tubular neighborhood.

If Γ is transversal to Δ in $M \times M$, we obtain a quick proof of the Lefschetz fixed point formula:

$$L(f) = \sum_{p, f(p)=p} \mathrm{sgn}\ \det(\mathrm{Id} - df_p).$$

Since $\Gamma \cap \Delta$ is a finite set of points, Theorem 2.1 and Poincaré duality give

$$L(f) \quad = \quad \int_\Delta \eta_\Gamma = \int_{M\times M} \eta_\Gamma \wedge \eta_\Delta = \int_{M\times M} \eta_{\Gamma\cap\Delta}$$

$$= \int_{\Gamma\cap\Delta} 1 = \sum_{p, f(p)=p} \pm 1.$$

Thus $L(f)$ is the sum of the orientations ± 1 of the fixed points p of f. By [9, p. 121], the orientation equals sgn $\det(\mathrm{Id} - df_p)$ in our sign convention. Note also that $L(f) = \int_{\Gamma\cap\Delta} 1$ implies that $L(f) = I(\Delta, \Gamma)$, the intersection number of Δ and Γ, so we have the three classical expressions for the Lefschetz number.

In [12], Mathai and Quillen obtain a geometric expression for the Thom class of an oriented even dimensional vector bundle (see [10] for other geometric representatives). Let E be a rank $n = 2m$ vector bundle over a manifold M, where E has an inner product and a compatible connection θ. Then a geometric representative MQ of the Thom class of E is given by

$$\mathrm{MQ} = \pi^{-m} e^{-|x|^2} \sum_{I, |I| \text{ even}} \epsilon(I, I')\mathrm{Pf}\left(\frac{1}{2}\Omega_I\right)(dx + \theta x)^{I'}, \tag{3}$$

where: x is an orthornormal fiber coordinate; Ω is curvature of the connection θ; Ω_I is the submatrix of Ω with respect to the multi-index I with entries in $\{1, 2, ..., n\}$; $\mathrm{Pf}(\frac{1}{2}\Omega_I)$ is the Pfaffian of $\frac{1}{2}\Omega_I$; I' denotes the complement of I in $\{1, 2, ..., n\}$; $\epsilon(I, I')$ is the sign of I, I' as a permutation (i.e. $dx^I \wedge dx^{I'} = \epsilon(I, I')dx^1 \wedge ... \wedge dx^n$); and

$$(dx + \theta x)^{I'} = (dx^{i_1} + \theta^{i_1}_{j_1} x^{j_1}) \wedge (dx^{i_2} + \theta^{i_2}_{j_2} x^{j_2}) \wedge ... \wedge (dx^{i_q} + \theta^{i_q}_{j_q} x^{j_q}),$$

with $I' = \{i_1, i_2, ..., i_q\}$. In the expression θx, θ denotes the connection one-forms of the connection for the frame $\{x^i\}$. The ordering of the elements of I' in $dx^{I'}$ is unimportant due to the $\epsilon(I, I')$ factor. For computations at a point $x \in M$, we will often assume that $\{x^i\}$ is a synchronous frame centered at x, in which case the connection one-forms θ vanish at x

Unlike the Euler characteristic, the Lefschetz number can be nonzero for odd dimensional manifolds, so we need to check that this formalism extends to bundles of odd rank.

Lemma 2.2 *(cf. [10, Ch. 4,§2]) Let E be an oriented odd rank n vector bundle with inner product over a manifold M, and let ∇ be a compatible connection on E with curvature Ω. Then*

$$U_M = \pi^{-\frac{n}{2}} e^{-|x|^2} \sum_{I, |I| \text{ even}} \epsilon(I, I')\mathrm{Pf}\left(\frac{1}{2}\Omega_I\right)(dx + \theta x)^{I'}$$

is a representative of the Thom class of E.

Proof: Denote E by E_M, and let E_{S^1} be the trivial bundle with the trivial connection over S^1. Equip $E_M \times E_{S^1}$ over $M \times S^1$ with the product connection. The Mathai-Quillen representative $\mathrm{MQ}_{M\times S^1} \in H^{n+1}(M \times S^1)$ of the Thom class of $E = E_M \times E_{S^1}$ is given by

$$\mathrm{MQ}_{M\times S^1} = \pi^{-(n+1)/2} e^{-|x|^2} \sum_{I,|I| \text{ even}} \epsilon(I, I')\mathrm{Pf}\left(\frac{1}{2}\Omega_I\right)(dx + \theta x)^{I'} ,$$

where θ is the connection one-form with respect to a product orthonormal frame $\{x^i\}$ of E, and $\Omega = \Omega_{M\times S^1}$ is the curvature of this connection. The curvature matrix for $M \times S^1$ is

$$\Omega = \begin{pmatrix} \Omega_M & 0 \\ 0 & 0 \end{pmatrix},$$

where Ω_M is the curvature matrix of E_M.

Recall that for an even-dimensional $k\times k$ skew-symmetric matrix ω, the Pfaffian is a homogeneous polynomial of degree $k/2$ in the entries of ω characterized up to sign by $\mathrm{Pf}^2(\omega) = \det(\omega)$. If Ω_I is a submatrix of Ω, then $\mathrm{Pf}\left(\frac{1}{2}\Omega_I\right) = 0$, unless Ω_I is a submatrix of Ω_M itself. Thus in the definition of $\mathrm{MQ}_{M\times S^1}$ we may assume that $n + 1 \notin I$, where $n + 1$ corresponds to the dt variable on S^1. Moreover, we have

$$\begin{aligned}(dx + \theta x)^{I'} &= (dx^{i'_1} + \theta^{i'_1}_{j_1} x^{j_1}) \wedge (dx^{i'_2} + \theta^{i'_2}_{j_2} x^{j_2}) \wedge \ldots \wedge (dt + \theta^{n+1}_{j_q} x^{j_q}) \\ &= (dx + \theta x)^{I'_M} \wedge dt,\end{aligned}$$

where $I' = I'_M \cup \{n + 1\} = \{i'_1, ..., i'_{q-1}, n + 1\}$ with I'_M the complement of $I - \{n + 1\}$ in $\{1, 2, ..., n\}$. Hence $\mathrm{MQ}_{M\times S^1}$ decomposes as follows:

$$\begin{aligned}\mathrm{MQ}_{M\times S^1} &= \pi^{-n/2} e^{-|x|^2} \sum_{I_M,|I_M| \text{ even}} \epsilon(I_M, I'_M)\mathrm{Pf}\left(\frac{1}{2}\Omega_{I_M}\right)(dx + \theta x)^{I'_M} \\ &\qquad \wedge(\sqrt{\pi})^{-1} e^{-t^2} dt \\ &= U_M \wedge U_{S^1},\end{aligned}$$

where

$$\begin{aligned}U_M &= \pi^{-\frac{n}{2}} e^{-x^2} \sum_{I_M,|I_M| \text{ even}} \epsilon(I_M, I'_M)\mathrm{Pf}\left(\frac{1}{2}\Omega_{I_M}\right)(dx + \theta x)^{I'_M}, \\ U_{S^1} &= \pi^{-1/2} e^{-t^2} dt.\end{aligned}$$

By Theorem 2.1 (iii), U_{S^1} represents the Thom class of E_{S^1}. Since $dU_{S^1} = 0$, we have $0 = d(\mathrm{MQ}_{M\times S^1}) = dU_M \wedge U_{S^1}$. U_{S^1} is non-zero, so $dU_M = 0$. Moreover, in each fiber $\int_{E_{S^1}} U_{S^1} = 1$, so in each fiber

$$1 = \int_{E_M \times E_{S^1}} U_M \wedge U_{S^1} = \int_{E_M} U_M.$$

By Theorem 2.1 (ii), U_M represents the Thom class of E_M. $\square$

The following elementary result is the basic integral formula for the Lefschetz number.

Theorem 2.3 *Let $f : M \to M$ be a smooth map of a closed, oriented, Riemannian manifold. Let Δ_ϵ be a tubular neighborhood of the diagonal in $M \times M$ of width ϵ, and let $\mathrm{MQ}_{\Delta_\epsilon}$ be the Mathai-Quillen form of the normal bundle to the diagonal, considered as a form supported in Δ_ϵ. Then the Lefschetz number is given by*

$$L(f) = (-1)^{\dim M} \int_M (\mathrm{Id}, f)^* \mathrm{MQ}_{\Delta_\epsilon}, \tag{4}$$

where $(\mathrm{Id}, f) : M \to \Gamma \subset M \times M$ is the graph map.

Proof: By Theorem 2.1(iv) and Poincaré duality, we have

$$\begin{aligned} L(f) &= \int_\Delta \eta_\Gamma = \int_{M\times M} \eta_\Gamma \wedge \eta_\Delta \\ &= (-1)^{\dim M} \int_\Gamma \eta_\Delta = (-1)^{\dim M} \int_{(\mathrm{Id},f)(M)} \mathrm{MQ}_{\Delta_\epsilon} \\ &= (-1)^{\dim M} \int_M (\mathrm{Id}, f)^* \mathrm{MQ}_{\Delta_\epsilon}, \end{aligned}$$

since (Id, f) is an orientation preserving diffeomorphism and hence of degree 1. $\square$

This formula generalizes the Chern-Gauss-Bonnet theorem. The Euler characteristic of an even dimensional Riemannian manifold M is given by

$$\chi(M) = L(\mathrm{Id}) = \int_M (\mathrm{Id}, \mathrm{Id})^* \mathrm{MQ}_{\Delta_\epsilon} = \int_M 0^* \mathrm{MQ}_{TM},$$

since a neighborhood of the zero section in TM is isomorphic to a tubular neighborhood of Δ under an isomorphism taking the zero section 0 to the graph map $(\mathrm{Id}, \mathrm{Id})$ of the identity. For the Levi-Civita connection θ, we have

$$0^*\mathrm{MQ}_{TM} = \pi^{-n/2}\mathrm{Pf}(\frac{1}{2}\Omega)$$

since $x = 0$ on M implies $0^*(dx + \theta x)^{I'} = 0$ if $I' \neq \emptyset$. Thus we obtain the Chern-Gauss-Bonnet theorem

$$\chi(M) = \frac{1}{(2\pi)^{n/2}} \int_M \mathrm{Pf}(\Omega).$$

Similarly, we find $\chi(M) = 0$ if dim M is odd.

Note that the support of the integrand in (4) is $\{x \in M : (x, f(x)) \in \Delta_\epsilon\}$.

2.2 The far point set

In [12], the formula $\chi(M) = \int_M s^*\mathrm{MQ}_{TM}$, where s is a section of TM, is modified by replacing s with ts for $t > 0$. As $t \to \infty$, the Hopf formula is recovered. At $t = 0$ the integrand becomes the Pfaffian of the curvature as above, yielding the Chern-Gauss-Bonnet formula. In this section, we deform f to a function f_t in the basic formula $L(f) = \int_M (\mathrm{Id}, f)^*\mathrm{MQ}_\Delta$. In §3, the Lefschetz fixed point formula is recovered as $t \to \infty$, as expected, since the Hopf formula can be derived from the fixed point formula. In contrast, the function f_t becomes discontinuous at $t = 0$ at those $x \in M$ for which $(x, f(x))$ is in the boundary of the tubular neighborhood of the diagonal. Thus this case is more complicated.

The maximum amount of information is obtained when the tube is as large as possible. As explained below, this occurs when the boundary of the vertical fiber of the tube at (x, x) is $\mathcal{C}_x$, the cut locus of x.

Recall that on a closed manifold, a geodesic $\gamma(t)$ is the minimal length curve joining $x = \gamma(0)$ to $\gamma(t)$ for $t \in [0, T]$ up to some maximal time T. The point $y = \gamma(T)$ is by definition in $\mathcal{C}_x$. In particular, if the graph of a smooth function $f : M \to M$ has the property that $f(x)$ is never on $\mathcal{C}_x$, then there is a unique minimal geodesic joining x to $f(x)$. Shrinking this geodesic gives a homotopy from f to the identity, and so the Lefschetz number satisfies $L(f) = \chi(M)$.

This indicates that the difference $L(f) - \chi(M)$ is controlled by the *far point set* of f

$$\mathrm{Far}(f) = \{x : f(x) \in \mathcal{C}_x\}. \tag{5}$$

However, this set is quite complicated in general.

In this subsection, we assume that $\mathrm{Far}(f)$ is finite. By studying the $t \to 0$ limit of the integral formula for $L(f)$, we prove the far point formula (1) and conclude that $|L(f) - \chi(M)| \le |\mathrm{Far}(f)|$ (Theorem 2.7), under a transversality assumption on f that holds for diffeomorphisms. In fact, for all but very special metrics, the transversality condition implies that $\mathrm{Far}(f)$ is infinite for diffeomorphisms with $L(f) \neq \chi(M)$.

Recall that the set of vectors $v \in T_xM$ such that the geodesic through x with tangent vector v is minimal up to time one form a topological ball B_x, whose boundary is called the cut locus of x in T_xM. The exponential map satisfies $\exp_x(\partial B_x) = \mathcal{C}_x$, and we always consider the exponential map $\exp_x : \bar{B}_x \to M$ to have domain $\bar{B}_x$.

We first construct the largest (topological) tubular neighborhood of the diagonal Δ in $M \times M$. A tubular neighborhood is given by points of the form $(\exp_{\bar{x}} v, \exp_{\bar{x}}(-v))$, for $v \in T_{\bar{x}}M, \bar{x} \in M$, and $|v|$ is small. For $x, y \in M$, y is said to be inside $\mathcal{C}_x$ if there is a unique minimal geodesic from x to y. Let $N_{\bar{x}} = \{\exp_{\bar{x}} v : \exp_{\bar{x}} v \text{ is inside } \mathcal{C}_{\exp_{\bar{x}}(-v)}\}$. Define $T \subset M \times M$ by $T = \{(\exp_{\bar{x}} v, \exp_{\bar{x}}(-v)) : \exp_{\bar{x}} v \in N_{\bar{x}}, \bar{x} \in M\}$. We call T the *cut locus tubular neighborhood.*

Lemma 2.4 *(i) T is a topological tubular neighborhood of the diagonal.*

(ii) $(x, y) \in T$ iff y is inside $\mathcal{C}_x$.

(iii) The vertical fiber $T \cap (\{x\} \times M)$ of T at (x, x) equals $\{x\} \times (M \setminus \mathcal{C}_x)$.

Proof: We prove (ii) first. The forward implication is from the definition of T. Conversely, if y is inside $\mathcal{C}_x$, then there is a unique minimal geodesic from x to y. Then $(x, y) = (\exp_{\bar{x}} v, \exp_{\bar{x}}(-v))$, where $\bar{x}$ is the midpoint of the geodesic. Thus $x, y \in N_{\bar{x}}$, so $(x, y) \in T$.

For (i), the standard argument that the interior of the cut locus is a topological sphere immediately extends to show that $\exp_x^{-1}(N_x) \subset T_xM$ is the interior of a topological sphere. This argument in turn extends to show that the radius of this sphere is a continuous function on the unit tangent sphere, which implies that $\exp^{-1}(T)$ is a topological disk bundle.

To finish the proof, we must show that exp on $M \times M$ is injective on $\{(v, -v)\} \subset \coprod_x (\exp_x^{-1}(N_x) \times \exp_x^{-1}(N_x))$. If not, there exists $\bar{x}, \bar{y}, v, w$ with $\alpha = \exp_{\bar{x}} v = \exp_{\bar{y}} w$ and $\beta = \exp_{\bar{x}}(-v) = \exp_{\bar{y}}(-w)$. By the definition of $N_{\bar{x}}, N_{\bar{y}}$, this gives two minimal geodesics from α to β, a contradiction.

For (iii), $(\exp_{\bar{x}} v, \exp_{\bar{x}}(-v))$ is in the vertical fiber over $(\exp_{\bar{x}} v, \exp_{\bar{x}} v)$, and at ∂T, $\exp_{\bar{x}}(-v) \in \mathcal{C}_{\exp_{\bar{x}} v}$. $\square$

We now define f_t for $t > 0$ by pushing $f(x)$ out the minimal geodesic joining x and $f(x)$ towards ∂T as $t \to \infty$ and pushing $f(x)$ towards x as $t \to 0$, if $(x, f(x)) \in T$, and fixing $f(x)$ otherwise. More precisely, we want

(1) $f_1 = f$,

(2) $f_t(x) = f(x)$ if $(x, f(x)) \notin T$ or if $f(x) = x$,

(3) $\lim_{t\to\infty} f_t(x) \in \partial T$ if $(x, f(x)) \in T$ but $f(x) \neq x$,

(4) $\lim_{t\to 0} f_t(x) = x$ if $(x, f(x)) \in T$.

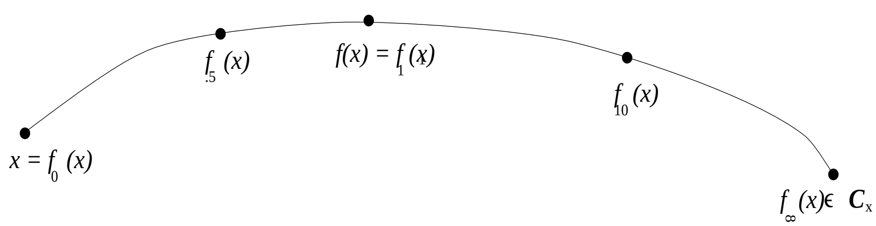

Figure 1: Deforming f along the minimal geodesic from x to $f(x)$ for the cut locus tubular neighborhood.

For motivation, think of T as diffeomorphic to the tangent bundle of M, and f as a vector field which blows up on $\{x : (x, f(x)) \notin T\}$. Then the family f_t is analagous to the family of sections ts, except that we freeze f "at infinity."

To construct f_t, fix a diffeomorphism $\mu : [0, 1) \to [0, \infty)$ with $\mu(0) = 0, \mu(1) = 1$ and such that the derivative of μ^{-1} grows at most polynomially. For the moment, let T denote any smooth tubular neighborhood of Δ given by geodesics of the form $(x, y) = (\gamma(t), \gamma(-t))$. For such (x, y), set

$$d_{x,y} = \min\{t : (\gamma(t), \gamma(-t)) \in \partial T\}.$$

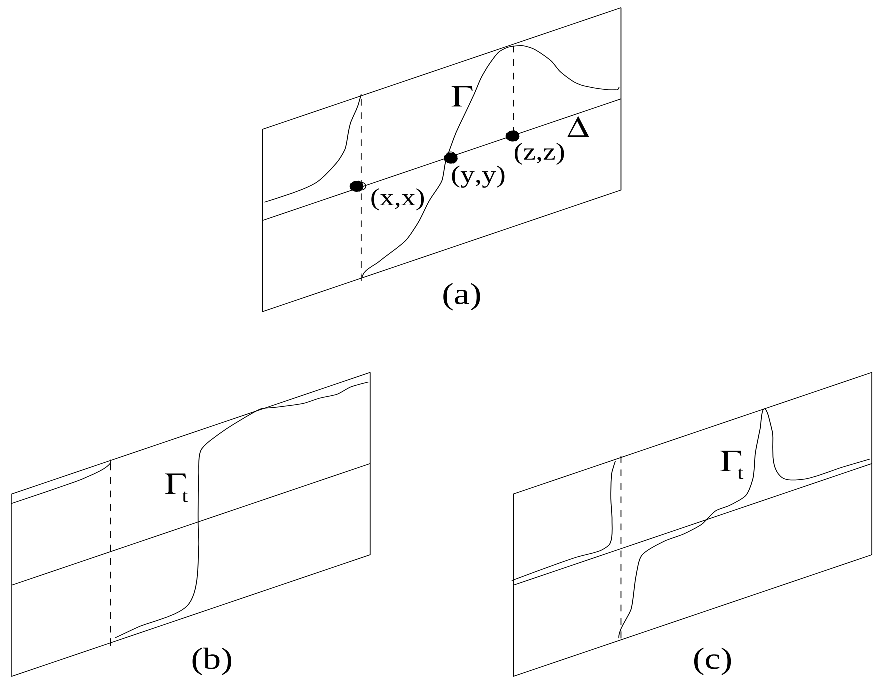

Figure 2: (a) The graph Γ of f inside the cut locus tubular neighborhood of the diagonal Δ for $M = S^1$. The top and bottom lines are the boundary of the cut locus tubular neighborhood. $\mathrm{Fix}(f) = \{y\}$ and $\mathrm{Far}(f) = \{x, z\}$. (b) The graph Γ_t of f_t for t large. The support of the Lefschetz integrand $(\mathrm{Id}, f_t)^*\mathrm{MQ}_\Delta$ approaches $\mathrm{Fix}(f) \times M$. (c) The graph Γ_t of f_t for $t \approx 0$. The support of $(\mathrm{Id}, f_t)^*\mathrm{MQ}_\Delta$ approaches a subset of $\mathrm{Fix}(f) \times M$.

For $x \in M$ and $t \in [0, \infty)$, define $t_x : M \to M$ by

$$t_x(y) = \begin{cases} \exp_x[\mu^{-1}(\mu(d_{x,y}^{-1}|v|)t)d_{x,y}\frac{v}{|v|}], & (x,y) \in T,\ y = \exp_x v,\ y \neq x, \\ y, & (x,y) \notin T, \\ x, & y = x. \end{cases}$$

Thus for x, y close but unequal, t_x pushes y towards ∂T as $t \to \infty$ along their minimal geodesic, but fixes y if it is far from x, as measured by T. The complicated expression for $t_x(y)$ in the first case ensures that $1_x(y) = y$.

For $f : M \to M$, define $f_t : M \to M$ by

$$f_t(x) = t_x(f(x)).$$

The maps f_t are smooth for $t > 0$. Note that $f_1(x) = \exp_x v = f(x)$ if $(x, f(x)) \in T$ and $f_1(x) = f(x)$ otherwise, so $f_1 = f$. Similarly, we have

$f_0(x) = x$ if $(x, f(x)) \in T$, and $f_0(x) = f(x)$ otherwise. Thus f_0 is discontinuous on $\{x : (x, f(x)) \in \partial T\}$.

We now examine the $t \to 0$ limit of pullbacks of Mathai-Quillen forms. Fix ϵ, and let $\mathrm{MQ}_\Delta = \mathrm{MQ}_{\Delta_\epsilon}$ be the Mathai-Quillen form on the ϵ-neighborhood of the diagonal. Then

$$\begin{aligned} L(f) &= \int_\Delta (\mathrm{Id}, f)^* \mathrm{MQ}_\Delta \\ &= \lim_{t\to 0} \int_\Delta (\mathrm{Id}, f_t)^* \mathrm{MQ}_\Delta \\ &= \lim_{t\to 0} \int_{\{(x,x):(x,f(x))\in T\}} (\mathrm{Id}, f_t)^* \mathrm{MQ}_\Delta \\ &\qquad + \lim_{t\to 0} \int_{\{(x,x):(x,f(x))\notin T\}} (\mathrm{Id}, f_t)^* \mathrm{MQ}_\Delta. \end{aligned}$$

Here we do not distinguish between (Id, f_t) as a map on M or on $M \times M$. Now $(\mathrm{Id}, f_t)^*(\mathrm{MQ}_\Delta)_{(x,x)} = (\mathrm{MQ}_\Delta)_{(x,f(x))} \circ (\mathrm{Id}, f_t)_* = 0$ if $(x, f(x)) \notin T$, so

$$L(f) = \lim_{t\to 0} \int_{\{(x,x):(x,f(x))\in T\}} (\mathrm{Id}, f_t)^* \mathrm{MQ}_\Delta.$$

An open exhaustion A_δ as $\delta \to 1$ of $\{(x,x) : (x, f(x)) \in T\}$ is constructed as follows. Note that $d(x,y)/d_{x,y} < 2|v|/|v| = 2$ for $(x,y) = (\exp_{\bar{x}} v, \exp_{\bar{x}}(-v))$. Fix $\delta < 1$, set

$$A_\delta = \{x : (x, f(x)) \in T, \frac{d(x, f(x))}{d_{x,f(x)}} \le 2\delta\},$$

and set $B_\delta = \{x : (x, f(x)) \in T\} \setminus A_\delta$.

Lemma 2.5

$$\lim_{t\to 0} (\mathrm{Id}, f_t)^* \mathrm{MQ}_\Delta = \mathrm{Pf}(\Omega)$$

uniformly on A_δ.

Proof: On A_δ, $f_t(y) \to y$ uniformly as $t \to 0$. Thus if $\gamma(s)$ is a short curve with $\gamma(0) = y, \dot\gamma(0) = w$, then $f_t(\gamma(s)) \to \gamma(s)$ uniformly as $t \to 0$, and

$$\begin{aligned} \lim_{t\to 0} (f_t)_*(w) &= \lim_{t\to 0} \lim_{s\to 0} \frac{f_t(\gamma(s)) - f_t(y)}{s} \\ &= \lim_{s\to 0} \lim_{t\to 0} \frac{f_t(\gamma(s)) - f_t(y)}{s} \\ &= \lim_{s\to 0} \frac{\gamma(s) - \gamma(0)}{s} = \dot\gamma(0) = w. \end{aligned}$$

This shows that

$$[(\mathrm{Id}, f_t)^*\mathrm{MQ}_\Delta]_y(v_1,\dots,v_n) = \\ (\mathrm{MQ}_\Delta)_{(y,f(y))}((v_1,(f_t)_*v_1),\dots,(v_n,(f_t)_*v_n))$$

converges uniformly in y to

$$(\mathrm{MQ}_\Delta)_{(y,y)}((v_1,v_1),\dots,(v_n,v_n)) = \mathrm{Pf}(\Omega)_y(v_1,\dots,v_n)$$

as $t \to 0$. □

By the lemma,

$$L(f) = \int_{A_\delta} \mathrm{Pf}(\Omega) + \lim_{t\to 0}\int_{B_\delta}(\mathrm{Id}, f_t)^*\mathrm{MQ}_\Delta.$$

Since $\mathrm{Pf}(\Omega)$ is smooth on M, we get

$$L(f) = \int_{\{x:(x,f(x))\in T\}} \mathrm{Pf}(\Omega) + \lim_{\delta\to 1}\lim_{t\to 0}\int_{B_\delta}(\mathrm{Id}, f_t)^*\mathrm{MQ}_\Delta. \tag{6}$$

We now check that this construction extends to the cut locus tubular neighborhood T of Lemma 2.4. Fix $\epsilon > 0$ and pick a smooth disk bundle $D^\epsilon \subset \nu_\Delta$ such that $T^\epsilon = \exp D^\epsilon$ is inside T and is within ϵ of filling T – i.e. for all $(v,-v) \in \partial D^\epsilon$, $d_{M\times M}((\exp v, \exp(-v)), (\exp(tv), \exp(-tv))) < \epsilon$, where t is the smallest positive number such that $(\exp tv, \exp(-tv)) \in \partial T$. To define the Mathai-Quillen form on T^ϵ, we choose a diffeomorphism $\alpha^\epsilon : \nu_\Delta \to D^\epsilon$ and pull back the Mathai-Quillen form MQ_ν from ν_Δ. As $\epsilon \to 0$, D^ϵ fills out a continuous disk bundle in ν_Δ, and we demand that for all $R > 0$, there exists $\epsilon_0 = \epsilon_0(R)$ such that $\alpha^\epsilon(B_R(\nu_\Delta))$ is constant for all $\epsilon < \epsilon_0$, where $B_R(\nu_\Delta)$ is the R-ball around the zero section in ν_Δ. For this choice of α^ϵ, it is immediate that

$$\mathrm{MQ}^0_\Delta(v_1,\dots,v_n) \equiv \lim_{\epsilon\to 0}[((\alpha^\epsilon)^{-1})^*\mathrm{MQ}_\nu(v_1,\dots,v_n)]$$

exists pointwise and is smooth, In fact, since MQ_ν decays exponentially at infinity in ν_Δ, the convergence is uniform. This yields

$$L(f) = \lim_{\epsilon\to 0}\int_M (\mathrm{Id}, f_t)^*\mathrm{MQ}^\epsilon_\Delta = \int_M (\mathrm{Id}, f_t)^*\mathrm{MQ}^0_\Delta.$$

Definition: f is *transverse to the cut locus* if (i) $\mathrm{Far}(f)$ consists of a finite set of points, and (ii) the graph Γ of f is transverse to $M \times \{f(x)\}$ for all $x \in \mathrm{Far}(f)$.

In Figure 2, Γ is transverse to $M \times \{f(y)\}$ but not to $M \times \{f(z)\}$.

Condition (ii) is equivalent to df_x being invertible, as $(v, df_x v) \in T(M \times \{f(x)\})$ implies $df_x v = 0$. In particular, a diffeomorphism of M satisfies (ii).

Let f be transverse to the cut locus, and assume for simplicity that $\mathrm{Far}(f) = \{x\}$. Then B_δ is an ϵ-ball $B_\epsilon(x)$ around x, and for $\mathrm{MQ}_\Delta = \mathrm{MQ}^0_\Delta$, we get as in (6)

$$\begin{aligned} L(f) &= \lim_{\epsilon\to 0}\lim_{t\to 0}\int_{M\setminus B_\epsilon(x)} (\mathrm{Id}, f_t)^*\mathrm{MQ}_\Delta + \lim_{\epsilon\to 0}\lim_{t\to 0}\int_{B_\epsilon(x)} (\mathrm{Id}, f_t)^*\mathrm{MQ}_\Delta \\ &= \lim_{\epsilon\to 0}\int_{M\setminus B_\epsilon(x)} \mathrm{Pf}(\Omega) + \lim_{\epsilon\to 0}\lim_{t\to 0}\int_{B_\epsilon(x)} (\mathrm{Id}, f_t)^*\mathrm{MQ}_\Delta \qquad (7) \\ &= \chi(M) + \lim_{\epsilon\to 0}\lim_{t\to 0}\int_{B_\epsilon(x)} (\mathrm{Id}, f_t)^*\mathrm{MQ}_\Delta, \end{aligned}$$

by Lemma 2.5.

Since f is transverse to the cut locus, df_x is inverible, so we may assume that $\Gamma|_{B_\epsilon(x)}$ is a graph over a neighborhood $U_\epsilon \subset \{x\} \times M$ containing $(x, f(x))$. For fixed $\epsilon' < \epsilon$, there exists $\delta = \delta(\epsilon')$ such that any δ perturbation of Γ in the C^1 topology is still a graph over a similar $U_{\epsilon'} \subset \{x\} \times M$. Also, for any sequence $\epsilon_n \to 0$, there exists a sequence $t_n \to 0$ such that the graph Γ_{t_n} of f_{t_n} is a $\delta_n = \delta(\epsilon_n)$ perturbation of Γ. Thus there is a set $U_n \subset f_{t_n}(B_{\epsilon_n}(x))$ such that Γ_{t_n} is a graph over U_n. Set $W_n = (f_{t_n})^{-1}(U_n) \cap B_{\epsilon_n}(x)$. Then

$$\begin{aligned} \lim_{\epsilon\to 0}\lim_{t\to 0}\int_{B_\epsilon(x)} (\mathrm{Id}, f_t)^*\mathrm{MQ}_\Delta &= \lim_{n\to\infty}\lim_{t\to 0}\Big[\int_{B_{\epsilon_n}(x)\setminus W_n} (\mathrm{Id}, f_t)^*\mathrm{MQ}_\Delta \\ &\qquad + \int_{W_n} (\mathrm{Id}, f_t)^*\mathrm{MQ}_\Delta\Big] \\ &= \lim_{n\to\infty}\int_{B_{\epsilon_n}(x)\setminus W_n} \mathrm{Pf}(\Omega) \qquad (8) \\ &\qquad + \lim_{n\to\infty}\lim_{t\to 0}\int_{W_n} (\mathrm{Id}, f_t)^*\mathrm{MQ}_\Delta \\ &= \lim_{n\to\infty}\lim_{t\to 0}\int_{W_n} (\mathrm{Id}, f_t)^*\mathrm{MQ}_\Delta. \end{aligned}$$

In summary, $L(f) - \chi(M)$ equals $\int_{\Gamma_t|_{W_n}} \mathrm{MQ}_\Delta$ in the limit as $n \to \infty, t \to 0$. (See Figure 3.)

The next technical lemma replaces f_t by a family of maps g_t deforming $f(y)$ towards x rather than towards y, for x, y close.

Lemma 2.6 *For $\mu > 0$, there exists a neighborhood $U = U_\mu$ of x such that for all $y_0 \in U$, there exists a unique minimal geodesic $\gamma_{f(y_0),x}$ from $f(y_0)$ to x which is μ close in the C^1 topology to the unique minimal geodesic $\gamma_{f(y_0),y_0}$ from $f(y_0)$ to y_0.*

PROOF: The lemma is obvious unless $f(y_0) \in \mathcal{C}_x$. In general, fix y_0 close to x and let y denote a point on the minimal geodesic from y_0 to x. Since $f(y_0) \notin \mathbf{C}_{y_0}$, we have $y_0 \notin \mathcal{C}_{f(y_0)}$, and in particular y_0 is not in the conjugate locus of $f(y_0)$. Thus the exponential map $\exp_{f(y_0)} : T_{f(y_0)}M \to M$ surjects onto some neighborhood of y_0. For y close to y_0, there is a unique minimal geodesic $\gamma_{f(y_0),y}$ from $f(y_0)$ to y, and the family of such geodesics is C^1 close. Now take a curve γ_ϵ which is a smoothed approximation to $\gamma_{f(y_0),y}$ followed by the minimal geodesic from y to x such that the length of γ_ϵ satisfies $\ell(\gamma_\epsilon) \leq d(f(y_0), y) + d(y, x) + \epsilon$. Parametrizing all curves by arclength, we see that for y_0 close enough to x, the new family of curves is still C^1 close. By the Ascoli theorem, a subsequence of this family converges in C^0 as $y \to x$ and as $\epsilon \to 0$ to a curve $\gamma_{f(y_0),x}$ from $f(y_0)$ to x of length $d(f(y_0), x)$–i.e. $\gamma_{f(y_0),x}$ is a minimal geodesic from $f(y_0)$ to x. Since $\gamma_{f(y_0),x}$ is smooth and since the tangent vectors $\exp^{-1}_{f(y_0)} y$ lie on the unit sphere in $T_{f(y_0)}M$, it follows easily that a subsequence of $\exp^{-1}_{f(y_0)} y$ converges to a vector v with $\exp_{f(y_0)}(sv) = \gamma_{f(y_0),x}$. By the smooth dependence of geodesics on initial conditions, the minimal geodesic $\gamma_{f(y_0),x}$ is C^1 close to the minimal geodesic from $f(y_0)$ to y, and hence C^1 close to $\gamma_{f(y_0),y_0}$.

This shows that along any radial geodesic r centered at x, there exists a distance $\delta = \delta(r)$ such that if y is a point on r with $d(x, y) < \delta$, then there is a minimal geodesic from $f(y)$ to x which is μ close to the minimal geodesic from $f(y)$ to y in the C^1 topolgy. A similar argument shows that we may take δ to be a continuous function of the radial direction. □

Fix μ close to zero, and pick n large enough so that the lemma applies to all $y \in W_n$. Define a family of maps $g_t : W_n \to M$, $t \in (0, 1]$, which are approximations to f_t as follows. For $x \in \mathrm{Far}(f)$, set $g_t(x) = f_t(x) = f(x)$. For $y \notin \mathrm{Far}(f)$ and $v_y = \exp^{-1}_{f(y)} y$, define α_t by $f_t(y) = \exp_{f(y)}(\alpha_t v_y)$. Now set $g_t(y) = \exp_{f(y)}(\alpha_t v_x)$, where $\exp_{f(y)}(sv_x)$ is the minimal geodesic from $f(y)$ to x just constructed. By the smooth dependence of geodesics on parameters, we see that $\mu \to 0$, f_t is arbitrarily C^1 close to g_t for all $y \in W_n$ and for all

$t \in (0,1]$. This implies by (7), (8)

$$
\begin{aligned}
L(f) - \chi(M) &= \lim_{n\to\infty}\lim_{t\to 0}\int_{W_n}(\mathrm{Id}, f_t)^*\mathrm{MQ}_\Delta \\
&= \lim_{n\to\infty}\lim_{t\to 0}\int_{W_n}(\mathrm{Id}, g_t)^*\mathrm{MQ}_\Delta \\
&\qquad + \lim_{n\to\infty}\lim_{t\to 0}\int_{W_n}[(\mathrm{Id}, f_t)^* - (\mathrm{Id}, g_t)^*]\mathrm{MQ}_\Delta \\
&= \lim_{n\to\infty}\lim_{t\to 0}\int_{W_n}(\mathrm{Id}, g_t)^*\mathrm{MQ}_\Delta \qquad (9)\\
&\qquad + \lim_{n\to\infty}\int_{W_n}\lim_{t\to 0}[(\mathrm{Id}, f_t)^* - (\mathrm{Id}, g_t)^*]\mathrm{MQ}_\Delta \\
&= \lim_{n\to\infty}\lim_{t\to 0}\int_{W_n}(\mathrm{Id}, g_t)^*\mathrm{MQ}_\Delta .
\end{aligned}
$$

Let $TM^\uparrow$ denote $\{0\}\times TM \subset T(M\times M)|_\Delta$, and let $\mathrm{MQ}_{TM^\uparrow}$ denote the Mathai-Quillen form of $TM^\uparrow$, considered as a form supported on the cut locus tubular neighborhood. Thus $\mathrm{MQ}_{TM^\uparrow} = (\exp^{-1})^*\beta^*\mathrm{MQ}$, where MQ is the Mathai-Quillen form on $TM^\uparrow$, exp is the exponential map from $TM^\uparrow$ to $M\times M$, and β is a homeomorphism from the neighborhood of zero in $TM^\uparrow$ with fiber $\exp_x^{-1}(M\setminus \mathcal{C}_x)$ to TM; this uses Lemma 2.4 (iii). As before, β is a limit of diffeomorphisms, and because of the decay of MQ, β may be treated as a diffeomorphism.

Note that $p_2^*\mathrm{MQ}_{TM^\uparrow} = \mathrm{MQ}_\Delta$, since (i) $p_2^*\mathrm{MQ}_{TM^\uparrow}$ is closed and (ii) for a fiber $F = \{(\exp_x(-v), \exp_x v) : v \in N_x\}$ of the cut locus tubular neighborhood,

$$
\begin{aligned}
\int_F p_2^*\mathrm{MQ}_{TM^\uparrow} &= \int_{p_2 F}\mathrm{MQ}_{TM^\uparrow} = \int_{M\setminus\mathcal{C}_x}\mathrm{MQ}_{TM^\uparrow} \\
&= \int_{\beta\exp_x^{-1}(M\setminus\mathcal{C}_x)}\mathrm{MQ} = \int_{T_xM^\uparrow}\mathrm{MQ} = 1. \qquad (10)
\end{aligned}
$$

Since $\Gamma_{t_n}|_{W_n}$ is a graph over U_n, the projection $p_2 : M\times M \to M$ onto the second factor restricts to a diffeomorphism $p_2 : \Gamma_{t_n}|_{W_n} \to U_n$ with sign sgn $p_2 = \pm 1$, which equals the sign of $p_2 : \Gamma \to M$ at x. Thus

$$
\begin{aligned}
\int_{W_n}(\mathrm{Id}, g_t)^*\mathrm{MQ}_\Delta &= (\mathrm{sgn}\ p_2)\int_{p_2(\mathrm{Id},g_t)W_n}(p_2^{-1})^*\mathrm{MQ}_\Delta \\
&= (\mathrm{sgn}\ p_2)\int_{\beta\exp_\cdot^{-1} g_t(W_n)}\mathrm{MQ} \qquad (11)
\end{aligned}
$$

where exp. denotes the exponential map from $TM^{\uparrow}|_{W_n}$ to $M \times M$. By (9), (11),

$$L(f) - \chi(M) = (\mathrm{sgn}\ p_2) \lim_{n\to\infty} \lim_{t\to 0} \int_{\beta \exp_x^{-1} g_t(W_n)} \mathrm{MQ} \equiv \alpha_x, \tag{12}$$

since $\exp_q^{-1}$ is C^1 close to $\exp_x^{-1}$ for q close to x.

$|\alpha_x|$ can be computed as follows. As $t \to 0$, $\beta \exp_x^{-1} g_t(W_n)$ approaches a cone in T_xM with cross section $\exp_x^{-1}(f(W_n)\cap\mathcal{C}_x)$, since $g_0(y) = x$ unless $f(y) \in \mathcal{C}_x$. (See Figure 4.) Thus in the $n \to \infty$ limit, $|\alpha_x|$ is given by integrating MQ over a cone with cross section $\beta \exp_x^{-1}(f(x))$. Radially project this cross section to a region Z_x on the unit sphere $S_x \subset T_xM$. Since the Mathai-Quillen form is radially symmetric in the fiber T_xM, we have

$$|\alpha_x| = \frac{|Z_x|}{|S_x|}, \tag{13}$$

where $|\cdot|$ is the $(n-1)$-dimensional measure on S_x. Z_x is also the radial projection of $\exp_x^{-1} f(x)$ onto S_x.

Theorem 2.7 (The far point formula) *Assume that f is transverse to the cut locus. For $x_i \in \mathrm{Far}(f)$, let Z_i be the projection of $\exp_{x_i}^{-1}(f(x_i))$ onto the unit sphere $S_i \subset T_{x_i}M$. Let $p_2 : \Gamma \to M \times M$ be the projection from the graph of f onto the second factor with sign p_i at x_i. Then*

$$L(f) - \chi(M) = \sum_i p_i \frac{|Z_i|}{|S_i|}.$$

In particular,

$$|L(f) - \chi(M)| \leq |\mathrm{Far}(f)| \tag{14}$$

for f transverse to the cut locus and for diffeomorphisms.

PROOF: For f transverse to the cut locus, (14) follows from $|Z_i|/|S_i| \leq 1$. A diffeomorphism satisfies condition (ii) for being transverse to the cut locus, so either condition (i) holds or $|\mathrm{Far}(f)| = \infty$. $\square$

Remark: (14) is trivially sharp for $f = \mathrm{Id}$. For any map on S^n with the standard metric, $\exp_{x_i}^{-1}(f(x_i))$ is a sphere in $T_{x_i}S^n$, so $Z_i = S_i$. Thus $L(f) -$

$\chi(S^n) = \sum_i p_i$. The maps $f : z \mapsto z^n$ on S^1 give examples where (14) is sharp for arbitrary $L(f)$. The two point suspension of f to S^2 is transverse to the cut locus and gives equality in (14), and iterating this procedure gives sharp maps in all dimensions.

In fact, nontrivial sharp maps exist only on spheres. Recall that $\exp_x^{-1} g_t(y)$ lies on the radial line joining $\exp_x^{-1} y$ to 0 in T_xM. By its definition, $|\alpha_x| = 1$ iff $\exp_x^{-1}(\mathrm{Id}, f)B_\epsilon(x)$ contains an interior collar of the cut locus in T_xM, as only in this case will $\lim_{n\to\infty} \lim_{t\to 0} \beta \exp_x^{-1}(\mathrm{Id}, g_t)W_n = T_xM^\dagger$. Letting ϵ shrink, we see that this collar condition occurs only if the cut locus of x in M is contained in an arbitrary neighborhood of $f(x)$–i.e. the cut locus of x in M is precisely $f(x)$. Thus M is homeomorphic to the one point compactification of $\exp_x^{-1}(M \setminus \{f(x)\})$, so $M \approx S^n$.

Corollary 2.8 *(i) Let $f : M \to M$ be a smooth map which is transverse to the cut locus. If $\mathrm{Far}(f) \neq \emptyset$ and $|L(f) - \chi(M)| = |\mathrm{Far}(f)|$, then M is homeomorphic to S^n.*

(ii) Let $f : M \to M$ be a smooth map which is Lefschetz (i.e. Γ is transverse to Δ) and transverse to the cut locus. Let $\mathrm{Fix}(f)$ be the fixed point set of f. Then

$$|\chi(M)| \leq |\mathrm{Fix}(f)| + |\mathrm{Far}(f)|,$$

with strict inequality if M is not homeomorphic to S^n and $\mathrm{Far}(f) \neq \emptyset$.

(ii) follows from $|\chi(M)| \leq |L(f) - \chi(M)| + |L(f)|$ and the Lefschetz fixed point theorem. As an application of (i), let f be transverse to the cut locus and have $L(f) \neq \chi(M)$. Then if $M \not\approx S^n$, either $\mathrm{Far}(f) = \emptyset$ or $|\mathrm{Far}(f)| \geq 2$. This fails on S^n for the suspension of $z \mapsto z^2$.

In general, we expect $\exp_x^{-1}(f(x))$ to be a small subset of $\exp_x^{-1}(\mathcal{C}_x)$, forcing $|Z_x|$ to be zero.

Definition: For $x, y \in M$, let Z_y be the radial projection of $\exp_x^{-1}(y) \cap \exp_x^{-1}(\mathcal{C}_x)$ onto the unit sphere in T_xM. The metric on M is *somewhere (resp. nowhere) spherelike* if $|Z_y| \neq 0$ for some x, y (resp. $|Z_y| = 0$ for all x, y).

A typical metric is nowhere spherelike, but it is easy to construct a somewhere sphere-like metric on any M, by considering M as the connect sum $M\#S^n$. Thus for any $f : M \to M$ and fixed $x \in M$, there is a metric on M such that $x \in \mathrm{Far}(f)$ and $|Z_x|/|S_x| \in [0, 1)$ is arbitrary.

Theorem 2.9 *(i) A metric of non-positive curvature on a manifold of dimension greater than one is nowhere spherelike.*

(ii) Let $f : M \to M$ be a diffeomorphism with $L(f) \neq \chi(M)$. If M is nowhere spherelike (viz. if M has non-positive curvature), then $|\mathrm{Far}(f)| = \infty$.

PROOF: (i) The map $\exp_x$ is a covering map for metrics of non-positive curvature, so the inverse image of any $y \in M$ is discrete in T_xM.

(ii) If f is not transverse to the cut locus, then $0 \neq L(f) - \chi(M) = \sum_{\mathrm{Far}(f)} 0$. Since condition (ii) is satisfied, (i) must fail. Thus $|\mathrm{Far}(f)| = \infty$. □

Examples: (i) This result applies to local diffeomorphisms. For example, on the flat torus T^2 with coordinates (θ, ψ), the maps $f(\theta, \psi) = (n\theta, m\psi)$, $(n, m) \in \mathbf{Z}^2$, is a local diffeomorphism with $L(f) = (1 - n)(1 - m)$, so for $n, m \neq 1$, $|\mathrm{Far}(f)| = \infty$. In fact, it is easy to check that $\mathrm{Far}(f)$ is a union of lines.

(ii) Let Σ^g be a genus $g > 1$ surface arranged symmetrically about a plane passing through the g holes. The diffeomorphism $f : \Sigma^g \to \Sigma^g$ given by reflection through this plane has $L(f) = 0$. For example, Σ^g can be the hyperelliptic curve $y^2 = \prod_{i=1}^{2g+1}(x - a_i)$ with a_i real and distinct, with f the involution $(x, y) \mapsto (x, -y)$. For any metric on Σ^g, either (i) f is not transverse to the cut locus and so $|\mathrm{Far}(f)| = \infty$, or (ii) the metric is somewhere sphere-like and $|\mathrm{Far}(f)| > 2g - 2$. Thus for any metric, $|\mathrm{Far}(f)| > 2g - 2$, and for most metrics (e.g. metrics of constant negative curvature) $\mathrm{Far}(f)$ is infinite.

(iii) Let $f : S^2 \to S^2$ be a holomorphic map of degree n. Then f is a branched covering and so Γ is transverse to $M \times \{f(x)\}$ except at the branch points B. As above, for any metric on S^2 with $f^{-1}(B) \cap \mathrm{Far}(f) = \emptyset$, we have $|\mathrm{Far}(f)| \geq |L(f) - \chi(S^2)| = n - 1$.

If $\mathrm{Far}(f) = \{x_1, \ldots, x_n\}$ is finite, there is an alternative way to express $L(f) - \chi(M)$ in terms of $\mathrm{Far}(f)$. Since the cut locus tubular neighborhood T is homeomorphic to TM, and diffeomorphic away from ∂T, we can consider f as a smooth vector field V_f on M with singularities at the x_i by setting $V_f(y) = \exp_y^{-1} f(y)$. At each fixed point x of f, the local Lefschetz number $L_x(f)$ equals the Hopf index $\mathrm{ind}_x(V_f)$ of V_f [9, p. 135]. We modify V_f by multiplying the vectors in a neighborhood of each x_i by a smooth function which is one on the boundary of the neighborhood and which vanishes to sufficiently high order at x_i. The modified vector field V_f' extends to a smooth vector field, also denoted V_f', on all of M with zeros at the fixed points of f and at the x_i. By the Hopf index formula,

$$\chi(M) = \sum_{\{x:V_f'(x)=0\}} \mathrm{ind}_x(V_f') = \sum_{\{x:f(x)=x\}} L_x(f) + \sum_i \mathrm{ind}_{x_i}(V_f'),$$

and so

$$L(f) - \chi(M) = -\sum_i \mathrm{ind}_{x_i}(V_f'). \tag{15}$$

By Theorem 2.7 and (15), a function transverse to the cut locus satisfies the curious equation

$$\sum_i p_i \frac{|Z_i|}{|S_i|} = -\sum_i \mathrm{ind}_{x_i}(V_f').$$

Since $|Z_i|/|S_i| \in [0,1)$ if M is not homeomorphic to S^n and $\mathrm{ind}_{x_i}(V_f') \in \mathbf{Z}$, this equation cannot hold if

$$\frac{p_i}{\mathrm{sgn\ ind}_{x_i}(V_f')} = \pm 1 \tag{16}$$

is independent of i. In particular, a local diffeomorphism satisfying (16) must have $|\mathrm{Far}(f)| = \infty$.

The following result gives a condition which guarantees (16). For a fixed $x \in M$, $\mathcal{C}_x$ is a closed subset of M of Hausdorff dimension at most $n-1$, since it is the pushforward of a set of Hausdorff dimension at most $n-1$ and since it is the complement of the image of an open set under $\exp_x$. Thus a small perturbation $f_1 = f_{1,x}$ of f will have $f_1(y) \notin \mathcal{C}_x$ for all y close to x, and for these y the map f_1 can be lifted to a map f' on a neighborhood V of $0 \in T_xM$:

$$f' : V \to T_xM, \quad f'(v) = \exp_x^{-1}(f_1(\exp_x v)).$$

Moreover, we may assume that df_1 is invertible at x and hence df' is invertible at 0.

Lemma 2.10 *If the perturbation $f_{1,x}$ can be chosen so that* $\mathrm{sgn}\det(\mathrm{Id} - (df_0')^{-1})$ *is constant for all $x \in M$, then (16) holds.*

Corollary 2.11 *(i) If M is not homeomorphic to S^n and if for a fixed metric g and diffeomorphism f, $\operatorname{sgn}\det(\mathrm{Id}-(df_0')^{-1})$ has constant sign for all $x \in M$, then either $|\mathrm{Far}(\tilde f)| = 0$ or $|\mathrm{Far}(\tilde f)| = \infty$ for all metrics sufficiently close to g and functions $\tilde f$ sufficiently close to f.*

(ii) An expanding map on a flat manifold with $L(f) \neq \chi(M)$ has $\operatorname{sgn}\det(\mathrm{Id} - (df_0')^{-1})$ of constant sign for all $x \in M$. In particular, transversality to the cut locus is not a generic condition.

For (ii), because the definition of f' depends continuously on g, the lemma and hence the corollary apply to metrics close to g and functions close to f.

Proof of the lemma: We use a prime to indicate lifts of objects to T_xM: e.g. $y' = \exp_x^{-1} y$, $e_i' = d(\exp_x^{-1})_y e_i$ for $e_i \in T_yM$. For simplicity of notation, we just denote f_1 by f. The index of V_f' at x can be computed in the coordinate patch $\exp_x^{-1}(M \setminus \mathcal{C}_x)$, so $\mathrm{ind}_x(V_f')$ equals degree of the map

$$\alpha : S' \subset T_xM \to T_{\cdot}T_xM \simeq T_xM,$$

$$y' = \exp_x^{-1} y \mapsto d(\exp_x^{-1})_y(\exp_y^{-1}(f(y))),$$

for S' a small sphere around $0 \in T_xM$. We use the fact that the degree of $f : \mathbf{R}^n \setminus \{0\} \to \mathbf{R}^n \setminus \{0\}$ equals the degree of $f/|f|$.

The degree of α is computed by the signed sum of the preimages $\{y_k'\} = \alpha^{-1}(z)$ for a generic $z \in \mathrm{Im}(\alpha)$, with the sign at $y' = y_k'$ the sign of the determinant of $d\alpha_{y'}$. Taking an orthonormal frame $e_1', \ldots, e_{n-1}'$ of S' at y', we need to compute the determinant of the vectors

$$d\alpha(e_i') = \left.\frac{d}{dt}\right|_{t=0} d(\exp_x^{-1})_{\exp_x(y'+te_i')} \exp^{-1}_{\exp_x(y'+te_i')} f(\exp_x(y'+te_i')) - te_i',$$

where $-te_i'$ is included to shift all vectors back to $T_{y'}T_xM$. Thus

$$d\alpha(e_i') = \left.\frac{d}{dt}\right|_{t=0} d(\exp_x^{-1})_{\exp_x(y'+te_i')} \exp^{-1}_{\exp_x(y'+te_i')} \exp_x f'(y'+te_i') - te_i'.$$

Since $d(\exp_x^{-1})_{\exp_x(y'+te_i')} \exp^{-1}_{\exp_x(y'+te_i')} \exp_x \to \mathrm{Id}$ as $y' \to 0$, $d\alpha(e_i')$ is closely approximated by

$$\left.\frac{d}{dt}\right|_{t=0} f'(y'+te_i') - te_i' = (df' - \mathrm{Id})(e_i').$$

Thus the sign of the determinant of $d\alpha_{y'}$ is the sign of the determinant of $df' - \mathrm{Id}$ on S'.

To compute p_i, we extend the frame on S' to $e'_1, \ldots, e'_n$ on $T_{y'}T_xM$ with e'_n the unit outward pointing vector. Then p_i is the sign of the determinant of $df'(e'_1), \ldots, df'(e'_n)$ with respect to an oriented frame $\{h'_i\}$ at $f'(y')$ which has $h'_1, \ldots, h'_{n-1}$ tangent to $f'(S')$ and with h'_n outward pointing. df' takes the form

$$\begin{pmatrix} df'(e'_i) & * \\ 0 & \langle df'(e'_n), h'_n \rangle \end{pmatrix}$$

where the upper left block has dimensions $(n-1)\times(n-1)$. Since e'_n is pointing in a direction of increasing distance from $0 \in T_xM$, $df'(e'_n)$ is pointing away from the cut locus in the tangent space, and so the lower right entry is negative. Thus $p_i = -\mathrm{sgn}\det(df'(e'_i))_{(n-1)\times(n-1)}$.

Since $\det(df' - \mathrm{Id}) = \det(df')\det(\mathrm{Id} - (df')^{-1})$, we have $p_i = -\mathrm{sgn}\det(df') = \mathrm{sgn}\det(df' - \mathrm{Id}) = \mathrm{sgn}\det d\alpha$ iff $\mathrm{sgn}\det(\mathrm{Id} - (df')^{-1})|_{S'} = -1$ and $p_i = -\mathrm{sgn}\det d\alpha$ iff $\mathrm{sgn}\det(\mathrm{Id} - (df')^{-1})|_{S'} = 1$. Under the hypothesis that sgn det $(\mathrm{Id} - (df'_0)^{-1})$ is constant in x, $\mathrm{sgn}\det(\mathrm{Id} - (df')^{-1})|_{S'}$ is also constant for small S'. □

2.3 A current on the far point set

We remarked at the beginning of the last subsection that $L(f) - \chi(M)$ is controlled by the far point set and made this statement more precise for functions transverse for the cut locus. In this subsection, we will treat general functions and find a singular current supported on $\mathrm{Far}(f)$ whose singular part evaluated at the function 1 gives $L(f) - \chi(M)$.

As in (6),

$$L(f) = \int_{M\setminus \mathrm{Far}(f)} \mathrm{Pf}(\Omega) + \lim_{\delta\to 1}\lim_{t\to 0}\int_{B_\delta} (\mathrm{Id}, f_t)^*\mathrm{MQ}^0_\Delta, \tag{17}$$

where we have replaced T in (6) with the cut locus tubular neighborhood and MQ_Δ with the corresponding MQ^0_Δ. Since $\mathrm{Far}(f)$ is closed, the first integral exists. By the Chern-Gauss-Bonnet theorem, we get

$$L(f) = \chi(M) - \int_{\mathrm{Far}(f)} \mathrm{Pf}(\Omega) + \lim_{\delta\to 1}\lim_{t\to 0}\int_M \chi_{B_\delta}\cdot(\mathrm{Id}, f_t)^*\mathrm{MQ}^0_\Delta. \tag{18}$$

We now define zero currents L^{f_t}, E, on M via their action on $g \in C^\infty(M)$:

$$\begin{aligned} L^{f_t}(g) &= \int_M g \cdot (\mathrm{Id}, f_t)^* \mathrm{MQ}^0_\Delta, \\ E(g) &= \int_M g \cdot \mathrm{Pf}(\Omega). \end{aligned}$$

We also set

$$\mathcal{C}^f(g) = -\int_{\mathrm{Far}(f)} g \cdot \mathrm{Pf}(\Omega) + \lim_{\delta \to 1} \lim_{t \to 0} \int_M g \cdot \chi_{B_\delta} \cdot (\mathrm{Id}, f_t)^* \mathrm{MQ}^0_\Delta,$$

whenever the right hand side exists. We define the limit of currents by pointwise convergence: $\lim_{t\to 0} L^{f_t} = L^0$ if $\lim_{t \to 0} L^{f_t}(g) = L^0(g)$ for all smooth g.

Lemma 2.12 *As a current,* $(\lim_{t\to 0} L^{f_t}) - \mathcal{C}^f$ *exists and equals* E*. In particular,* $\lim_{t\to 0} L^{f_t}(g)$ *exists whenever* $\mathrm{supp}\ g \cap \mathrm{Far}(f) = \emptyset$.

PROOF: We have

$$L^{f_t}(g) = \int_M g \cdot (\mathrm{Id}, f_t)^* \mathrm{MQ}^0_\Delta = \int_{A_\delta} g \cdot (\mathrm{Id}, f_t)^* \mathrm{MQ}^0_\Delta + \int_{B_\delta} g \cdot (\mathrm{Id}, f_t)^* \mathrm{MQ}^0_\Delta,$$

and so

$$\begin{aligned} \lim_{t\to 0} L^{f_t}(g) - \lim_{\delta \to 1} \lim_{t \to 0} \int_{B_\delta} g \cdot (\mathrm{Id}, f_t)^* \mathrm{MQ}^0_\Delta &= \int_{M \setminus \mathrm{Far}(f)} g \cdot \mathrm{Pf}(\Omega) \\ &= E(g) - \int_{\mathrm{Far}(f)} g \cdot \mathrm{Pf}(\Omega), \end{aligned}$$

as in (17). This gives the first statement. For the second statement, if $\mathrm{supp}\ g \cap \mathrm{Far}(f) = \emptyset$, then $g \cdot \chi_{B_\delta} = 0$ for $\delta \approx 1$, and so $\mathcal{C}^f(g) = 0$ for such g. □

In view of this lemma, we think of L^0 as a singular current, with $\mathcal{C}^f$ the singular part of L^0 and E the finite part. Note that $L^{f_t}(1) = L(f)$ for all t. This gives:

Proposition 2.13 *For a smooth function* $f : M \to M$ *on a closed, oriented, Riemannian manifold, there exists a canonical singular part* $\mathcal{C}^f$ *to* $L^0 = \lim_{t\to 0} L^{f_t}$*, with supp* $\mathcal{C}^f \subset \mathrm{Far}(f)$*. Moreover, we have*

$$L(f) - \chi(M) = \mathcal{C}^f(1).$$

Remarks:

(1) This discussion can be carried over to the degree of f, defined by $\deg(f) = \int_M f^*\omega / \int_M \omega$ for any top degree form ω. By its homotpy invariance, the degree of f is one if the graph of f never intersects $\mathcal{C}_x$, so we expect that a singular 0-current, with singular part supported on $\mathrm{Far}(f)$, computes $\deg(f)-1$. Taking ω to be the volume form dvol of the Riemannian metric on M, we get

$$\begin{aligned}
\mathrm{vol}(M)\cdot\deg(f) &= \lim_{t\to 0}\int_M (f_t)^*\mathrm{dvol} \\
&= \int_{M\setminus \mathrm{Far}(f)} \mathrm{dvol} + \lim_{\delta\to 1}\lim_{t\to 0}\int_M \chi_{B_\delta}(f_t)^*\mathrm{dvol}.
\end{aligned}$$

Setting

$$\mathcal{D}^f(g) = -\int_{\mathrm{Far}(f)} g\cdot \mathrm{dvol} + \lim_{\delta\to 1}\lim_{t\to 0}\int_M g\cdot\chi_{B_\delta}\cdot (f_t)^*\mathrm{dvol},$$

whenever the right hand side exists, we see that the support of $\mathcal{D}^f$ is contained in $\mathrm{Far}(f)$, and that

$$\deg(f) - 1 = \frac{\mathcal{D}^f(1)}{\mathrm{vol}(M)}.$$

(2) In (1) and in the previous section, f is compared to (the homotopy class of) the identity map. We can also compare f to a constant map $c(x) = x_0$. In this case the Lefschetz number (resp. degree) of f is 1 (resp. 0) if the graph of f misses $\mathbf{C}_{x_0}$. Again there are singular 0-currents, with singular part supported on $\mathbf{C}_{x_0}$, which measure $L(f)-1$ and $\deg(f)$.

(3) Finally, f can also be compared to a fixed map f_0. We obtain

$$L(f) - L(f_0) = \mathcal{E}^{f,f_0}(1),$$

where the singular current $\mathcal{E}^{f,f_0}$ has singular part supported on $\{x : f_0(x) \in \mathcal{C}_{f(x)}\}$. There is a similar result for degrees. As a well known example, if $M = S^n$ and f, f_0 have different Lefschetz numbers (equiv. different degrees), then there exists $x\in S^n$ such that $f(x), f_0(x)$ are antipodal.

(4) As in the introduction to §2, for a vector field s we have $\chi(M) = \int_M (ts)^*\mathrm{MQ}_{TM}$ for all t. Let B_ϵ be the ϵ-neighborhood of the zero set of s. Then by the uniform decay of $(ts)^*\mathrm{MQ}_{TM}$ off B_ϵ,

$$\chi(M) = \lim_{\epsilon\to 0}\lim_{t\to\infty}\int_{B_\epsilon}(ts)^*\mathrm{MQ}_{TM}.$$

Define a family of Euler currents by

$$E_s^t(g) = \int_M g \cdot (ts)^* \mathrm{MQ}_{TM}.$$

Then $E_s^t(1) = \chi(M)$ and

$$\lim_{t\to\infty} E_s^t = \lim_{\epsilon\to 0}\lim_{t\to\infty} \int_{B_\epsilon} (ts)^* \mathrm{MQ}_{TM}$$

as zero currents. Thus the singular current $\lim_{t\to\infty} E_s^t$ is supported on the zero set of s. If the zero set consists of nondegenerate points $x_1, \ldots, x_n$, the singular part vanishes, and

$$\lim_{t\to\infty} E_s^t = \sum_i \pm\delta_{x_i},$$

as in [12, §8]. The sign is determined by the Hopf index at x_i, which recovers the Hopf index formula at $g = 1$.

3 Topological deformations of the Lefschetz integral

In §2 the $t \to 0$ limit of the geometric formula for the Lefschetz number was studied. In this section, we show by topological arguments that the $t \to \infty$ behavior gives a new proof of the Lefschetz fixed point/submanifold formula. (We use the geometry of geodesics only to define tubular neighborhoods, but this can be replaced by a jet bundle argument.)

In this section, ν_X^Y denotes the normal bundle of X in Y.

To state the fixed submanifold theorem, let $f : M \to M$ be a smooth map of a closed oriented m-manifold M, and assume that the fixed point set of f consists of the disjoint union of smooth submanifolds N_j of dimension n_j. Let N be one such component, and let ν be the quotient bundle $\nu = TM/TN$ over N. Since df preserves the subbundle TN, it induces a map df_ν on ν.

We assume the non-degeneracy condition $\det(\mathrm{Id} - df_\nu) \neq 0$ (also known as clean intersection), i.e. f leaves infinitesimally fixed only directions tangent to N.

If we put df_n, $n \in N$, in Jordan canonical form, TN will be the span of eigenvectors with eigenvalues 1, and ν is isomorphic to the span of the generalized eigenvectors for the remaining eigenvalues. This induces a natural splitting of $TM\big|_N \simeq TN \oplus \nu$. A choice of Riemannian metric on M gives an identification of ν with ν_N^M, and df_ν with a map on ν_N^M.

Theorem 3.1 *Let $f : M \to M$ be a smooth non-degenerate map of a closed oriented m-manifold M, whose fixed point set consists of the disjoint union of submanifolds $N_1, N_2, ..., N_r$. Then*

$$L(f) = \sum_{j=1}^{r} \mathrm{sgn}(\det(\mathrm{Id} - df_\nu))\chi(N_j).$$

For simplicity, we assume that the fixed point set of f consists of a single submanifold N of dimension n. Let Δ_M^ϵ be an ϵ-neighborhood of the diagonal Δ_M $M \times M$. Choose $\epsilon > 0$ small enough so there exists a unique minimal geodesic from x to y, for all $(x, y) \in \Delta_M^\epsilon$.

For technical reasons, rather than using the family f_t of functions of §2.2, we construct a family of diffeomorphisms $F_t : M \times M \to M \times M$, for $t > 0$, with $F_1 = \mathrm{Id}$, which pushes out fibers of $\nu_{\Delta_M}^{M\times M}$, while fixing Δ_M and $M \times M - \Delta_M^\epsilon$. Let $(x, y) \in \Delta_M^\epsilon$ and consider the geodesic γ in $M \times M$ from $(\bar{x}, \bar{x}) \in \Delta_M$ to (x, y), where $\bar{x}$ is the midpoint of the geodesic α between x and y in M. Setting $\dot{\alpha}(\bar{x}) = v = v(x, y) \in T_{\bar{x}}M$, we have $\dot{\gamma}(\bar{x}, \bar{x}) = (-v, v)$. For $v \neq 0$, define a diffeomorphism $\lambda_v(t) : [0, \infty) \to [0, \frac{\epsilon}{|v|})$ with $\lambda_v(1) = 1$, which is smooth in v, and set $\lambda_0(t) = 0$. Define $F_t : M \times M \to M \times M$ by:

$$F_t(x, y) = \begin{cases} (x, y), & (x, y) \notin \Delta_M^\epsilon, \\ \exp_{(\bar{x},\bar{x})}(\lambda_{v(x,y)}(t) \cdot \exp^{-1}_{(\bar{x},\bar{x})}(x, y)), & (x, y) \in \Delta_M^\epsilon. \end{cases}$$

As in §2.1, we have

$$L(f) = (-1)^{\dim M} \int_\Gamma \eta_{\Delta_M}^{M\times M},$$

where $\eta_{\Delta_M}^{M\times M}$ is the Poincaré dual of Δ_M in $M \times M$. Since F_t is homotopic to the identity,

$$\begin{aligned} (-1)^{\dim M} L(f) &= \int_\Gamma \eta_{\Delta_M}^{M\times M} = \lim_{t\to\infty} \int_\Gamma F_t^* \eta_{\Delta_M}^{M\times M} = \lim_{t\to\infty} \int_{(\mathrm{Id},f)(\Delta_M)} F_t^* \eta_{\Delta_M}^{M\times M} \\ &= \lim_{t\to\infty} \int_{(\mathrm{Id},f)(\Delta_N^\delta)} F_t^* \eta_{\Delta_M}^{M\times M} = \lim_{t\to\infty} \int_{F_t\circ(\mathrm{Id},f)(\Delta_N^\delta)} \eta_{\Delta_M}^{M\times M}, \quad (19) \end{aligned}$$

where Δ_N^δ is a δ-neighborhood of Δ_N in Δ_M, for δ small enough. This uses

$$\lim_{t\to\infty} \int_{(\mathrm{Id},f)(\Delta_M \setminus \Delta_N^\delta)} F_t^* \eta_{\Delta_M}^{M\times M} = 0,$$

as $F_t^*\eta_{\Delta_M}^{M\times M}$ decays uniformly as $t\to\infty$ on $\Delta_M\setminus\Delta_N^\delta$, since $d(x,f(x))$ and hence $|v|$ has positive minimum on M minus a δ-neighborhood of N.

Let $\pi:\Delta_N^\delta\to\Delta_N$ be the projection given by the identification of Δ_N^δ with a δ-neighborhood of the zero section of $\nu_{\Delta_N}^{\Delta_M}$.

The next lemma uses the non-degeneracy hypothesis.

Lemma 3.2 *If $df_\nu-\mathrm{Id}$ is invertible at $n\in N$, then*

$$T_{(n,n)}\Big[F_t\circ(\mathrm{Id},f)(\pi^{-1}(n,n))\Big]\cap T_{(n,n)}\Delta_N=\{0\}.$$

Proof: If $T_{(n,n)}\Big[F_t\circ(\mathrm{Id},f)(\pi^{-1}(n,n))\Big]\cap T_{(n,n)}\Delta_N\neq 0$, there exists $0\neq(q,q)\in T_{(n,n)}\pi^{-1}(n,n)$ with $q\perp N$ such that $dF_t(q,df_\nu q)=dF_t\circ(\mathrm{Id},df)(q,q)\in T_{(n,n)}\Delta_N$.

We split $(q,df_\nu q)\in T_{(n,n)}M\times M$ into its components in $T\Delta_M$ and in $\nu_{\Delta_M}^{M\times M}$. Since dF_t leaves vectors in $T\Delta_M$ unchanged and stretches vectors in the normal bundle by a λ factor, we get

$$\begin{aligned} dF_t(q,df_\nu q) &= dF_t\Big[\Big(\frac{q+df_\nu q}{2},\frac{q+df_\nu q}{2}\Big)+\Big(\frac{q-df_\nu q}{2},\frac{-q+df_\nu q}{2}\Big)\Big]\\ &= \Big(\frac{q+df_\nu q}{2},\frac{q+df_\nu q}{2}\Big)+\lambda(t)\Big(\frac{q-df_\nu q}{2},\frac{-q+df_\nu q}{2}\Big)\\ &= \Big(\frac{1+\lambda(t)}{2}q+\frac{1-\lambda(t)}{2}df_\nu q,\frac{1-\lambda(t)}{2}q+\frac{1+\lambda(t)}{2}df_\nu q\Big),\end{aligned}$$

for $\lambda(t)=\lambda_w(t)$ with $w=(q-df_\nu q)/2$. Note that by hypothesis, $w\neq 0$ and so $\lambda(t)\neq 0$.

We have $dF_t\circ(\mathrm{Id},df)(q,q)=(v,v)$ for some $v\in T_nN$, so

$$\Big(\frac{1+\lambda(t)}{2}q+\frac{1-\lambda(t)}{2}df_\nu q\Big)-\Big(\frac{1-\lambda(t)}{2}q+\frac{1+\lambda(t)}{2}df_\nu q\Big)=v-v=0.$$

This implies $\lambda(t)(\mathrm{Id}-df_\nu)q=0$. Since $q\neq 0$, $\lambda(t)\neq 0$, this contradicts that $\mathrm{Id}-df_\nu$ is invertible. $\square$

Define $E_{n,t}\subset T_{(n,n)}(M\times M)$ by

$$E_{n,t}=T_{(n,n)}\Big[F_t\circ(\mathrm{Id},f)(\pi^{-1}(n,n))\Big],$$

and note the decomposition

$$T(M\times M)\Big|_{\Delta_M} \simeq T\Delta_M \oplus \nu_{\Delta_M}^{M\times M} \simeq T\nu_{\Delta_M}^{M\times M}.$$

Let

$$\tilde{\pi} : T(M\times M)\Big|_{\Delta_M} \to \nu_{\Delta_M}^{M\times M}$$

be the projection to $\nu_{\Delta_M}^{M\times M}$. By Lemma 3.1, $\tilde{\pi}$ has no kernel on $E_{n,t}$ and hence is an isomorphism of $E_{n,t}$ to a vector subspace $H_{n,t} \subset \nu_{\Delta_M}^{M\times M}$. Let

$$\beta_{n,t} : E_{n,t} \to F_t \circ (\mathrm{Id}, f)(\pi^{-1}(n,n))$$

be the diffeomorphism given by the exponential map. Actually, $\beta_{n,t}$ is a diffeomorphism on a neighborhood of 0 in $E_{n,t}$, whose radius goes to infinity as $t \to \infty$.

Thus, $\tilde{\pi} \circ \beta_{n,t}^{-1} : F_t \circ (\mathrm{Id}, f)(\pi^{-1}(n,n)) \to \tilde{H}_{n,t} \subset H_{n,t}$ is a diffeomorphism onto its image $\tilde{H}_{n,t}$, where $\tilde{H}_{n,t}$ is an arbitrarily large ball in $H_{n,t}$, for large t. Then

$$\begin{aligned}
(-1)^{\dim M} L(f) &= \lim_{t\to\infty} \int_{F_t\circ(\mathrm{Id},f)(\Delta_N^\delta)} \eta_{\Delta_M}^{M\times M} \\
&= \lim_{t\to\infty} [\deg(\tilde{\pi}\circ\beta_t^{-1})^{-1}]^{-1} \\
&\qquad \cdot \int_{(\tilde{\pi}\circ\beta_t^{-1})F_t\circ(\mathrm{Id},f)(\Delta_N^\delta)} ((\tilde{\pi}\circ\beta_t^{-1})^{-1})^* \eta_{\Delta_M}^{M\times M} \\
&= \lim_{t\to\infty} \deg(\tilde{\pi}) \int_{\cup_n \tilde{H}_{n,t}} ((\tilde{\pi}\circ\beta_t^{-1})^{-1})^* \eta_{\Delta_M}^{M\times M},
\end{aligned}$$

where $\beta_t : \cup_n E_{n,t} \to F_t \circ (\mathrm{Id}, f)(\Delta_N^\delta)$ is given by $\beta_{n,t}$ on each $E_{n,t}$. This uses $\deg(\tilde{\pi}\circ\beta_t^{-1})^{-1} = (\deg\,\tilde{\pi})^{-1} = \deg\,\tilde{\pi}$, as β_t is an orientation preserving diffeomorhism and $\tilde{\pi}$ is an isomorphism.

Let $H_t = \cup_n H_{n,t}$ be the subbundle of $\nu_{\Delta_M}^{M\times M}$ over Δ_N with fiber $H_{n,t}$ over (n,n). We obtain

$$(-1)^{\dim M} L(f) = \lim_{t\to\infty} \deg(\tilde{\pi}) \int_{H_t} ((\tilde{\pi}\circ\beta_t^{-1})^{-1})^* \eta_{\Delta_M}^{M\times M}.$$

Since the $E_{n,t}$ are getting more "vertical" as $t \to \infty$, $\tilde{\pi}\circ\beta_t^{-1} \to \pm\mathrm{Id}$ and $H_{n,t} \to H_{n,\infty}$, where $H_{n,\infty}$ is the vector subspace of $(\nu_{\Delta_M}^{M\times M})_{(n,n)}$ spanned by the projection of vectors in $H_{n,t}$ into $\nu_{\Delta_M}^{M\times M}$, for any t. Set $H_\infty = \cup_n H_{n,\infty}$ with projection map $p : H_\infty \to \Delta_N$. Then H_∞ is also a subbundle of $\nu_{\Delta_M}^{M\times M} \to \Delta_N$, and

$$\begin{aligned}(-1)^{\dim M} L(f) &= \lim_{t\to\infty} \deg(\tilde\pi) \int_{H_t} ((\tilde\pi \circ \beta_t^{-1})^{-1})^* \eta_{\Delta M}^{M\times M} \\ &= \deg(\tilde\pi) \int_{H_\infty} \eta_{\Delta_M}^{M\times M}. \end{aligned} \tag{20}$$

By Theorem 2.1, $\eta_{\Delta_M}^{M\times M}$ and $\Phi(\nu_{\Delta_M}^{M\times M})$, the Thom class of $\nu_{\Delta_M}^{M\times M}$ considered as a class on $M \times M$, can be represented by the same form. Since we do not distinguish between the integral of a cohomology class and the integral of a representative form, we have

$$\begin{aligned}\int_{H_\infty} \eta_{\Delta_M}^{M\times M} &= \int_{H_\infty} \Phi(\nu_{\Delta_M}^{M\times M}) = \int_{H_\infty} \Phi(\nu_{\Delta_M}^{M\times M}) \wedge p^* 1 \\ &= \int_{\Delta_N} p_*\Phi(\nu_{\Delta_M}^{M\times M}) \wedge 1 = \int_{\Delta_N} p_*\Phi(\nu_{\Delta_M}^{M\times M}),\end{aligned} \tag{21}$$

where the push forward formula for integration over the fiber [1, Prop. 6.15] is used between the third and fourth terms.

Let $H_\infty^\perp$ be the orthogonal (or any) complement of H_∞ in $\nu_{\Delta_M}^{M\times M}$. By [1, Prop. 6.19], we have $\Phi(\nu_{\Delta_M}^{M\times M}) = \Phi(H_\infty) \wedge \Phi(H_\infty^\perp)$. It is easy to check that

$$p_*\Phi(\nu_{\Delta_M}^{M\times M}) = p_*(\Phi(H_\infty) \wedge \Phi(H_\infty^\perp)) = p_*(\Phi(H_\infty)) \wedge \Phi(H_\infty^\perp),$$

since $\Phi(H_\infty^\perp)$ vanishes in H_∞ directions. Thus (21) becomes

$$\int_{\Delta_N} p_*\Phi(\nu_{\Delta_M}^{M\times M}) = \int_{\Delta_N} p_*(\Phi(H_\infty)) \wedge \Phi(H_\infty^\perp) = \int_{\Delta_N} \Phi(H_\infty^\perp), \tag{22}$$

as $p_*\Phi(H_\infty) = 1$ since $\Phi(H_\infty)$ integrates to one in each fiber.

We claim that

$$\nu_{\Delta_M}^{M\times M}\Big|_{\Delta N} \simeq \nu_{\Delta_N}^{N\times N} \oplus \nu_{\Delta_N}^{\Delta_M}.$$

Indeed, the metric on M is chosen so that

$$TM\big|_N = TN \oplus \nu_N^M. \tag{23}$$

$\nu_{\Delta_N}^{N\times N}$ is isomorphic to TN by the map $(v, -v) \mapsto v$. Similarly, $\nu_{\Delta_M}^{M\times M} \simeq T\Delta_M \simeq TM$. Finally, we trivially have $\nu_{\Delta_N}^{\Delta_M} \simeq \nu_N^M$. Plugging these terms into (23) gives the claim.

Thus we have the bundle isomorphisms $\nu_{\Delta_N}^{\Delta_M} \simeq E_t \simeq H_t \simeq H_\infty$, and by the claim we have $H_\infty^\perp \simeq \nu_{\Delta_N}^{N\times N}$. By (21), (22), we have

$$\begin{aligned}
\int_{H_\infty} \eta_{\Delta_M}^{M\times M} &= \int_{\Delta_N} \Phi(H_\infty^\perp) = \int_{\Delta_N} \Phi(\nu_{\Delta_N}^{N\times N}) = \int_{\Delta_N} \eta_{\Delta_N}^{N\times N} \\
&= I(\Delta_N, \Delta_N) = \chi(N),
\end{aligned} \tag{24}$$

where the self-intersection number of Δ_N appears as in §2.1. Combining (20), (24) gives the Lefschetz formula up to sign:

$$L(f) = (-1)^{\dim M} \deg(\tilde{\pi})\chi(N). \tag{25}$$

To compute the degree of $\tilde{\pi} : E_{n,t} \to H_{n,t} \simeq \nu_{\Delta_N}^{\Delta M}$, we pick θ and α, positively oriented bases for $E_{n,t} \subset T_{(n,n)}\Gamma$ and $H_{n,t} \simeq (\nu_{\Delta_N}^{\Delta M})_{(n,n)}$ respectively, and compute the sign of the determinant of the matrix of $\tilde{\pi}$ with respect to θ, α.

There exists a positively oriented basis for T_nM, $(v_1, ..., v_n, w_{n+1}, ..., w_m)$, with $v_i \perp w_j$, such that $v_1, ..., v_n \in T_nN$, $df_nv = v$ and $w_1, ..., w_{m-n} \in \nu_N^M$, $df_nw = df_{\nu_n}w$. A positively oriented basis for $T_{(n,n)}\Gamma$ is then

$$\{(v_1, v_1), ..., (v_n, v_n), (w_1, df_\nu w_1), ..., ..., (w_m, df_\nu w_{m-n})\},$$

and a positively oriented basis for $E_{n,t}$ is

$$\theta = \{(w_1, df_\nu w_1), ..., ..., (w_m, df_\nu w_{m-n})\},$$

since $E_{n,t} \simeq (\nu_{\Delta_N}^{\Delta_M})_{(n,n)}$. A positively oriented basis for $H_{n,t} \simeq (\nu_{\Delta_N}^{\Delta_M})_{(n,n)}$ is

$$\alpha = \{(-w_1, w_1), ..., (-w_{m-n}, w_{m-n})\}.$$

As in Lemma 3.1, the vectors in θ decompose into

$$(w_i, df_\nu w_i) = \Big(\frac{w_i + df_\nu w_i}{2}, \frac{w_i + df_\nu w_i}{2}\Big) + \Big(\frac{w_i - df_\nu w_i}{2}, \frac{-w_i + df_\nu w_i}{2}\Big).$$

Hence

$$\begin{aligned}
\deg\, \tilde{\pi} &= \text{sgn}\ \det\left\{(-w_i, w_i) \mapsto \left(\frac{w_i - df_\nu w_i}{2}, \frac{-w_i + df_\nu w_i}{2}\right)\right\} \\
&= \text{sgn}\ \det\left\{(-w_i, w_i) \mapsto (df_\nu - \text{Id})(-w_i, w_i)\right\} \\
&= \text{sgn}\ \det(df_\nu - \text{Id}).
\end{aligned}$$

Since the right hand side of (25) vanishes if dim N is odd, we assume dim N is even. (25) becomes

$$\begin{aligned} L(f) &= (-1)^{\dim M}\mathrm{sgn}(\det(df_\nu - \mathrm{Id}))\chi(N) \\ &= (-1)^{\dim M}(-1)^{\dim M - \dim N}\mathrm{sgn}(\det(\mathrm{Id} - df_\nu))\chi(N) \\ &= \mathrm{sgn}(\det(\mathrm{Id} - df_\nu))\chi(N), \end{aligned}$$

which concludes the proof of Theorem 3.1.

If the graph of f is transversal to the diagonal, the fixed point set reduces to a finite number of isolated fixed points $n_1, n_2, ..., n_r$, and the Lefschetz fixed point formula is easily recovered. For let n be one such isolated fixed point. Then H_∞ reduces to $H_{n,\infty}$, the fiber over (n, n) in Δ_N and $\int_{H_{n,\infty}} \Phi(\nu_{\Delta_M}^{M\times M}) = 1$. df_ν is just df_n and $\deg(\tilde{\pi}) = \mathrm{sgn}\ \det(df_n - \mathrm{Id})$. So (20), (21) give the fixed point formula

$$L(f) = (-1)^{\dim M}\sum_{i=1}^{r}\deg(\tilde{\pi}_i)\int_{H_{n_i,\infty}}\Phi(\nu_{\Delta_M}^{M\times M}) = \sum_{i=1}^{r}\mathrm{sgn}\ \det(\mathrm{Id} - df_{n_i}).$$

4 Local expressions for the Lefschetz integral

In this section we calculate the integrand in the Lefschetz integral formula Theorem 2.3 in local coordinates. In §4.1 the integrand is computed explicitly for flat metrics and checked on a simple example. In §4.2, the case of general metrics is discussed, and the integrand is given explicitly for constant curvature metrics. At the end, a geometric proof of the Lefschetz submanifold formula is sketched.

4.1 Local expressions for flat manifolds

On a flat manifold, there exists a local orthornomal frame $\{x^i\}$ for which the connection and the curvature forms vanish. The Mathai-Quillen form for the normal bundle to the diagonal is given by

$$\mathrm{MQ}_{\nu_\Delta} = \pi^{-n/2}e^{-|x|^2}dx^1\wedge ...\wedge dx^n, \tag{26}$$

where x is the fiber coordinate.

We need an explicit diffeomorphism $\alpha : \Delta_\epsilon \to \nu_\Delta$ between an ϵ-neighborhood of the diagonal and the normal bundle to compute $\mathrm{MQ}_{\Delta_\epsilon} = \alpha^*\mathrm{MQ}_{\nu_\Delta}$. Even though the exponential map is trivial near the diagonal, its use avoids confusion between normal vectors and points of $M \times M$.

Fix $(x, y) \in \Delta_\epsilon$. Since the normal bundle consists of vectors of the form $(-v, v)$, there exists $(\bar{x}, \bar{x}) \in \Delta$ such that $(x, y) = \exp_{(\bar{x},\bar{x})}(-v, v) = (\exp_{\bar{x}}(-v), \exp_{\bar{x}} v)$. Thus $\bar{x}$ is the midpoint of the geodesic $\exp_{\bar{x}}(tv)$, $t \in [-1, 1]$ from x to y. This gives a diffeomorphism $\eta : U \to \Delta_\epsilon$ for U the ϵ-neighborhood of the zero section in ν_Δ:

$$\eta(v, -v)_{(\bar{x},\bar{x})} = (\exp_{\bar{x}}(v), \exp_{\bar{x}}(-v)).$$

Let $\rho : [0, \epsilon) \to [0, \infty)$ be a fixed diffeomorphism with $\rho(0) = 0$, $\rho'(0) = 0$. Extend ρ to take on values ∞ outside of $[0, \epsilon)$.

In the product metric, $d((x, y), (\bar{x}, \bar{x})) = d(x, y)/\sqrt{2}$, so $\Delta_\epsilon = \{(x, y) \in M \times M : d(x, y) < \sqrt{2}\epsilon.\}$. Thus $\beta : U \to \nu_\Delta$ given by

$$\beta(v, -v)_{(\bar{x},\bar{x})} = \begin{cases} \left(\rho\left(\frac{d(x,y)}{\sqrt{2}}\right)\frac{v}{|v|}, -\rho\left(\frac{d(x,y)}{\sqrt{2}}\right)\frac{v}{|v|}\right), & v \neq 0, \\ (\bar{x}, \bar{x}), & v = 0, \end{cases}$$

is a diffeomorphism and $\alpha = \beta \circ \eta^{-1} : \Delta_\epsilon \to \nu_\Delta$ is our desired map:

$$\alpha(x, y) = \begin{cases} \left((\bar{x}, \bar{x}), \left(\rho\left(\frac{d(x,y)}{\sqrt{2}}\right)\frac{v}{|v|}, -\rho\left(\frac{d(x,y)}{\sqrt{2}}\right)\frac{v}{|v|}\right)\right), & x \neq y, \\ ((x, x), 0), & x = y, \end{cases}$$

where $(x, y) = (\exp_{\bar{x}}(v), \exp_{\bar{x}}(-v))$, $|v| = d(x, y)/\sqrt{2}$. (Strictly speaking, α is only a homeomorphism at the diagonal.)

In the flat case, η can be treated as identity map, α reduces to $\beta : \mathbf{R}^n \to \mathbf{R}^n$, with

$$\beta(v) = \begin{cases} \rho(|v|)\frac{v}{|v|}, & v \neq 0, \\ 0, & v = 0, \end{cases}$$

and

$$\mathrm{MQ}_{\Delta_\epsilon} = \alpha^*\mathrm{MQ}_{\nu_\Delta} = \beta^*\mathrm{MQ}_{\nu_\Delta}.$$

By (26), computing this last term reduces to calculating $\beta^*\mathrm{dvol}$, which is easiest in polar coordinates. A short calculation gives $\beta_{*v}(\partial_r) = \rho'(|v|)\partial_r$, $\beta_{*v}(\partial_{\theta^i}) = \partial_{\theta^i}$, and so $(\beta^*\partial_r)_v = \rho'(|v|)dr$. Similarly, $\beta^*d\theta^i = d\theta^i$. Thus

$$\begin{aligned} \mathrm{MQ}_{\Delta_\epsilon} &= \beta^*\mathrm{MQ}_{\nu_\Delta} \\ &= \beta^*(\pi^{-n/2}e^{-r^2}r^{n-1}dr \wedge d\theta^1 \wedge \ldots \wedge d\theta^{n-1}) \end{aligned}$$

$$
\begin{aligned}
&= \pi^{-n/2}e^{-\rho^2(|v|)}\rho(|v|)^{n-1}\rho'(|v|)dr\wedge d\theta^1\wedge\ldots\wedge d\theta^{n-1}\\
&= \pi^{-n/2}e^{-\rho^2(\frac{d(x,y)}{\sqrt2})}\rho'\left(\frac{d(x,y)}{\sqrt2}\right)\left(\frac{\rho\left(\frac{d(x,y)}{\sqrt2}\right)}{\frac{d(x,y)}{\sqrt2}}\right)^{n-1}\mathrm{dvol}.
\end{aligned}
$$

The last step is to calculate

$$
\begin{aligned}
(\mathrm{Id},f)^*\mathrm{MQ}_{\Delta_\epsilon} &= (\mathrm{Id},f)^*\Bigg[\pi^{-n/2}e^{-\rho^2(\frac{d(x,y)}{\sqrt2})}\rho'\left(\frac{d(x,y)}{\sqrt2}\right)\left(\frac{\rho\left(\frac{d(x,f(x))}{\sqrt2}\right)}{\frac{d(x,f(x))}{\sqrt2}}\right)^{n-1}\\
&\qquad\cdot\mathrm{dvol}_{\alpha(x,y)}\Bigg]\\
&= \pi^{-n/2}e^{-\rho^2(\frac{d(x,f(x))}{\sqrt2})}\rho'\left(\frac{d(x,f(x))}{\sqrt2}\right)\\
&\qquad\cdot\left(\frac{\rho\left(\frac{d(x,f(x))}{\sqrt2}\right)}{\frac{d(x,f(x))}{\sqrt2}}\right)^{n-1}(\mathrm{Id},f)^*(\mathrm{dvol}_{\alpha(x,y)}).
\end{aligned}
$$

Here $\mathrm{dvol}_{\alpha(x,y)}$ is the volume element on the normal bundle, considered as a form near the diagonal.

Since $(\mathrm{Id},f)^*\mathrm{MQ}_{\Delta_\epsilon}$ vanishes if $(x,f(x))$ is not in the tubular neighborhood, we may assume there is a unique minimal geodesic from x to y. Let (x^i) be flat coordinates near x, and let (y^i) be flat coordinates at y given by parallel translating the ∂_{x^i} along the geodesic. Then

$$
\mathrm{dvol}_{\alpha(x,y)} = \bigwedge_{i=1}^{n}\left(\frac{-dx^i+dy^i}{\sqrt2}\right),
$$

since the normal fiber $\nu_{(\bar x,\bar x)}$ at $(\bar x,\bar x)$ consists of vectors of the form $(-v,v)$. In the (x^i), (y^i) coordinates, we may write $f=(f^1,\ldots,f^n)$. Then

$$
\begin{aligned}
(\mathrm{Id},f)^*\mathrm{dvol}_{\alpha(x,y)} &= (\mathrm{Id},f)^*\bigwedge_{i=1}^{n}\left(\frac{-dx^i+dy^i}{\sqrt2}\right) = 2^{-n/2}\bigwedge_{i=1}^{n}(-dx^i+df^i)\\
&= 2^{-n/2}\bigwedge_{i=1}^{n}\left(-dx^i+\frac{\partial f^i}{\partial x^j}dx^j\right)\\
&= 2^{-n/2}\bigwedge_{i=1}^{n}\left[\left(\frac{\partial f^i}{\partial x^i}-1\right)dx^i+\sum_{i\neq j}\frac{\partial f^i}{\partial x^j}dx^j\right].
\end{aligned}
$$

The transformation $df \circ \| - \mathrm{Id}$, where $\|$ denotes parallel translation from $f(x)$ to x along their geodesic, has entries

$$a_{ij} = \begin{cases} (\partial f_i/\partial x^i) - 1, & i = j, \\ \partial f_i/\partial x^j, & i \neq j, \end{cases}$$

so

$$\begin{aligned} (\mathrm{Id}, f)^* \mathrm{dvol}_{\alpha(x,y)} &= 2^{-n/2} \bigwedge_{i=1}^{n} \sum_{j=1}^{n} a_{ij} dx^j \\ &= 2^{-n/2} \det(df \circ \| - \mathrm{Id}) \mathrm{dvol}_M, \end{aligned}$$

and so

$$\begin{aligned} (\mathrm{Id}, f)^* \mathrm{MQ}_{\Delta_\epsilon} &= (\mathrm{Id}, f)^* \alpha^* \mathrm{MQ}_{\nu_\Delta} \\ &= (2\pi)^{-n/2} e^{-\rho^2 (\frac{d(x,f(x))}{\sqrt{2}})} \rho' \left(\frac{d(x, f(x))}{\sqrt{2}} \right) \qquad (27) \\ &\quad \cdot \left(\frac{\rho\left(\frac{d(x,f(x))}{\sqrt{2}}\right)}{\frac{d(x,f(x))}{\sqrt{2}}} \right)^{n-1} \det\left(df \circ \| - \mathrm{Id}\right) \mathrm{dvol}_M. \end{aligned}$$

Since $(-1)^n \det(df \circ \| - \mathrm{Id}) = \det(\mathrm{Id} - df \circ \|)$, Theorem 2.3 and (27) yield:

Theorem 4.1 *Let $f : M \to M$ be a smooth map of a closed, oriented, flat n-manifold M. Pick $\epsilon > 0$ sufficiently small, and let $\rho : [0, \epsilon) \to [0, \infty)$ be an orientation preserving diffeomorphism. Set $\rho(t) = \infty$ for $t \geq \epsilon$. Then the Lefschetz number of f is given by*

$$\begin{aligned} L(f) &= \frac{1}{(2\pi)^{n/2}} \int_M e^{-\rho^2 \left(\frac{d(x,f(x))}{\sqrt{2}} \right)} \rho' \left(\frac{d(x, f(x))}{\sqrt{2}} \right) \left(\frac{\rho\left(\frac{d(x,f(x))}{\sqrt{2}}\right)}{\frac{d(x,f(x))}{\sqrt{2}}} \right)^{n-1} \\ &\qquad \cdot \det\left(\mathrm{Id} - df \circ \|\right) \mathrm{dvol}_M. \end{aligned}$$

For f the identity map, $\det(\mathrm{Id} - df \circ \|)$ vanishes and the theorem gives $\chi(M) = L(\mathrm{Id}) = 0$ as expected. Of course, the $\sqrt{2}$ factor in the integrand can be incorporated into the diffeomorphism ρ.

Example: Let $f : S^1 \to S^1$ be given by $f(z) = z^n$, so $L(f) = 1 - n$.

For $\epsilon = \pi/2\sqrt{2}$, let $\alpha : \Delta_\epsilon \to \nu^{S^1\times S^1}_{\Delta_{S^1}}$ be the diffeomorphism

$$\alpha(\theta_1,\theta_2) = \left(\frac{\theta_1+\theta_2}{2}, \rho\left(\frac{\theta_1-\theta_2}{\sqrt{2}}\right)\right),$$

where (θ_1,θ_2) are the coordinates on $S^1\times S^1$ and $\rho : \left(-\frac{\pi}{2\sqrt{2}}, \frac{\pi}{2\sqrt{2}}\right) \to (-\infty,\infty)$ is an orientation preserving diffeomorphism given by a fixed odd function ρ. We set $\rho(x) = \infty,\ -\infty$ if $x > \pi 2\sqrt{2},\ x < -\frac{\pi}{2\sqrt{2}}$, respectively.

The condition $\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}dx = 1$ implies

$$\frac{1}{\sqrt{\pi}}\int_{-\frac{\pi}{2\sqrt{2}}}^{\frac{\pi}{2\sqrt{2}}} \rho'(\theta)e^{-\rho^2(\theta)}d\theta = 1, \quad \int_0^{\frac{\pi}{2\sqrt{2}}} \rho'(\theta)e^{-\rho^2(\theta)}d\theta = \frac{\sqrt{\pi}}{2}.$$

The graph of f, drawn on $[0,2\pi]\times[0,2\pi]$, consists of n line segments $\theta_2 = n\theta_1 - 2(k-1)\pi,\ k = 1,2,\dots,n$. Since the upper and lower limits of the tubular neighborhoods are given by $\theta_2 = \theta_1 \pm \frac{\pi}{2}$, it is easy to check that Γ is in the tubular neighborhood iff

$$\frac{(4k-5)\pi}{2(n-1)} \le \theta_1 \le \frac{(4k-3)\pi}{2(n-1)},$$

for $k = 2, ..., n-1$, or

$$0 \le \theta_1 \le \frac{\pi}{2(n-1)}, \qquad \frac{(4n-5)\pi}{2(n-1)} \le \theta_1 \le 2\pi,$$

for the first and last segment, respectively. Thus the integrand in Theorem 4.1 becomes

$$\left(\frac{n-1}{\sqrt{2\pi}}\right)e^{-\rho^2(\frac{(n-1)\theta-(k-1)2\pi}{\sqrt{2}})}\rho'\left(\frac{(n-1)\theta-(k-1)2\pi}{\sqrt{2}}\right)d\theta, \tag{28}$$

since df is multiplication by n. This gives

$$\begin{aligned} L(f) \quad = \quad & -\frac{1}{\sqrt{\pi}}\int_0^{\frac{\pi}{2(n-1)}} e^{-\rho^2(\frac{(n-1)\theta}{\sqrt{2}})}\rho'\left(\frac{(n-1)\theta}{\sqrt{2}}\right)d\theta \\ & -\sum_{k=2}^{n-1}\left[\frac{1}{\sqrt{\pi}}\int_{\frac{(4k-5)\pi}{2(n-1)}}^{\frac{(4k-3)\pi}{2(n-1)}} \left(\frac{n-1}{\sqrt{2}}\right)e^{-\rho^2\left(\frac{(n-1)\theta}{\sqrt{2}}-(k-1)\pi\sqrt{2}\right)} \right. \\ & \left. \rho'\left(\frac{(n-1)\theta-(k-1)2\pi}{\sqrt{2}}\right)d\theta\right] \end{aligned}$$

$$-\frac{1}{\sqrt{\pi}}\int_{\frac{(4n-5)\pi}{2(n-1)}}^{2\pi}\left(\frac{n-1}{\sqrt{2}}\right)e^{-\rho^2\left(\frac{(n-1)(\theta-2\pi)}{\sqrt{2}}\right)}\rho'\left(\frac{(n-1)(\theta-2\pi)}{\sqrt{2}}\right)d\theta.$$

Under the change of variables $\lambda = [(n-1)\theta/\sqrt{2}] - (k-1)\pi\sqrt{2},\ k = 1,\dots,n$, the first and last integrals become $1/2$, and the integrals under the sum become 1. Thus

$$L(f) = -\frac{1}{2} - \sum_{k=2}^{n-1} 1 - \frac{1}{2} = 1-n.$$

Theorem 4.1 can also be used to estimate $L(f)$ for flat manifolds.

Proposition 4.2 *Let $f: M \to M$ be a smooth map of a closed, oriented, flat n-manifold M. Then*

$$|L(f)| \le \frac{C}{(2\pi)^{n/2}}(\|df\|+1)^n$$

for some constant C independent of n.

PROOF: We may choose ρ so that $\lim_{z\to\epsilon} e^{-\rho^2(z)}\rho'(z)(\rho(z)/z)^{n-1} = 0$. Hence there exists $C' > 0$ such that $0 \le e^{-\rho^2(z)}\rho'(z)(\rho(z)/z)^{n-1} \le C'$. Note also that for $v \in T_{f(x)}M$,

$$|(\mathrm{Id} - df\circ\|)(v)| \le (\|df| \ +1)|v|,$$

since parallel translation in an isometry. Thus $|\det(\mathrm{Id} - df\circ\|)| \le (\|df\|+1)^n$. By Theorem 2.2, we have

$$\begin{aligned}|L(f)| &\le \frac{1}{(2\pi)^{n/2}}\int_M \left|e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})}\rho'\left(\frac{d(x,f(x))}{\sqrt{2}}\right)\left(\frac{\rho\left(\frac{d(x,f(x))}{\sqrt{2}}\right)}{\frac{d(x,f(x))}{\sqrt{2}}}\right)^{n-1}\right| \\ &\qquad \cdot\left|\det(\mathrm{Id} - df\circ\|)\right|\mathrm{dvol} \\ &\le \frac{C'}{(2\pi)^{n/2}}\mathrm{vol}(M)(\|df\|+1)^n.\end{aligned}$$

$C'\cdot\mathrm{vol}(M)$ is bounded above under a scaling $g \mapsto \lambda g$ of the metric, as $\rho(z) \mapsto \rho(\lambda^{-1/2}z)$, so $e^{-\rho^2(z)}\rho(z)^{n-1}$ stays bounded, ρ' scales by $\lambda^{-1/2}$, $z^{1-n} = (d(x,f(x))/\sqrt{2})^{1-n}$ scales by $\lambda^{(1-n)/2}$, and dvol scales by $\lambda^{n/2}$. Thus $C\cdot\mathrm{vol}(M)$ is bounded above by a constant independent of $\dim(M)$. □

The appendix contains a similar result for arbitrary manifolds via Hodge theory.

4.2 Local expressions for arbitrary metrics

In this subsection we calculate the local expression for the integrand in Theorem 2.3 for an arbitrary Riemanninan metric. Unlike the flat case, the exponential map is nontrivial, and the Jacobi fields which measure the deviation of the exponential map from the identity enter the computations.

A tubular neighborhood Δ_ϵ of the diagonal Δ in $M \times M$ is diffeomorphic to a neighborhood of the zero section in $\nu_\Delta = \nu_\Delta^{M\times M}$, which in turn is diffeomorphic to a neighborhood of zero in TM. The Levi-Civita connection on M determines the space H_M of horizontal vectors on TTM, while the space V_M of vertical vectors is independent of the connection. The Mathai-Quillan form MQ_{TM} is written in terms of horizontal and vertical vectors, so we have to identify the corresponding horizontal and vertical vectors in the tube in order to compute $\mathrm{MQ}_{\Delta_\epsilon}$.

Let α be the isomorphism from the neighborhood in ν_Δ to the tube: for $\nu_\Delta = \{(v, -v) : v \in TM\}$, we have $\alpha(v, -v) = (\exp_{\bar{x}}(v), \exp_{\bar{x}}(-v))$ at $(\bar{x}, \bar{x}) \in \Delta$. As before, the radius of the tube is chosen small enough so that there exists a unique minimal geodesic between x and y whenever (x, y) is in the tube. For $(x, f(x)) \in \Delta_\epsilon$, recall from §4.1 that $\bar{x}$ is the midpoint of the unique minimal geodesic γ from x to $f(x)$ and $|v| = d(x, f(x))/2$ in M's metric.

Pick an orthonormal frame $\{Y_i\}$ at $\bar{x}$. Let $\beta : TM \to \nu_M$ be the bundle isomorphism $\beta(v_x) = (v_x, -v_x)$. The horizontal space H in the tube is by definition be $d(\alpha\beta)(H_M)$, and the vertical space V in the tube is $d(\alpha\beta)(V_M)$. Define vectors $X_i, \widetilde{X}_i$ at x by

$$\begin{aligned} X_i &= d(\exp_{\bar{x}})_v(Y_i), \\ \widetilde{X}_i &= d(\exp_{\cdot} \|v)_{\bar{x}}(Y_i), \end{aligned} \tag{29}$$

where in the first line Y_i is trivially translated to a vector in $T_v T_{\bar{x}} M$, and in the second line $\|v$ denotes the parallel translation of v along a curve in M with tangent vector Y_i. Similarly define vectors $Z_i, \widetilde{Z}_i$ at $f(x)$ by replacing v in (29) with $-v$. If we parametrize γ from $\bar{x}$ to x as $\gamma(t)$, then $\widetilde{X}_i$ is the endpoint of a Jacobi field J with $J(0) = Y_i$ – i.e. J is the variation vector field of the family of geodesics $\gamma_s(t) = \exp_{\eta(s)}(t\|v)$, where $\dot{\eta}(0) = Y_i$ and $t \in [0, 1]$. Similarly, X_i is the endpoint of a Jacobi field J, the variation vector field of the family of geodesics $\gamma_s(t) = \exp_{\bar{x}}(t(v + sY_i))$, which has $(\nabla J)(0) = Y_i$ (cf. [7, Cor. 3.46]). Similar remarks apply to $Z_i, \widetilde{Z}_i$.

Lemma 4.3 *The vertical space V at $(x, f(x))$ is spanned by $\{(-X_i, Z_i)\}$ and the horizontal space H is spanned by $\{(\widetilde{X}_i, \widetilde{Z}_i)\}$.*

PROOF: Set $\delta = \alpha\beta$. A vertical vector at $v \in T_{\bar{x}}M$ is a tangent vector Y to a curve $\eta(t) \subset T_x M$ with $\eta(0) = v, \dot{\eta}(0) = Y$. Then

$$\begin{aligned} d\delta_v(Y) &= \left.\frac{d}{dt}\right|_{t=0} (\exp_{\bar{x}} \eta(t),\ \exp_{\bar{x}}(-\eta(t))) \\ &= \left(\left.\frac{d}{dt}\right|_{t=0} \exp_{\bar{x}} \eta(t), \left.\frac{d}{dt}\right|_{t=0} \exp_{\bar{x}}(-\eta(t)) \right) \\ &= (d(\exp_{\bar{x}})_v Y,\ d(\exp_{\bar{x}})_v(-Y)). \end{aligned}$$

Thus the vertical space at $(x, f(x)) = (\exp_{\bar{x}} v, \exp_{\bar{x}}(-v))$ is spanned by

$\{(d(\exp_{\bar{x}})_v(Y_i), d(\exp_{\bar{x}})_{-v}(Y_i))\}$.

Let $\|v = \|_y v$ denote the parallel translation of v along radial geodesics centered at $\bar{x}$. Then $\|v$ is parallel at $\bar{x}$, and the horizontal vectors at v are spanned by

$$\left.\frac{d}{dt}\right|_{t=0} \|_{\exp_{\bar{x}}(tY_i)} v.$$

Thus the horizontal vectors at $(x, f(x))$ are spanned by

$$\begin{aligned} &\left.\frac{d}{dt}\right|_{t=0} \delta(\|_{\exp_{\bar{x}}(tY_i)} v) \\ &\qquad = \left.\frac{d}{dt}\right|_{t=0} (\exp_{\exp_{\bar{x}}(tY_i)} \|_{\exp_{\bar{x}}(tY_i)} v, \exp_{\exp_{\bar{x}}(tY_i)} \|_{\exp_{\bar{x}}(tY_i)}(-v)) \\ &\qquad = (\widetilde{X}_i, \widetilde{Z}_i). \end{aligned}$$

$\square$

Remarks: 1) The lemma shows that at $(x, f(x))$,

$$\begin{aligned} V &= (-d(\exp_{\bar{x}})_v, d(\exp_{\bar{x}})_{-v}) V_M, \\ H &= (d(\exp \|v)_{\bar{x}}, d(\exp -\|v)_{\bar{x}}) H_M. \end{aligned}$$

2) $X_i, \widetilde{X}_i, Z_i, \widetilde{Z}_i$ are just parallel translations of Y_i if M is flat.

3) Vertical vectors at $(x, f(x))$ are those pairs of vectors in $T_x M \times T_{f(x)} M$ which are endpoints of a Jacobi field along γ which vanishes at $\bar{x}$. Horizontal vectors are pairs of vectors which are endpoints of a Jacobi field along γ whose covariant derivative vanishes at $\bar{x}$.

Lemma 4.4 *Let $(X,Z)\in T_{(x,f(x))}(M\times M)$. Let Y be the unique Jacobi field along γ with $Y(x)=X$, $Y(f(x))=Z$. Let X_1, resp. Z_1 be the values at x, resp. $f(x)$ of the Jacobi field Y_1 along γ given by $Y_1(\bar{x})=0$, $\frac{DY_1}{dt}(\bar{x})=\frac{DY}{dt}(\bar{x})$. Let $\widetilde{X}_1$, resp. $\widetilde{Z}_1$ be the values at x, resp. $f(x)$ of the Jacobi field Y_2 along γ given by $Y_2(\bar{x})=Y(\bar{x})$, $\frac{DY_2}{dt}(\bar{x})=0$. Then $(X,Z)=(X_1,Z_1)+(\widetilde{X}_1,\widetilde{Z}_1)$ is the decomposition of (X,Z) into vertical and horizontal vectors.*

PROOF: By Remark 3, $(X_1,Z_1),(\widetilde{X}_1,\widetilde{Z}_1)$ are vertical and horizontal vectors respectively. Since the Jacobi equation is linear, the endpoints of the Jacobi field Y_1+Y_2 are $X_1+\widetilde{X}_1, Z_1+\widetilde{Z}_1$. Since Y_1+Y_2 has the same position and velocity vectors as Y as $\bar{x}$, we must have $X=X_1+\widetilde{X}_1, Z=Z_1+\widetilde{Z}_1$. □

Let $\rho:[0,\epsilon)\to[0,\infty)$ be a diffeomorphism with $\rho(0)=\rho'(0)=0$, and and extend ρ to a smooth radial surjection $\rho: B_\epsilon(0)\to\mathbf{R}^n$ on the ϵ-ball around $0\in\mathbf{R}^n$. Let MQ_ν be the Mathai-Quillen form of the normal bundle $\nu=\nu_\Delta$ and let $\mathrm{MQ}_{\Delta_\epsilon}=(\exp^{-1})^*\rho^*\mathrm{MQ}_\nu$ be the corresponding Mathai-Quillen form on $M\times M$. Here we abbreviate $(\exp^{-1},\exp^{-1})$ to just $\exp^{-1}$.

In (3), the vertical coordinates are denoted by x^i and the horizontal coordinates are hidden in Ω_I. For the calculations on TM, we need to make the horizontal coordinates y^i explicit and take care not to confuse them with the vertical coordinates. So let $\{x^i\}$ be a synchronous orthonormal frame centered at $\bar{x}$. In each fiber of ν, we take the orthonormal polar coordinate frame $\{(-x^i,x^i)\}$, with $\{x^i\}=\{x^1=\partial_r, r^{-1}\partial_{\theta^i}\}$, away from the origin. These frames do not agree at the origin in each fiber, but the formulas below will be smooth at the origin. $\rho'(0)=0$ implies $(\rho^*dx^i)_{v=0}=0$, and for $v\neq 0$, $\rho^*dx^1_v=\rho^*dr_v=\rho'(|v|)dr_v$, $\rho^*d\theta^i_v=d\theta^i_v$. Thus for $v\neq0$,

$$[(\exp^{-1})^*\rho^*dx^i]_v(\alpha)=\begin{cases}\rho'(|(\exp^{-1})_*\alpha|)(\exp^{-1})^*dx^i_v(\alpha), & \text{if } i=1,\\ \frac{\rho(|v|)}{|v|}(\exp^{-1})^*dx^i(\alpha), & \text{if } i\neq 1.\end{cases}$$

If $\{y^i\}$ is another synchronous frame centered at $\bar{x}$ (possibly equal to $\{x^i\}$), then the horizontal lifts of y^i into $T\nu$ are orthonormal in the metric on $T\nu$ induced by the metric on M. (Since $\rho_*(y^i)=y^i$, we have

$$[(\exp^{-1})^*\rho^*dy^i](\alpha)=(\widetilde{X}_i,\widetilde{Z}_i)^\#(\alpha),$$

where $(\widetilde{X}_i,\widetilde{Z}_i)^\#$ is the cotangent vector dual to $(\widetilde{X}_i,\widetilde{Z}_i)$.) If $(\mathrm{Id},f)^*(\mathrm{MQ}_{\Delta_\epsilon})_{(x,x)}$ $=D\ \mathrm{dvol}_{(x,x)}$, then

$$D=(\mathrm{Id},f)^*(\mathrm{MQ}_{\Delta_\epsilon})_{(x,x)}\left(\frac{(y^1,y^1)}{\sqrt{2}},\dots,\frac{(y^n,y^n)}{\sqrt{2}}\right).$$

Thus

$$D = \frac{e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})}}{(2\pi)^{n/2}} \sum_{I,I'} \epsilon(I,I')[(\exp_{\bar{x}}^{-1})^*\rho^*]\big(\mathrm{Pf}(\Omega_I)$$
$$\wedge dx^{I'}\big)((y^1, f_*y^1), \ldots, (y^n, f_*y^n)).$$

Write

$$(y^i, f_*y^i) = P_V^i + P_H^i \tag{30}$$

for the decomposition of (y^i, f_*y^i) into vertical and horizontal vectors as in the lemma. Let Σ_n be the permutation group on $\{1, \ldots, n\}$. Then

$$D = \frac{e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})}}{(2\pi)^{n/2}} \sum_{I,I'} \frac{\epsilon(I,I')}{|I|!\,|I'|!} \sum_{\mu\in\Sigma_n} (\exp_{\bar{x}}^{-1})^*\mathrm{Pf}(\Omega_I)(P_H^{\mu_1}, \ldots, P_H^{\mu_{|I|}})$$
$$\cdot\widetilde{\rho}_{I'}\left(\frac{d(x,f(x))}{\sqrt{2}}\right) d\widetilde{x}^{I'}\left(P_V^{\mu_{|I|+1}}, \ldots, P_V^{\mu_n}\right),$$

where

$$\widetilde{\rho}_{I'}\left(\frac{d(x,f(x))}{\sqrt{2}}\right) = \begin{cases} \rho'(\frac{d(x,f(x))}{\sqrt{2}})\left[\frac{\rho(d(x,f(x))/\sqrt{2})}{d(x,f(x))/\sqrt{2}}\right]^{|I|'-1}, & \text{if } 1 \in I', \\ \left[\frac{\rho(d(x,f(x))/\sqrt{2})}{d(x,f(x))/\sqrt{2}}\right]^{|I|'}, & \text{if } i_1 \notin I'. \end{cases}$$

Here $d\widetilde{x}^{I'} = (\exp_{\bar{x}}^{-1})^* dx^i$, and we have used $\rho^*\ \mathrm{Pf}(\Omega_I) = \mathrm{Pf}(\Omega_I)$, since this Pfaffian is a horizontal form. Note that $\widetilde{\rho}_{I'}(0) = 1$ since $\rho(x) = \mathrm{o}(x^2)$.

Define $n \times n$ matrices $A = A_x$, $B = B_x$ by

$$(\exp_{\bar{x}}^{-1})_* P_H^i = A_j^i y^j, \quad (\exp_{\bar{x}}^{-1})_* P_V^i = B_j^i x^j, \tag{31}$$

where strictly speaking the last term is $(-B_j^i x^j, B_j^i x^j)$.

Examples: At a fixed point $x = f(x)$, the decomposition of (q, f_*q) into vertical and horizontal components is given by

$$(q, f_*q) = \left(\frac{q - f_*q}{2}, \frac{-q + f_*q}{2}\right) + \left(\frac{q + f_*q}{2}, \frac{q + f_*q}{2}\right),$$

since $(\exp^{-1})_* = \mathrm{Id}$ at a fixed point. Since vertical (resp. horizontal) vectors on the diagonal are of the form $(-v, v)$ (resp. (v, v)), A is the matrix of $\frac{1}{2}(df + \mathrm{Id})$ and B is the matrix of $\frac{1}{2}(df - \mathrm{Id})$.

It follows easily from Remark 2 that on a flat manifold, $A = \frac{1}{2}(\| \circ df + \mathrm{Id})$, $B = \frac{1}{2}(\| \circ df - \mathrm{Id})$ for arbitrary x.

Thus

$$\begin{aligned} D &= \frac{e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})}}{(2\pi)^{n/2}} \sum_{I,I'} \frac{\epsilon(I,I')}{|I|!\,|I'|!} \sum_{\mu\in\Sigma_n} (\mathrm{sgn}\ \mu) A^{\mu(1)}_{j_1} \cdot \ldots \cdot A^{\mu(|I|)}_{j_{|I|}} \\ &\quad \cdot \mathrm{Pf}(\Omega_I)_{\bar{x}}(y^{j_1}, \ldots, y^{j_{|I|}}) \\ &\quad \cdot \widetilde{\rho}'_{I'}\left(\frac{d(x,f(x))}{\sqrt{2}}\right) B^{\mu(|I|+1)}_{k_1} \cdot \ldots \cdot B^{\mu(n)}_{k_{|I'|}} dx^{I'}(x^{k_1}, \ldots, x^{k_{|I'|}}). \quad (32) \end{aligned}$$

(We use summation convention for the j and k indices.) The right hand side of (32) vanishes unless $I' = \{k_1, \ldots, k_{|I'|}\} \equiv K$, and

$$\sum_{K,K=I'} B^{\mu(|I|+1)}_{k_1} \cdot \ldots \cdot B^{\mu(n)}_{k_{|I'|}} dx^{I'}(x^{k_1}, \ldots, x^{k_{|I'|}}) = \det(B^{\mu(|I|+q)}_{i'_s}),$$

for $q, s = 1, \ldots, |I'|$. Here $I' = \{i'_1, \ldots, i'_{|I'|}\}$, with $i'_1 < \ldots < i'_{|I'}$. We denote this determinant by $\det(B^{\mu}_{I'})$. For $I = \{i_1, \ldots, i_{|I|}\}$, with $i_1 < \ldots < i_{|I|}$, we have

$$\begin{aligned} \mathrm{Pf}(\Omega_I) &= c_{|I|} \sum_{\sigma,\tau\in\sum_{|I|}} (\mathrm{sgn}\ \sigma)(\mathrm{sgn}\ \tau) R_{i_{\sigma(1)} i_{\sigma(2)} i_{\tau(1)} i_{\tau(2)}} \cdot \ldots \\ &\quad \cdot R_{i_{\sigma(|I|-1)} i_{\sigma(|I|)} i_{\tau(|I|-1)} i_{\tau(|I|)}} dy^1 \wedge \ldots \wedge dy^{|I|}, \end{aligned}$$

with $c_{|I|} = (-1)^{|I|/2}[2^{|I|}(|I|/2)!]^{-1}$ [12, (1.3)]. (The $(-1)^{|I|/2}$ reflects our sign convention on curvature.) As above, the right hand side of (32) vanishes unless $I = \{j_1, \ldots, j_{|I|}\} \equiv J$. Summing over such J produces the term $\det(A^{\mu(t)}_{i_k}) \equiv \det(A^{\mu}_I)$, for $t, k = 1, \ldots, |I|$.

Thus

$$\begin{aligned} D &= \frac{e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})}}{(2\pi)^{n/2}} \sum_{I,I'} \sum_{\mu\in\Sigma_n} c_{|I|} \frac{\epsilon(I,I')}{|I|!\,|I'|!} (\mathrm{sgn}\ \mu) \det(A^{\mu}_I) \det(B^{\mu}_{I'}) \\ &\quad \cdot \widetilde{\rho}'_{I'}\left(\frac{d(x,f(x))}{\sqrt{2}}\right) \\ &\quad \cdot \sum_{\sigma,\tau\in\Sigma_{|I|}} (\mathrm{sgn}\ \sigma)(\mathrm{sgn}\ \tau) R_{i_{\sigma(1)} i_{\sigma(2)} i_{\tau(1)} i_{\tau(2)}} \cdot \ldots \cdot R_{i_{\sigma(|I|-1)} i_{\sigma(|I|)} i_{\tau(|I|-1)} i_{\tau(|I|)}}. \end{aligned}$$

Since $\int_M dy^1 \wedge \ldots \wedge dy^n = 2^{-n/2} \int_\Delta d(y^1, y^1) \wedge \ldots \wedge d(y^n, y^n)$, we have

Theorem 4.5 *Let $f : M \to M$ and fix $\epsilon > 0$ such that $(x, y) \in \Delta_M^\epsilon$ implies the existence of a unique minimal geodesic between x and y. For $x \in M$, define matrices A, B by (30), (31), provided $(x, f(x))$ is in the ϵ-neighborhood of the diagonal; otherwise set $A = B = 0$. Then*

$$\begin{aligned} L(f) &= \frac{(-1)^{\dim M}}{(2\pi)^{n/2}} \int_M e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})} \sum_{I,I'} c_{|I|} \frac{\epsilon(I,I')}{|I|!\,|I'|!} \sum_{\mu \in \Sigma_n} (\text{sgn } \mu) \det(A_I^\mu) \\ &\quad \cdot \det(B_{I'}^\mu) \widetilde{\rho}'_{I'} \left(\frac{d(x, f(x))}{\sqrt{2}} \right) \\ &\quad \cdot \sum_{\sigma,\tau \in \Sigma_{|I|}} R_{i_{\sigma(1)} i_{\sigma(2)} i_{\tau(1)} i_{\tau(2)}} \cdot \ldots \cdot R_{i_{\sigma(|I|-1)} i_{\sigma(|I|)} i_{\tau(|I|-1)} i_{\tau(|I|)}} \text{dvol}. \end{aligned}$$

This formula can be checked in some cases. At a fixed point, $\rho^* dx^i = 0$, so the only contribution to the integrand comes from $I = \{1, \ldots, n\}$, $I' = \emptyset$. Thus the integrand is

$$\det\left(\frac{1}{2}(df + \text{Id}) \right) \text{Pf}(\Omega).$$

In particular, for $f = \text{Id}$ and $n = \dim M$ is odd, the integrand vanishes and $\chi(M) = L(\text{Id}) = 0$. If n is even, the theorem reduces to

$$\chi(M) = \frac{(-1)^{n/2}}{(8\pi)^{n/2}(n/2)!} \int_M \text{Pf}(\Omega),$$

the Chern-Gauss-Bonnet theorem.

The other extremal case occurs when M is flat. Because $R_{ijkl} = 0$, the only contribution to the integrand occurs when $I = \emptyset$. Then $c_0 = 1$ and $(\text{sgn } \mu) \det(B_{I'}^\mu) = \det(|| \circ df - \text{Id})$, so the integrand is

$$\frac{1}{(2\pi)^{n/2}} e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})} \rho' \left(\frac{d(x, f(x))}{\sqrt{2}} \right) \left(\frac{\rho\left(\frac{d(x,f(x))}{\sqrt{2}}\right)}{\frac{d(x,f(x))}{\sqrt{2}}} \right)^{n-1} \det(\text{Id} - || \circ df),$$

which agrees with the flat case formula, since parallel translation is an isometry.

Example: We determine the integrand in the Lefschetz formula for M an oriented surface of constant Gaussian curvature -1. At the end we indicate the changes for constant curvature manifolds in general.

We first determine the horizontal and vertical components of a vector $(q, f_*q) \in T_{(x,f(x))}(M \times M)$. Assume there exists a unique minimal geodesic γ joining x to $f(x)$ with midpoint $\bar{x}$. Let $|\dot{\gamma}| = 1$, and let α be the unit normal to γ determined by the orientation. Set $d = d(x, f(x))$.

Let J be a Jacobi field along γ. Plugging $J(t) = a(t)\dot{\gamma} + b(t)\alpha$ into the Jacobi equation $D^2J/dt^2 + R(\dot{\gamma}, J)\dot{\gamma} = 0$ and using $\langle R(\dot{\gamma}, J)\dot{\gamma}, \dot{\gamma}\rangle = 0$, $\langle R(\dot{\gamma}, \alpha)\dot{\gamma}, \alpha\rangle = -1$ yields $\ddot{a} = 0$, $\ddot{b} - b = 0$. Thus $J(t) = (c_0 + c_1 t)\dot{\gamma} + (d_1 \sinh(t) + d_2 \cosh(t))\alpha$. Imposing the boundary conditions $J(0) = q$, $J(d) = f_*q$ gives

$$J(t) = \left(q_1 + \left(\frac{w_1 - q_1}{d}\right) t\right) \dot{\gamma} + \left[\left(\frac{w_2 - q_2 \cosh(d)}{\sinh(d)}\right) \sinh(t) + q_2 \cosh(t)\right] \alpha,$$

where $q = q_1\dot{\gamma} + q_2\alpha$, $f_*q = w_1\dot{\gamma} + w_2\alpha$. In particular

$$\begin{aligned}
J\left(\frac{d}{2}\right) &= \left(\frac{q_1 + w_1}{2}\right)\dot{\gamma} + \left[\left(\frac{w_2 - q_2\cosh(d)}{\sinh(d)}\right)\sinh\left(\frac{d}{2}\right)\right. \\
&\qquad \left. + q_2 \cosh\left(\frac{d}{2}\right)\right]\alpha \\
&= \left(\frac{q_1 + w_1}{2}\right)\dot{\gamma} + \left(\frac{w_2 + q_2}{2\cosh(\frac{d}{2})}\right)\alpha, \\
\frac{DJ}{dt}\left(\frac{d}{2}\right) &= \left(\frac{w_1 - q_1}{d}\right)\dot{\gamma} + \left(\frac{w_2 - q_2}{2\sinh(\frac{d}{2})}\right)\alpha,
\end{aligned}$$

The Jacobi fields J_1, J_2 determined by $J_1(d/2) = 0$, $(DJ_1/dt)(d/2) = (DJ/dt)(d/2)$ and $J_2(d/2) = J(d/2)$, $(DJ_2/dt)(d/2) = 0$ are given by

$$\begin{aligned}
J_1(s) &= \left(\frac{w_1 - q_1}{d}\right) s\dot{\gamma} + \left(\frac{w_2 - q_2}{2\sinh(d/2)}\right)\sinh(s)\alpha, \\
J_2(s) &= \left(\frac{q_1 + w_1}{2}\right)\dot{\gamma} + \left(\frac{w_2 + q_2}{2\cosh(d/2)}\right)\cosh(s)\alpha,
\end{aligned}$$

where $s = 0$ corresponds to $\bar{x}$. Evaluating J_1, J_2 at $s = \pm d/2$ gives the decomposition of (q, f_*q) into vertical and horizontal components:

$$\begin{aligned}
(q, f_*q)_{\text{vert}} &= \left(\left(\frac{-w_1 + q_1}{2}\right)\dot{\gamma} - \left(\frac{w_2 - q_2}{2}\right)\alpha,\right. \\
&\qquad \left.\left(\frac{w_1 - q_1}{2}\right)\dot{\gamma} + \left(\frac{w_2 - q_2}{2}\right)\alpha\right) \\
(q, f_*q)_{\text{hor}} &= \left(\left(\frac{q_1 + w_1}{2}\right)\dot{\gamma} + \left(\frac{w_2 + q_2}{2}\right)\alpha,\right. \\
&\qquad \left.\left(\frac{q_1 + w_1}{2}\right)\dot{\gamma} + \left(\frac{w_2 + q_2}{2}\right)\alpha\right).
\end{aligned}$$

Let $(x, f(x)) = (\exp_{\bar{x}} v, \exp_{\bar{x}}(-v))$, so $v = -(d/2)\dot{\gamma}$ at $\bar{x}$. We now determine $(\exp_{\bar{x}}^{-1})_*^{(2)}$: $T_{(x,f(x))}(M \times M) \to T_{(v,-v)}T_{(\bar{x},\bar{x})}(M \times M)$, where $(\exp_{\bar{x}}^{-1})_*^{(2)}$ is shorthand for $(-(\exp_{\bar{x}}^{-1})_*, (\exp_{\bar{x}}^{-1})_*)$. For a vertical vector $\beta = \beta_1\dot{\gamma} + \beta_2\alpha \in T_v T_{\bar{x}} M$, with $\dot{\gamma}, \alpha$ trivially parallel translated to v,

$$(\exp_{\bar{x}})_{*,v}(\beta) = \frac{d}{ds}\bigg|_{s=0} \exp_{\bar{x}}(v + s\beta),$$

which is the value at x of the Jacobi field J along γ with $J(\bar{x}) = 0$, $(DJ/dt)(\bar{x}) = 2\beta/d$, since $|v| = d/2$. Solving for J as above, we get

$$(\exp_{\bar{x}})_{*,v}(\beta) = \beta_1\dot{\gamma} + \frac{2}{d}\beta_2 \sinh\left(\frac{d}{2}\right)\alpha.$$

Thus

$$(\exp_{\bar{x}}^{-1})_*^{(2)}(q, f_*q)_{\mathrm{vert}} =$$
$$\left(\left(\frac{-w_1 + q_1}{2}\right)\dot{\gamma} - \frac{d(w_2 - q_2)}{4\sinh(\frac{d}{2})}\alpha, \left(\frac{-w_1 + q_1}{2}\right)\dot{\gamma} - \frac{d(w_2 - q_2)}{r\sinh(\frac{d}{2})}\alpha\right).$$

Similarly, for a horizontal vector $\delta = \delta_1\dot{\gamma} + \delta_2\alpha$, where $\dot{\gamma}, \alpha$ now denote the horizontal lifts of $\dot{\gamma}, \alpha$ to $T_v T_{\bar{x}} M$, we have

$$(\exp_{\bar{x}})_{*,v}(\delta) = \delta_1\dot{\gamma} + \delta_2 \cosh\left(\frac{d}{2}\right)\alpha,$$

so

$$(\exp_{\bar{x}}^{-1})_*^{(2)}(q, f_*q)_{\mathrm{hor}} =$$
$$\left(\left(\frac{q_1 + w_1}{2}\right)\dot{\gamma} + \left(\frac{w_2 + q_2}{2\cosh(\frac{d}{2})}\right)\alpha, \left(\frac{q_1 + w_1}{2}\right)\dot{\gamma} + \left(\frac{w_2 + q_2}{2\cosh(\frac{d}{2})}\right)\alpha\right).$$

To determine the matrix B, we have to express $(\exp_{\bar{x}}^{-1})_*^{(2)}(q, f_*, q)$, for $q = \dot{\gamma}, \alpha$, in polar coordinates at $(v, -v)$ in $\nu_{(\bar{x},\bar{x})}$. The radial vector at $(v, -v)$ is

$$r = \left(\frac{v}{|v|\sqrt{2}}, \frac{-v}{|v|\sqrt{2}}\right) = \left(\frac{v\sqrt{2}}{d}, \frac{v\sqrt{2}}{d}\right),$$

and the unit angular vector "$r^{-1}\partial_\theta$" is $(-\alpha/\sqrt{2}, \alpha/\sqrt{2})$. Note that $(\dot{\gamma}, -\dot{\gamma}) = -\sqrt{2}r$. For $f_*\dot{\gamma} = w_{11}\dot{\gamma} + w_{12}\alpha, f_*\alpha = w_{21}\dot{\gamma} + w_{22}\alpha$, we get

$$(B^i_j) = \begin{pmatrix} -\frac{\sqrt{2}}{2}(-w_{11} + 1) & \frac{\sqrt{2}dw_{12}}{4\sinh(d/2)} \\ \frac{\sqrt{2}}{2}w_{21} & \frac{\sqrt{2}d(w_{22}-1)}{4\sinh(d/2)} \end{pmatrix}.$$

Thus

$$\begin{aligned}
\det B &= \frac{d}{4\sinh(\frac{d}{2})}((w_{11}-1)(w_{22}-1)-w_{12}w_{21}) \\
&= \frac{d}{\sinh(\frac{d}{2})}\det\left(\frac{1}{2}(\| \circ df - \mathrm{Id})\right).
\end{aligned}$$

Similarly,

$$(A^i_j) = \begin{pmatrix} -\frac{w_{11}+1}{\sqrt{2}} & \frac{-\sqrt{2}w_{12}}{2\cosh(d/2)} \\ \frac{-w_{21}}{\sqrt{2}} & \frac{-\sqrt{2}(w_{22}+1)}{2\cosh(\frac{d}{2})} \end{pmatrix}$$

and

$$\det A = \frac{2}{\cosh(\frac{d}{2})}\det\left(\frac{1}{2}(\| \circ df + \mathrm{Id})\right).$$

We now plug this information into the Lefschetz formula. Note that $I = \{1,2\}$ or $I = \emptyset$ and that $R_{1212} = 1$ in our convention. We obtain

$$\begin{aligned}
L(f) &= \frac{1}{2\pi}\int_M e^{-\rho^2(\frac{d}{\sqrt{2}})}\left[\left(\frac{2}{\cosh(\frac{d}{2})}\right)\frac{(-1)\cdot 4}{4\cdot 2!}\det\left(\frac{1}{2}(\| \circ df + \mathrm{Id})\right)\right. \\
&\qquad \left. + \rho'\left(\frac{d}{\sqrt{2}}\right)\frac{\rho(d/\sqrt{2})}{d/\sqrt{2}}\left(\frac{d}{\sinh(\frac{d}{2})}\right)\cdot\frac{1}{2!}\det\left(\frac{1}{2}(\| \circ df - \mathrm{Id})\right)\right] dA.
\end{aligned}$$

In the first line, there are factors of $c_{\{1,2\}} = -1/4$, $|I|! = 2$, and

$\sum_{\sigma,\tau\in\Sigma_2} R_{\sigma(1)\sigma(2)\tau(1)\tau(2)} = 4$. In the second line, $c_\emptyset = 1$ and $|I'|! = 2$. Thus we obtain

Proposition 4.6 *Let M be an oriented surface of constant curvature -1. Then*

$$\begin{aligned}
L(f) &= \frac{1}{2\pi}\int_M e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})}\left[\frac{-\det(\frac{1}{2}(\| \circ df_x + \mathrm{Id}))}{\cosh(\frac{d(x,f(x))}{2})}\right. \\
&\qquad \left. + \rho'\left(\frac{d(x,f(x))}{\sqrt{2}}\right)\left(\frac{\rho(d(x,f(x))/\sqrt{2})}{\sqrt{2}\sinh(\frac{d(x,f(x))}{2})}\right)\det\left(\frac{1}{2}(\| \circ df_x - \mathrm{Id})\right)\right] dA.
\end{aligned}$$

It is straightforward to extend this result to higher dimensional constant curvature spaces. The integrand in Proposition 4.6 now involves a sum over I, I'. The general term inside the brackets is $c_{|I|}\epsilon(I, I')/[|I|!|I'|!]$ times

$$\sum_{\mu\in\Sigma_n}(\text{sgn }\mu)\frac{2^{|I|/2}\det(\frac{1}{2}(\| \circ df_x + \text{Id}))_I \det(\frac{1}{2}(\| \circ df_x - \text{Id}))_{I'}\widetilde{\rho}_{I'}(d(x, f(x))/\sqrt{2})}{[\cosh(d(x, f(x))/2)]^{|I|-1}\cdot[\sinh(d(x, f(x))/2)]^{|I'|-1}}$$

for negative curvature -1. For constant curvature 1, cosh, sinh are replaced by cos, sin, and there is an extra factor of $(-1)^{|I|}$ due to $R_{ijij} = -1$.

Remark: We sketch a geometric proof of the Lefschetz fixed submanifold formula based on Theorem 4.5. Assume that the metric on M is a product near a fixed point submanifold N. If the submanifold is given by $\{x^{k+1} = \ldots = x^n = 0\}$ in local coordinates, then as $t \to \infty$, the integrand for $L(f)$ concentrates on a tubular neighborhood of the fixed point, and the only contribution to the integrand comes from $I = \{1, \ldots, k\}$, since the curvature term vanishes otherwise due to the product metric. Converting back to rectangular coordinates in the normal fiber as in the topological proof eliminates the $\tilde{\rho}_{I'}$ factor and introduces a factor of sgn $\det(df_\nu - \text{Id})$. Since $f = \text{Id}$ in submanifold directions, $\det(\frac{1}{2}(d(f_t)+\text{Id})^\mu_I) = 1$. Thus the integral splits into the curvature integral over N, yielding $\chi(N)$, and a normal integral, which gives sgn $\det(d(f_t)_\nu - \text{Id})$. In the $t \to \infty$ limit, $d(f_t)_\nu - \text{Id}$ in the normal fiber goes to the identity map, so its determinant becomes one. Plugging these terms into the integrand in Theorem 4.5 gives the Lefschetz fixed submanfiold formula.

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A Hodge theoretic techniques

As mentioned in §4.1, the upper bound for the Lefschetz number of a flat manifold can be extended to arbitrary metrics. Using sectional curvature bounds to control the Jacobi fields and the curvature tensor, one can extract an upper bound from the integral formula Theorem 4.5 in terms of the sectional curvature. In contrast, there is a Hodge theory argument which constructs a better upper bound in terms of Ricci curvature.

Let $\mathbf{N} = \mathbf{N}(n, C, D, V)$ be the class of Riemannian n-manifolds (M, g) with Ricci curvature $\mathrm{Ric} \geq C$, $\mathrm{diam}(M) \leq D$ and $\mathrm{vol}(M) \geq V$.

Proposition A.1 *There exist constants $C(k, n)$ and $D = D(\mathbf{N})$ such that for all $(M, g) \in \mathbf{N}$,*

$$|L(f)| \leq 1 + D \sum_{k=1}^{n} C(k, n) \binom{n}{k} \beta_k \cdot \sup_{x \in M} |df_x|_\infty^k,$$

where β_k is the k^{th} Betti number of M.

Before the proof, we compare two norms for differential forms. For $\alpha \in \Lambda^k T_x^* M$, we have the L^2 (Hodge) norm $|\alpha|_2^2 = *(\alpha \wedge *\alpha)$ and the L^∞ norm

$$|\alpha|_\infty = \sup_{v \in (T_x M)^{\otimes k} \setminus \{0\}} \frac{|\alpha(v)|}{|v|},$$

where $v = v_1 \otimes \ldots \otimes v_k$ has norm $|v| = \prod |v_i|$. Here we consider α as a linear functional on $(T_xM)^{\otimes k}$. Of course, there exists $C = C(g)$ such that $C^{-1}|\alpha|_\infty \leq |\alpha|_2 \leq C|\alpha|_\infty$, but we want this constant to depend only on k, n.

Lemma A.2 *There exists a constant $C(k,n)$ such that*

$$\binom{n}{k}^{-1/2} |\alpha|_2 \leq |\alpha|_\infty \leq C(k,n)|\alpha|_2.$$

PROOF: Let $\{\theta^i\}$ be an orthonormal basis of T_x^*M with dual basis $\{X_i\}$ of T_xM. For $\alpha = \alpha_I \theta^I$, we have

$$|\alpha|_\infty \geq \frac{|(\alpha_I\theta^I)(X_{i_1} \otimes \ldots \otimes X_{i_k})|}{|X_{i_1} \otimes \ldots \otimes X_{i_k}|} = |\alpha_{I_0}|,$$

where $I_0 = (i_1, \ldots, i_k)$. Thus

$$|\alpha|_\infty \geq \sup_I |\alpha_I| \geq \binom{n}{k}^{-1/2} \left(\sum_I |\alpha_I|^2\right)^{1/2} = \binom{n}{k}^{-1/2} |\alpha|_2.$$

For the other estimate,

$$|\alpha|_\infty^2 \leq \sup_{v = v_1 \otimes \ldots \otimes v_k \neq 0} \frac{\sum_I |\alpha_I|^2 |\theta^I(v_1 \otimes \ldots \otimes v_k)|^2}{|v_1 \otimes \ldots \otimes v_k|^2}.$$

For fixed $I_0 = (i_1, \ldots, i_k)$ and $v_1 = a_1^{j_1} X_{j_1}, \ldots, v_k = a_k^{j_k} X_{j_k}$, we have

$$|\theta^{I_0}(v_1 \otimes \ldots \otimes v_k)| \leq \sum_{\substack{j_1, \ldots, j_k \\ \{j_1, \ldots, j_k\} = I_0}} |a_1^{j_1} \cdot \ldots \cdot a_k^{j_k}|.$$

Thus

$$\begin{aligned}
|\alpha|_\infty^2 &\leq \sup_{v \neq 0} \frac{\sum_{I_0} |\alpha_{I_0}|^2 \sum_{\{j_1, \ldots, j_k\} = I_0} |a_1^{j_1} \cdot \ldots \cdot a_k^{j_k}|^2 \cdot k!}{|v_1 \otimes \ldots \otimes v_k|^2} \\
&= \sup_{v \neq 0} \frac{\sum_{I_0} |\alpha_{I_0}|^2 \sum_{\{j_1, \ldots, j_k\} = I_0} |a_1^{j_1} \cdot \ldots \cdot a_k^{j_k}|^2 \cdot k!}{\prod_{q=1}^k (\sum_{l_q} (a_q^{l_q})^2)} \\
&= \sup_{v \neq 0} \sum_{I_0} |\alpha_{I_0}|^2 \left[\frac{k!}{\binom{n}{k}} \frac{\sum_{\{j_1, \ldots, j_k\} = I_0} |a_1^{j_1} \cdot \ldots \cdot a_k^{j_k}|^2}{\prod_{q=1}^k (\sum_{l_q} (a_q^{l_q})^2)} \right].
\end{aligned}$$

For fixed I_0, the term inside the square brackets is a scale invariant function on $\mathbf{R}^{nk} = \{(a_i^j) : i = 1, \ldots, k,\ j = 1, \ldots, n\}$ and so is bounded above by $C'(k,n)$ independent of I_0. Thus

$$|\alpha|_\infty^2 \leq \frac{k!}{\binom{n}{k}} C'(k,n) \sum_I |\alpha_I|^2 = (C(k,n))^2 |\alpha|_2^2.$$

$\square$

Proof of the Proposition: Let $\{\omega_k^i\}$ be an L^2-orthonormal basis of harmonic k-forms. The trace of $f^* : H^k(M;\mathbf{R}) \to H^k(M;\mathbf{R})$ is $\sum_i \langle f^*\omega_k^i, \omega_k^i\rangle$, so

$$\begin{aligned} |L(f)| &\leq \sum_{k,i} \left| \langle f^*\omega_k^i, \omega_k^i \rangle \right| \leq \sum_{k,i} \|f^*\omega_k^i\| \\ &= \sum_{k,i} \left[\int_M |(f^*\omega_k^i)_x|_2^2 \mathrm{dvol}(x) \right]^{1/2}, \end{aligned} \tag{33}$$

by Cauchy-Schwarz. Here $\|\alpha\|^2 = \int_M \alpha \wedge *\alpha$ is the global L^2 norm. When $k = 0$, we have $\|f^*\omega_0^1\| = \|\omega_0^1\| = 1$.

By (33) and the lemma, we have

$$|L(f)| \leq 1 + \sum_{k=1}^n \sum_i \binom{n}{k} \mathrm{vol}^{1/2}(M) \sup_{x \in M} |(f^*\omega_k^i)_x|_\infty. \tag{34}$$

Now

$$|(f^*\omega)_x|_\infty = \sup_{v \neq 0} \frac{|(f^*\omega)_x (v_1 \otimes \ldots \otimes v_k)|}{|v_1 \otimes \ldots \otimes v_k|} = \sup_{v \neq 0} \frac{|\omega_{f(x)}(f_*v_1 \otimes \ldots \otimes f_*v_k)|}{|v_1 \otimes \ldots \otimes v_k|},$$

where $f_* = df$. Since the last term vanishes if $f_*v_i = 0$ for some i, we assume $f_*v_i \neq 0$. Then

$$\begin{aligned} |(f^*\omega)_x|_\infty &= \sup_{v \neq 0} \frac{|\omega_{f(x)}(f_*v_1 \otimes \ldots \otimes f_*v_k)|}{|f_*v_i \otimes \ldots \otimes f_*v_k|} \cdot \frac{|f_*v_1 \otimes \ldots \otimes f_*v_k|}{|v_1 \otimes \ldots \otimes v_k|} \\ &\leq |\omega_{f(x)}|_\infty \cdot \sup_{v \neq 0} \frac{\prod_i |f_*v_i|}{\prod_i |v_i|} \\ &\leq |\omega_{f(x)}|_\infty \cdot \sup_{v \neq 0} \frac{\prod_i |df_x|_\infty |v_i|}{\prod_i |v_i|} \\ &\leq |\omega_{f(x)}|_\infty |df_x|_\infty^k. \end{aligned}$$

By (34) and the lemma, we get

$$\begin{aligned}
|L(f)| &\leq 1+\sum_{k=1}^{n}\binom{n}{k}\mathrm{vol}^{1/2}(M)\sum_{i}\sup_{x\in M}|df_x|_\infty^k\cdot|(\omega_k^i)_{f(x)}|_\infty. \\
&\leq 1+\sum_{k=1}^{n}\binom{n}{k}\mathrm{vol}^{1/2}(M)\sum_{i}\sup_{x\in M}|df_x|_\infty^k\cdot C(k,n)|(\omega_k^i)_{f(x)}|_2.
\end{aligned}$$

By [2], [11], there is an explicit constant $D_1(\mathbf{N})$ such that for all $x \in M$,

$$|(\omega_k^i)_x|_2 \leq D_1(\mathbf{N})\|\omega_k^i\| = D_1(\mathbf{N}).$$

Thus

$$|L(f)| \leq 1+\sum_{k=1}^{n}\binom{n}{k}\beta_k\cdot\mathrm{vol}^{1/2}(M)\cdot D_1(\mathbf{N})C(k,n)\sup_{x\in M}|df_x|_\infty^k.$$

Finally, $\mathrm{vol}(M)$ is bounded above on $\mathbf{N}$ by standard comparison theorems. $\square$

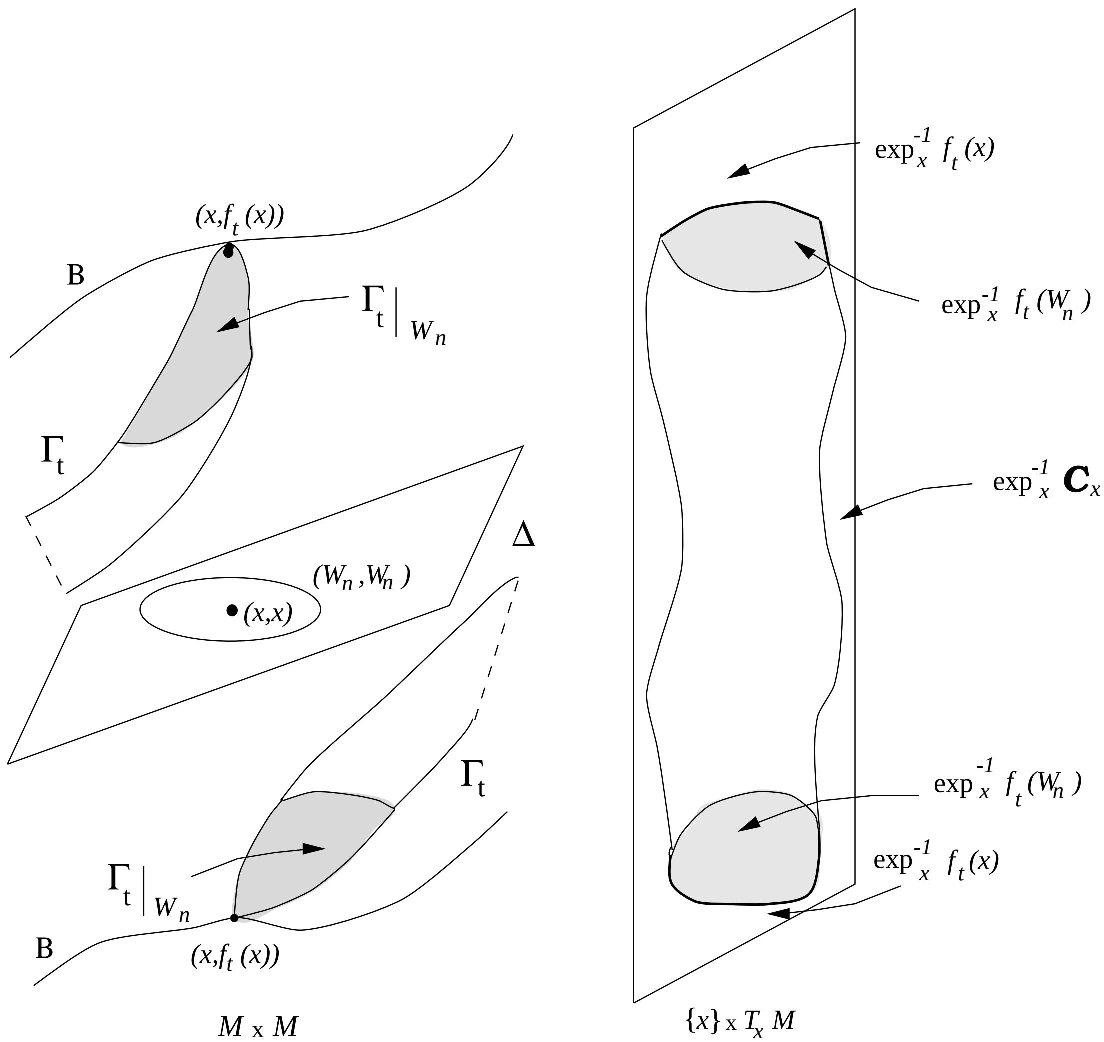

Figure 3: On the left, the graph Γ_t of f_t inside $M \times M$ and its restriction to W_n near $x \in \mathrm{Far}(f)$. B is the boundary of the cut locus tubular neighborhood. According to (8), $L(f) - \chi(M)$ equals $\int_{\Gamma_t|_{W_n}} \mathrm{MQ}_\Delta$ in the limit as $n \to \infty, t \to 0$. After a pullback via the exponential map, $L(f) - \chi(M)$ becomes an integral over the shaded region in the vertical tangent space $\{x\} \times T_x M$ on the right.

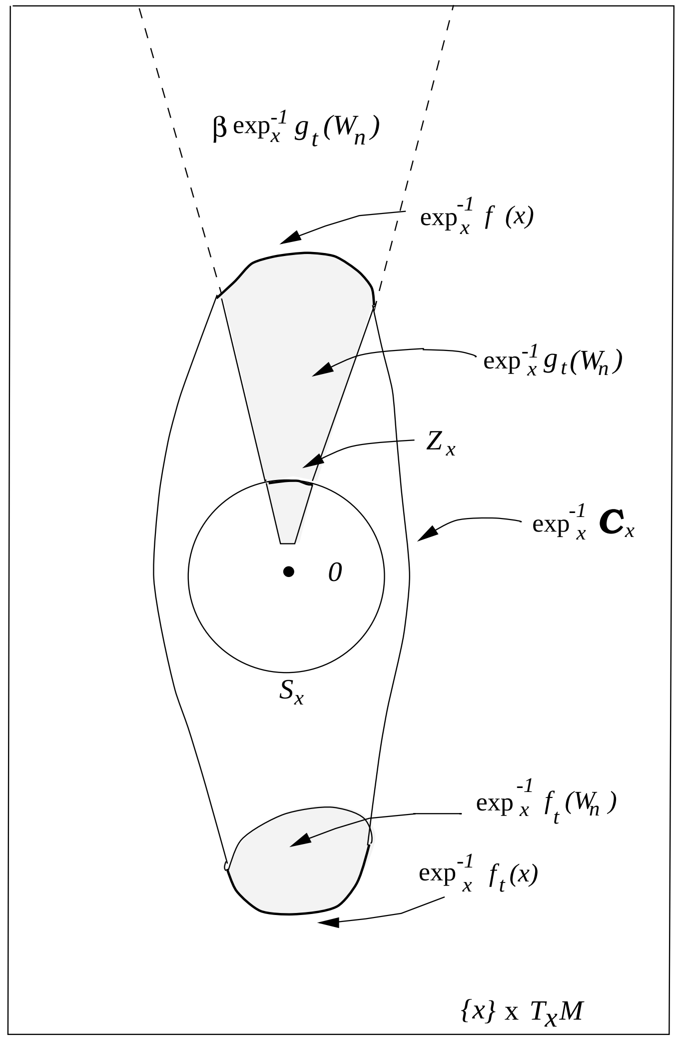

Figure 4: The deformation of $exp_x^{-1} f(x)$ via g_t. On the bottom is one component of $\exp_x^{-1} f(x)$ and its neighborhood inside the cut locus in the tangent space as in Figure 3. On the top is another component as well as its deformation via g_t and its pullback via β to all of $\{x\} \times T_xM$. Z_x is the intersection of this region (and the corresponding region from the bottom) with a small sphere S_x, and the far point invariant α_x is the solid angle ratio $|Z_x|/|S_x|$.