Bounded and L^2 Harmonic Forms on Universal Covers

K. D. Elworthy

Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK, kde@maths.warwick.ac.uk

Xue-Mei Li

Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA, xmli@math.uconn.edu

Steven Rosenberg

Department of Mathematics, Boston University, Boston, MA 02215, USA, sr@math.bu.edu

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1 Introduction

In this paper we relate certain curvature conditions on a complete Riemannian manifold to the existence of bounded and L^2 harmonic forms. In the case where the manifold is the universal cover of a compact manifold, we obtain topological and geometric information about the compact manifold. Many of these results can be thought of as differential form analogues of Myers' theorem that a complete manifold with strictly positive Ricci curvature is compact with finite fundamental group. We also give pinching conditions on certain sums of sectional curvatures which imply Bochner-type vanishing theorems for harmonic forms. In particular, we construct a compact manifold with planes of negative sectional curvature at each point and which satisfies the hypothesis of our vanishing theorem. We believe this is the first explicit example of a vanishing theorem for the Laplacian on forms on a compact manifold which allows negative curvature everywhere. Finally, we remark on related spectral gap estimates for the Laplacian on forms on complete manifolds.

In more detail, recall that Bochner's theorem states that there are no L^2 harmonic p-forms on a complete manifold M if the curvature term $\mathcal{R}^p$ in the Weitzenböck decomposition $\Delta^p = \nabla^*\nabla + \mathcal{R}^p$ for the Laplacian on p-forms is positive. In particular, if M is compact, then $\mathcal{R}^p > 0 \Rightarrow H^p(M; \mathbf{R}) = 0$. This result has the disadvantage that

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$\mathcal{R}^p$ is a complicated curvature expression whose geometric/topological significance is unclear, except for $\mathcal{R}^1 = \mathrm{Ric}$, the Ricci curvature. In general, even manifolds with positive sectional curvature need not have $\mathcal{R}^p > 0$ for $1 < p < n-1$.

We have previously made some progress in this area. In [9], we showed that $\mathcal{R}^p$ need only be strongly stochastically positive, or s. s. p., (as defined in §2) for Bochner's theorem to hold. In practice, this allows $\mathcal{R}^p$ to be negative on a set of small volume if M is compact. We also showed that $\mathcal{R}^2$ s. s. p. implies $\pi_2(M)$ is a torsion group. In [11], we showed that the positivity of $\mathcal{R}^p$ on primitive vectors is equivalent to the positivity of a sum of sectional curvatures, and we used this to obtain vanishing theorems for $H_p(M;\mathbf{Z})$ for manifolds isometrically immersed in $\mathbf{R}^N$. Finally, in [17] the second author showed that the hypothesis of Myers' theorem can be extended to the strong stochastic positivity of the Ricci curvature.

In this paper, we extend these results in several directions. First, in contrast to the usual Jacobi field proof, the proof of Myers' theorem in [17] comes down to showing the existence of an L^2 harmonic function on the universal cover of M. In §2, we extend this argument from functions to forms. The main analytic result (Theorem 2.1) states that if $\mathcal{R}^{p\pm 1}$ is s. s. p., and if M has a bounded harmonic p-form, then M has an L^2 harmonic p-form. In fact, all the results in this section carry over to the Bismut-Witten Laplacian, and are stated in this generality.

As consequences, we give a series of results indicating the geometric significance of $\mathcal{R}^2$. We show that the universal cover of a compact manifold M with $\mathcal{R}^2$ s. s. p. and with a nonconstant bounded harmonic function admits a nonconstant harmonic function of bounded energy (Proposition 2.1). This implies that a compact manifold cannot admit both a metric with $\mathcal{R}^2$ s. s. p. and a metric with pinched negative curvature (Theorem 2.3). (There are corresponding results for p-forms.) Moreover, if M, N are compact manifolds with $\pi_1(M)$ nonamenable and $\pi_1(N)$ infinite, then $M \times N$ admits no metric with $\mathcal{R}^2$ s. s. p. (Theorem 2.4). We also show that a 4-manifold with nonamenable fundamental group and $\chi(M) > -2$ has no metric with $\mathcal{R}^2$ s. s. p. (Proposition 2.2); in particular, no 4-manifold of negative curvature admits such a metric.

In §3, we prove that $\mathcal{R}^p > 0$ if there exists $A = A(x) > 0$ such that

$$CA < \sum_{i=1}^{p} \sum_{j=p+1}^{n} K(v^i, v^j) \equiv \sum_p < A, \tag{1.1}$$

for all orthonormal bases $\{v^1, \ldots, v^n\}$ of T_xM. Here $C = C(n,p)$ is an explicit constant, and $K(v^i, v^j)$ is the sectional curvature of the (v^i, v^j)-plane. In particular, a manifold with this pinching estimate has no L^2 harmonic p-forms.

Note that the sum of curvatures in (1.1) is $\mathrm{Ric}(v^1, v^1)$ if $p = 1$. In general, this sum of curvatures is precisely the sum in [11] mentioned above (which in turn is based on [15]). Combined with Theorem 2.3, this shows that a compact manifold cannot admit both a metric of pinched negative curvature and a metric with $\sum_2$ pinched as in (1.1). We also show that the metric product $M = \Sigma \times S^4$ of a surface Σ of constant negative curvature with the 4-sphere satisfies the pinching estimate above with $p = 3$. As mentioned above, this example is significant because it has planes of negative sectional curvature at each point.

Finally, in §4 we give estimates for the spectral gap at zero for the (Bismut-Witten) Laplacians on forms. These estimates come from the L^2 analogues of the L^∞ estimates of §2.

2 Bounded harmonic forms and L^2 harmonic forms

In this section we relate the positivity of the Weitzenböck term for the Laplacian on forms to the existence of bounded and L^2 harmonic forms. We give applications of this result to the geometry and topology of compact manifolds.

Let (M, g) be a complete Riemannian manifold and $h : M \to \mathbf{R}$ a smooth function. The case $h \equiv 0$ is of particular interest. Let δ^h be the adjoint of the exterior derivative d on forms with respect to the measure $\mu(dx) = \mathrm{e}^{2h}\mathrm{dvol}_g$. The Bismut-Witten Laplacian $\Delta^h = (d + \delta^h)^2$ is a self-adjoint operator on L^2 forms with respect to $\mu(dx)$, see [16]. This Laplacian restricts to an operator $\Delta^{h,q}$ on q-forms, and has the Weitzenböck decomposition

$$\Delta^{h,q} = \nabla^*\nabla + \mathcal{R}^{h,q}.$$

In particular, $\mathcal{R}^{h,1} = \mathrm{Ric} - 2 \cdot \mathrm{Hess}(h)$, where $\mathrm{Hess}(h) = -\nabla^2 h$ is the Hessian of h. We use the convention $\mathcal{R}^{h,-1} = 0$, $\mathcal{R}^{h,n+1} = 0$. Let $\underline{\mathcal{R}}^{h,q}(x)$ be the infimum of $\mathcal{R}^{h,q}(v)$ over all unit q-covectors $v \in \Lambda^q T_x^* M$. We will omit the superscripts h, q if the context is clear.

Let $\{x_t\}$ be a path continuous diffusion process with $-\Delta^h = -\Delta^{h,0}$ as generator, i.e. a h-Brownian motion, starting from x_0. Assume that the h-Brownian motion does not explode, or equivalently that the associated Bismut-Witten heat semigroup P_t^h is conservative: $P_t^h 1 \equiv 1$. For $P_t^{h,q} = \mathrm{e}^{-\frac{1}{2}t\Delta^{h,q}}$ the heat semigroup on $L^2\Lambda^q(M, \mu(dx))$ defined by the spectral theorem, it is known that for each q-form $\phi \in L^\infty \cap L^2$,

$$P_t^{h,q}\phi(v_0) = \mathbf{E}\phi\left(W_t^{h,q}(v_0)\right), \tag{2.1}$$

provided $\underline{\mathcal{R}}^{h,q}$ is bounded from below. Here $W_t^{h,q}(v_0)$ is the solution to the stochastic covariant differential equation along paths of $\{x_t\}$:

$$\begin{cases} \frac{DW_t^{h,q}}{\partial t} & = & -\frac{1}{2}\mathcal{R}^{h,q}(W_t^{h,q}(v_0)) \\ W_0^{h,q}(v_0) & = & v_0, \end{cases} \tag{2.2}$$

for $v_0 \in \Lambda^q T_{x_0} M$; we use the metric g to let $\mathcal{R}^{h,q}$ act on $\Lambda^q T_{x_0} M$. This easily implies

$$|W_t^{h,q}(v_0)| \le \mathrm{e}^{-\frac{1}{2}\int_0^t \underline{\mathcal{R}}^{h,q}(x_s)ds}. \tag{2.3}$$

Let

$$\underline{\mathcal{R}}_q(x_0) = \int_0^\infty \mathbf{E} e^{-\frac{1}{2}\int_0^t \underline{\mathcal{R}}^{h,q}(x_s)ds} dt,$$

and let $H = H^{h,q}$ be the h-harmonic projection on the space of $L^2\Lambda^q(M, \mu(dx))$. Thus for such ϕ, $H\phi = \lim_{t\to\infty} P_t^{h,q}\phi$. Let $C_c^\infty\Lambda^q = C_c^\infty\Lambda^q T^*M$ denote the space of smooth compactly supported q-forms, and let $|\phi|_{L^p}$ denoted the L^p norm of $|\phi|$.

LEMMA 2.1 *Let $q \in \{2, \ldots, n-2\}$. Assume that the h-Brownian motion does not explode and that $\mathcal{R}^{h,q-1}$ and $\mathcal{R}^{h,q+1}$ are bounded below. For $\phi, \psi \in C_c^\infty\Lambda^q$, we have*

$$\begin{aligned} \left|\int_M \langle H\phi - \phi, \psi\rangle \mu(dx)\right| &\leq \frac{1}{2}\left[\sup_{x_0\in Supp(\phi)} \underline{\mathcal{R}}_{q+1}(x_0)\right] |d\psi|_\infty \cdot |d\phi|_{L^1} \\ &\quad + \frac{1}{2}\left[\sup_{x_0\in Supp(\phi)} \underline{\mathcal{R}}_{q-1}(x_0)\right] |\delta^h\psi|_\infty \cdot |\delta^h\phi|_{L^1}. \end{aligned} \tag{2.4}$$

PROOF. For $P_t = P_t^{h,q}$, we have

$$\int_M \langle H\phi - \phi, \psi\rangle \mu(dx) = \lim_{t\to\infty} \int_M \langle P_t\phi - \phi, \psi\rangle \mu(dx). \tag{2.5}$$

On the other hand,

$$\begin{aligned} \int_M \langle P_t\phi - \phi, \psi\rangle \mu(dx) &= -\frac{1}{2}\int_M \int_0^t \langle \Delta^h(P_s\phi), \psi\rangle \mu(dx) \\ &= \frac{1}{2}\int_0^t \int_M \langle d\phi, P_s(d\psi)\rangle \mu(dx)\, ds \\ &\quad + \frac{1}{2}\int_0^t \int_M \langle \delta^h\phi, P_s(\delta^h\psi)\rangle \mu(dx)\, ds. \end{aligned}$$

Under the assumptions of the lemma, $|P_s(d\psi)|(x_0) \leq |d\psi|_\infty \cdot \mathbf{E}|W_s^{h,q+1}|_{x_0}$ and $|P_s(\delta^h(\psi)|(x_0) \leq |\delta\psi|_\infty \cdot \mathbf{E}|W_s^{h,q-1}|_{x_0}$. So by (2.3) we have

$$\begin{aligned} &\left|\int_M \langle P_t\phi - \phi, \psi\rangle \mu(dx)\right| \\ &\quad \leq \frac{1}{2}|d\psi|_\infty \cdot \sup_{x_0\in supp(\phi)} \int_0^\infty \mathbf{E} e^{-\frac{1}{2}\int_0^u \underline{\mathcal{R}}^{h,q+1}(x_s)ds} du \cdot |d\phi|_{L^1} \\ &\qquad + \frac{1}{2}|\delta^h\psi|_\infty \cdot \sup_{x_0\in supp(\phi)} \int_0^\infty \mathbf{E} e^{-\frac{1}{2}\int_0^u \underline{\mathcal{R}}^{h,q-1}(x_s)ds} du \cdot |\delta^h\phi|_{L^1}, \end{aligned}$$

and the required inequality follows from (2.5). ■

For one-forms the corresponding result is:

Lemma 2.2 *Assume that the h-Brownian motion does not explode. Then for* $\phi, \psi \in C_c^\infty \Lambda^1 T^*M$,

$$\int_M \langle H\phi - \phi, \psi\rangle \mu(dx) \leq \frac{1}{2}\left[\sup_{x_0 \in Supp(\phi)} \underline{\mathcal{R}}_2(x_0)\right] |d\psi|_\infty \cdot |d\phi|_{L^1} + \frac{1}{2} \sup_{x_0 \in supp(\phi)} \left| \int_0^\infty P_s(\delta^h \phi) ds \right| \cdot |\delta^h \psi|_{L^1}$$

The main application of these estimates is to the existence of harmonic forms. By a *bounded* C^1 *h-harmonic q-form* ($q \neq 0$), we mean a bounded q-form with $d\phi = 0$ and $\delta^h \phi = 0$. For $q = 0$, a bounded harmonic function f is given by the usual definition $\Delta^{h,0} f = 0$. Note that a bounded solution of $\Delta^{h,q}\phi = 0$ need not be a bounded h-harmonic form. By an L^2 *h-harmonic q-form* we mean an L^2 solution of $\Delta^{h,q}\phi = 0$. An L^2 h-harmonic q-form ϕ satisfies $d\phi = \delta^h\phi = 0$. The following result is based on techniques going back to [4].

Theorem 2.1 *Let* $q \in \{2, 3, \ldots, n-2\}$. *Assume that h-Brownian motion does not explode. Suppose* $\mathcal{R}^{h,q-1}$ *and* $\mathcal{R}^{h,q+1}$ *are bounded from below with both* $\sup_{x \in K} \underline{\mathcal{R}}_{q+1}(x)$ *and* $\sup_{x\in K} \underline{\mathcal{R}}_{q-1}(x)$ *finite for all compact sets* $K \subset M$. *If there exists a nontrivial bounded h-harmonic q-form, then there exists a nontrivial* L^2 *harmonic q-form.*

Proof. Let ψ be a nontrivial bounded h-harmonic q-form. Take a sequence of functions $\{h_n\}$ on M with $0 \leq h_n \leq 1$, $|\nabla h_n| \leq \frac{1}{n}$ and $h_n \to 1$ as $n \to \infty$, see [4], [12]. Define $\psi_n = h_n \cdot \psi$. Then $d\psi_n = dh_n \wedge \psi$ and $\delta^h \psi_n = -\mathrm{i}_{\nabla h_n}\psi$, and so

$$|d\psi_n|_\infty \leq \frac{1}{n}|\psi|_\infty, \quad |\delta^h \psi_n|_\infty \leq \frac{1}{n}|\psi|_\infty.$$

Thus by Lemma 2.1,

$$\left|\int_M \langle H\phi - \phi, \psi_n\rangle \mu(dx)\right| \leq c\frac{|\psi|_\infty}{n}\left[|d\phi|_{L^1} + |\delta^h\phi|_{L^1}\right],$$

where c is a constant depending on the support of ϕ. If there are no L^2 h-harmonic q-forms, then $H\phi \equiv 0$ for all $\phi \in C_c^\infty \Lambda^q$. Thus

$$\lim_{n\to\infty} \int_M \langle -\phi, \psi_n\rangle \mu(dx) = -\int_M \langle \phi, \psi\rangle \mu(dx),$$

since ψ is bounded and ϕ has compact support. So

$$\int_M \langle \phi, \psi \rangle \mu(dx) = 0$$

for any $\phi \in C_c^\infty \Lambda^q$. Therefore ψ has to be zero, contradicting the hypothesis. ■

Definition: A function $f \in C^1(M)$ is *strongly stochastically positive (s. s. p.)* if

$$\varlimsup_{t\to\infty} \frac{1}{t} \sup_{x_0 \in K} \log \mathbf{E}\left(e^{-\frac{1}{2}\int_0^t f(x_s)ds}\right) < 0$$

for each compact set K. We say that $\mathcal{R}^p$ is s. s. p. if the function $\underline{\mathcal{R}}^p$ is s. s. p..

Remarks: (1) Strong stochastic positivity is equivalent to $\Delta^0 + f > 0$ on $L^2(M, g)$ if M is compact [8]. In particular, a positive function on a compact manifold is strongly stochastically positive. More generally, if f is positive except on a set of small volume in the sense of [9], [23], then f is s. s. p. Note that a function is s. s. p. on any manifold M, then its pullback is s. s. p. on any cover of M, since Brownian motion on the cover projects to Brownian motion on the base. Thus if $\Delta^0 + \underline{\mathcal{R}}^p > 0$ on a compact manifold M, then $\mathcal{R}^p$ is s. s. p. on the universal cover $\widetilde{M}$ with respect to the pullback metric. Thus the condition $\mathcal{R}^p$ is s. s. p. on $\widetilde{M}$ generalizes the condition $\mathcal{R}^p > 0$ on M.

(2) By semigroup domination [8], $\mathcal{R}^{h,q}$ s. s. p. implies the vanishing of the spaces of L^2 h-harmonic q-forms and bounded h-harmonic q-forms. Note also that $\mathcal{R}^{h,q}$ s. s. p. implies that $\sup_{x\in K} \underline{\mathcal{R}}_q(x)$ is finite, for K compact in M. Thus the hypotheses in Theorem 2.1 on this quantity can be replaced by assuming $\mathcal{R}^{h,q\pm1}$ is s. s. p.

In summary, *all hypotheses in this paper of the form "$\mathcal{R}^p$ is* s. s. p. *on the compact manifold M" can be replaced by "$\mathcal{R}^p > 0$ on M."*

We now give some geometric applications of the last theorem. Let $H^k(M)$ and $L^2H^k(M)$ denote the spaces of harmonic and L^2 h-harmonic k-forms (or harmonic k-forms in the case of $h \equiv 0$). Of course, if M is compact then $H^k(M) \simeq H^k(M; \mathbf{R})$.

A manifold is said to have *negative curvature with pinching constant k* if its sectional curvatures K satisfy $-1 \le K < -k < 0$.

COROLLARY 2.1 *Let M be a compact n-manifold. Suppose $\mathcal{R}^{q\pm1} \equiv \mathcal{R}^{0,q\pm1}$ are strongly stochastic positive for some q with $q \in \{2, \ldots, n-2\}$ and $q \neq n/2$. If $H^q(M) \neq 0$, then M admits no metric of negative curvature with pinching constant $q^2/(n-q-1)^2$.*

PROOF. As noted in Remark (1), strong stochastic positivity is preserved in passing to a cover. In particular, the universal cover $\widetilde{M}$ of M has $\mathcal{R}^{q\pm1}$ s. s. p. Since $H^q(M; \mathbf{R}) \neq 0$, there is a nontrivial harmonic q-form on M, which lifts to a bounded harmonic form on $\widetilde{M}$. By Theorem 2.1, $L^2H^q(\widetilde{M}) \neq 0$.

Suppose M also admits a metric with negative curvature pinched as in the hypothesis. Then $L^2H^q(\widetilde{M}) = 0$, with respect to this new metric [10]. However the vanishing of $L^2H^q(\widetilde{M})$ is independent of the Riemannian metric on M [1], which gives a contradiction. ∎

Remark: Theorem 2.1 generalizes the corresponding result for $q = 0$ in [17]. In particular, since every manifold has a bounded harmonic function, namely 1, the universal cover $\widetilde{M}$ of a compact manifold M with strongly stochastic positive Ricci curvature has an L^2 harmonic function. Such functions are constant, which implies that $\widetilde{M}$ has finite volume. Thus M must be finite volume and have finite fundamental group. This extends Myers' theorem, which states that a complete manifold with $\mathrm{Ric} > \epsilon > 0$ is compact with finite fundamental group.

Next we discuss the corresponding result for differential one-forms. Recall that our diffusion process is transient if

$$\sup_{x\in K}\int_0^\infty P_t f(x)dt < \infty$$

for each f with compact support, see e.g. [3].

THEOREM 2.2 *Assume the h-Brownian motion does not explode and is transient. Suppose* $\underline{\mathcal{R}}^{h,2}$ *is bounded from below and* $\sup_{x\in K}\underline{\mathcal{R}}_2(x) < \infty$ *for each compact subset* K. *If there exists a nontrivial bounded h-harmonic one-form, then there exists a nontrivial* L^2 *h-harmonic one-form.*

The proof is similar to that of Theorem 2.1. This leads to the following stronger result:

PROPOSITION 2.1 *Let* M *be a compact manifold admitting a metric with* $\mathcal{R}^2$ *strongly stochastic positive. If either*

(i) $\widetilde{M}$ *admits a nonconstant bounded harmonic function with respect to the pullback metric, or*

(ii) $H^1(M;\mathbf{R}) \neq 0$ *and Brownian motion is transient on* $\widetilde{M}$,
then $L^2H^1(\widetilde{M}) \neq 0$, *and* $\widetilde{M}$ *admits a harmonic function of finite energy.*

THEOREM 2.3 *A compact n-manifold cannot admit a metric of* $[1-(n-2)^{-2}]$*-pinched negative curvature and a metric with* $\mathcal{R}^2$ *strongly stochastically positive.*

PROOF OF THE THEOREM: By [10], the pinching condition on the negatively curved metric g_1 implies $L^2H^1(\widetilde{M},\tilde{g}_1) = 0$, where $\tilde{g}_1$ is the pullback of g_1 to $\widetilde{M}$. A negatively curved manifold M has $\lambda_0(\widetilde{M}) > 0$ by McKean's estimate [19], where λ_0 is the bottom of the spectrum for Δ^0. By Brooks' theorem [5], $\pi_1(M)$ must be nonamenable. By [18], the nonamenability of $\pi_1(M)$ implies $\widetilde{M}$ has a nonconstant bounded harmonic function f, which can be assumed to be positive, with respect to any pullback metric, in this case the pullback of the metric g_2 with $\mathcal{R}^2$ s. s. p. The the condition $L^2H^1(\widetilde{M}) = 0$ is independent of the metric on M, since the spaces of L^2 harmonic forms are isomorphic to the L^2 closed forms modulo the closure of the L^2 exact forms, and hence are quasi-isometry invariants of $\widetilde{M}$. Thus $L^2H^1(\widetilde{M}, \tilde{g}_2) = 0$. This contradicts the proposition. ∎

Proof of the Proposition: For (i), it is well known that the existence of a nontrivial bounded harmonic function f implies the transience of Brownian motion on $\widetilde{M}$ with respect to the pullback metric. Thus $\omega = df$ is a nonzero one-form satisfying $d\omega = \delta\omega = 0$. Moreover, ω is bounded by Harnack's inequality [25] or by Bismut's formula in [7]. Thus Theorem 2.2 implies $L^2H^1(\widetilde{M}) \neq 0$. Moreover, f has $\Delta f = 0$ and has finite energy $\|df\|_{L^2}$.

For (ii), note that $H^1(M; \mathbf{R}) \neq 0$ implies that M has a nontrivial harmonic one-form, which pulls back to a bounded harmonic one-form on $\widetilde{M}$. By Theorem 2.2, $\widetilde{M}$ has an L^2 harmonic one-form ω. Since $H^1(\widetilde{M}; \mathbf{R}) = 0$, we have $\omega = df$ for some function f on $\widetilde{M}$. Thus $0 = \delta\omega = \Delta f$, and f has finite energy as before. ∎

As an example of the sharp nature of this result, let $M = S^3 \times S^1$. Since the standard metric on S^3 has $\mathcal{R}^2 > 0$, the product of the standard metrics has $\mathcal{R}^2 > 0$ by an easy calculation. M has a harmonic one-form, which lifts to a bounded harmonic one-form on $\widetilde{M} = S^3 \times \mathbf{R}$. However, $L^2H^1(\widetilde{M}) = 0$ by direct calculation. Brownian motion on $\widetilde{M}$ is recurrent, so the hypothesis on the transience of Brownian motion cannot be omitted from Theorem 2.2.

As another application of Theorem 2.2, we have

Theorem 2.4 *Let M, N be compact manifolds with $\pi_1(M)$ nonamenable and $\pi_1(N)$ infinite. Then $M \times N$ admits no metric with $\mathcal{R}^2$ strongly stochastically positive.*

Proof: The universal covers $\widetilde{M}, \widetilde{N}$ have no L^2 harmonic functions since $\pi_1(M), \pi_1(N)$ are infinite. The Künneth formula for L^2 harmonic forms thus implies $L^2H^1(\widetilde{M} \times \widetilde{N}) = 0$. However, $\pi_1(M \times N)$ is nonamenable, so as above $\widetilde{M} \times \widetilde{N}$ has a nontrivial bounded harmonic one-form with respect to any pullback metric. The theorem follows as above. ∎

Remark: As pointed out to us by J. Lott, the Künneth formula for the Laplacian on L^2 forms on a product manifold follows from the discussion of the spectrum of an operator of the form $A \otimes 1 + 1 \otimes B$ in [22, Vol. I, Thm. VIII.33]. Namely, this theorem implies that the zero spectrum of the Laplacian on a product manifold is given by the expected combintations of the zero spectra on the individual factors.

We now consider these results for 4-manifolds. In this dimension, the positivity (resp. s. s. p.) of $\mathcal{R}^2$ is equivalent to the positivity (resp. s. s. p.) of the curvature operator on complex isotropic two-planes [21]. The Euler characteristic of such manifolds is constrained as follows:

Proposition 2.2 *Let M be a compact, oriented 4-manifold.*

(i) If $\mathcal{R}^2$ is strongly stochastically positive and $\pi_1(M)$ is finite, then M is diffeomorphic to S^4.

(ii) If $\mathcal{R}^2$ is strongly stochastically positive and $\pi_1(M)$ is infinite, then $\chi(M) \leq 0$. In particular, a compact 4-manifold cannot admit a metric of negative curvature and a metric with $\mathcal{R}^2$ strongly stochastically positive.

(iii) If $\mathcal{R}^2$ is strongly stochastically positive and $\pi_1(M)$ is nonamenable, then $\chi(M) \leq -2$.

Proof: (i) By [9], $\mathcal{R}^2$ strongly stochastically positive implies $\widetilde{M}$ is a homotopy sphere and hence is diffeomorphic to S^4 by Freedman's solution to the Poincaré conjecture. Since $\chi(\widetilde{M})/\chi(M)$ equals the order of the covering map from $\widetilde{M}$ to M, the covering map is either $2-1$ or $1-1$. Since M is oriented, only the $1-1$ case is allowed.

(ii) By the L^2-index theorem [1], $\chi(M)$ equals $L^2\chi(\widetilde{M})$, the L^2 Euler characteristic of $\widetilde{M}$. Since $\pi_1(M)$ is infinite, $0 = L^2H^0(\widetilde{M}) \cong L^2H^4(\widetilde{M})$. Since $\widetilde{M}$ has $\mathcal{R}^2$ strongly stochastically positive for the pullback metric, $L^2H^2(\widetilde{M}) = 0$, and so $0 \geq L^2\chi(\widetilde{M})$.

By Avez's theorem [2], the Euler characteristic of a negatively curved 4-manifold is positive. Of course, such a manifold has infinite fundamental group.

(iii) As in the proofs of Proposition 2.1 and Theorem 2.3, the nonamenability of $\pi_1(M)$ implies the existence of a nonconstant bounded harmonic function f (and hence the transience of Brownian motion) and the existence of a bounded harmonic one-form df. By Theorem 2.2, $0 \neq L^2H^1(\widetilde{M}) \cong L^2H^3(\widetilde{M})$, so $\chi(M) = L^2\chi(\widetilde{M}) < 0$. Since $H^2(M;\mathbf{R}) = 0$, $\chi(M)$ is even, so $\chi(M) \leq -2$. ∎

S^4 with its standard metric shows that $\pi_1(M)$ must be infinite in (ii). For an example of (iii) not covered by (ii), let N be a compact hyperbolic 3-manifold with $H^1(N;\mathbf{R}) = 0$; examples of such manifolds occur in Dehn surgery on the figure eight knot complement in S^3. Set $M = N \times S^1$. Then $\chi(M) = 0$ and $\pi_1(M)$ is nonamenable, so M admits no metric with $\mathcal{R}^2$ strongly stochastically positive. Note that $H^2(M;\mathbf{R}) = 0$ and $\pi_2(M) = 0$. Thus neither (i), (ii) nor the Micallef-Moore result [20] ($\mathcal{R}^2$ pointwise positive implies $\pi_2(M^4) = 0$) shows that M admits no such metric.

The proposition above is valid for 6-manifolds (except for the part using Avez's theorem). The manifolds $N\times N$ and $N\times S^3$, for N as above, do not admit metrics with $\mathcal{R}^2$ strongly stochastically positive, although this cannot be seen from the L^2-index theorem alone.

3 Pinching and positivity of $\mathcal{R}^p$

In this section we will give conditions that guarantee stochastic spectral positivity of $\mathcal{R}^p$.

Definition: We say that a Riemannian manifold M is C-*p*-*pinched* if for each $x \in M$ there exists $A = A(x) > 0$ such that for each orthonormal basis $\{v^1, \ldots, v^n\}$ of T_xM,

$$CA < \sum_{i=1}^{p} \sum_{j=p+1}^{n} K(v^i, v^j) < A. \tag{3.1}$$

If $C = C(n,p)$ has explicit dependence on p, then we will just say that M is C-pinched.

Here $K(v^i, v^j)$ is the sectional curvature of the plane spanned by (the duals of) v^i, v^j. We will denote the sum of sectional curvatures just by $\sum_p$ if the context is clear.

The purpose of this section is to compute $C = C(n,p)$ such that C-pinching implies M^n has $\mathcal{R}^p > 0$. Note that a compact manifold is 0-1-pinched iff it has positive Ricci curvature, so we are not interested in pinching theorems for 1-forms. Since $*\mathcal{R}^p = \mathcal{R}^{n-p}*$ for the Hodge star operator $*$, we are similarly uninterested in pinching results for $(n-1)$-forms.

THEOREM 3.1 *Assume* $p \neq 1, n-1$. *Set*

$$C(n,p) = \frac{\frac{1}{2}p(n-p) + \frac{4}{3}\binom{p}{2}\binom{n-p}{2}}{1 + \frac{1}{2}p(n-p) + \frac{4}{3}\binom{p}{2}\binom{n-p}{2}}.$$

If M *is* $C(n,p)$-*pinched, then* $\mathcal{R}^p > 0$. *In particular,* $C(n,p)$-*pinching implies* $H^p(M;\mathbf{R}) = 0$ *if* M *is compact.*

Before we begin the proof, we review some curvature tensor manipulations, both as motivation and for use in a later example. Let $K(i,j)$ denote the sectional curvature of the 2-plane spanned by v^i, v^j, and similarly let $K(i+j,k)$ denote the sectional curvature for the 2-plane spanned by $v^i + v^j, v^k$. We have $R_{ijkl} = \langle R(v^i, v^j)v^k, v^l\rangle$, where we fix our sign convention by $K(1,2) = -R_{1212} = 1$ for an orthonormal frame on S^2. We set $R_{(i+j)k(s+t)l} = \langle R(\frac{v^i+v^j}{\sqrt{2}}, v^k)\frac{v^s+v^t}{\sqrt{2}}, v^l\rangle$, for $i \neq j, s \neq t$. From

$$\begin{aligned} -K(i, l+j) &\equiv \langle R(v^i, \frac{v^l+v^j}{\sqrt{2}})v^i, \frac{v^l+v^j}{\sqrt{2}}\rangle \\ &= \frac{1}{2}[R_{ilil} + R_{ijij} + R_{ilij} + R_{ijil}], \end{aligned}$$

for $l \neq j$, we get

$$R_{ijil} = \begin{cases} -K(i,j), & j = l \\ \frac{1}{2}K(i,l) + \frac{1}{2}K(i,j) - K(i, l+j), & j \neq l. \end{cases} \tag{3.2}$$

We also know

$$2R_{(i-k)j(i-k)l} = R_{ijil} - R_{ijkl} - R_{kjil} + R_{kjkl}. \tag{3.3}$$

Switching i and j in (3.3), subtracting the result from (3.3) and using the Bianchi identity, we obtain

$$3R_{ijkl} = -2R_{(i-k)j(i-k)l} + 2R_{(j-k)i(j-k)l} + R_{ijil} + R_{kjkl} - R_{jijl} - R_{kikl}. \tag{3.4}$$

Note that combining (3.2) and (3.4) gives the well known expression for R_{ijkl} in terms of sectional curvatures.

We now carry these calculations over to $\mathcal{R}^p$.

LEMMA 3.1 *Let J, K be multi-indices of length p.*

(i) If $|J \cap K| < p-2$, then $\langle \mathcal{R}^p v^J, v^K \rangle = 0$.

(ii) If I is a multi-index of length $p-2$ and i, j, k, l are distinct indices not in I, then

$$\langle R^p(v^i \wedge v^j \wedge v^I), v^k \wedge v^l \wedge v^I \rangle = 2R_{ijkl}. \tag{3.5}$$

(iii) If I is a multi-index of length $p-2$, if i, j, l are distinct indices not in I, and if $B < \sum_p < A$, then

$$|\langle \mathcal{R}^p(v^i \wedge v^j \wedge v^I), v^i \wedge v^l \wedge v^I \rangle| \le \frac{1}{2}(A-B). \tag{3.6}$$

PROOF: Let a^i denote interior multiplication by v^i, and let $(a^i)^*$ denote the adjoint action of wedging with v^i. Then $\mathcal{R}^p = R_{ijkl}(a^i)^* a^j (a^k)^* a^l$ [24]. In particular, $\mathcal{R}^p v^J = a_K v^K$ has $a_K = 0$ if J, K differ by more than two indices. This proves (i).

For (ii), it is immediate that for fixed p, q, r, s we have

$$\langle R_{pqrs}(a^p)^* a^q (a^r)^* a^s (v^i \wedge v^j \wedge v^I), v^k \wedge v^l \wedge v^I \rangle = 0,$$

unless $\{p, r\} = \{k, l\}, \{q, s\} = \{i, j\}$. Now a direct calculation of the four possibilities for p, q, r, s gives

$$\begin{aligned} \langle R^p(v^i \wedge v^j \wedge v^I), v^k \wedge v^l \wedge v^I \rangle &= R_{ljki} + R_{jkli} - R_{likj} + R_{kilj} \\ &= 2R_{ijkl}, \end{aligned}$$

by the symmetries of curvature tensor.

For (iii), recall that

$$\langle \mathcal{R}^p(v^1 \wedge \ldots \wedge v^p), v^1 \wedge \ldots \wedge v^p \rangle = \sum_{i=1}^{p} \sum_{j=p+1}^{n} K(v^i, v^j), \tag{3.7}$$

where $\{v^1, \ldots, v^n\}$ is an orthonormal basis extending $\{v^1, \ldots, v^p\}$ [11]. We have

$$\begin{aligned} &2\langle \mathcal{R}^p(v^i \wedge v^j \wedge v^I), v^i \wedge v^l \wedge v^I \rangle \\ &= \langle \mathcal{R}^p[v^i \wedge \frac{v^j + v^l}{\sqrt{2}} \wedge v^I], v^i \wedge \frac{v^j + v^l}{\sqrt{2}} \wedge v^I \rangle \\ &\quad - \langle \mathcal{R}^p[v^i \wedge \frac{v^j - v^l}{\sqrt{2}} \wedge v^I], v^i \wedge \frac{v^j - v^l}{\sqrt{2}} \wedge v^I \rangle. \end{aligned} \tag{3.8}$$

From the hypothesis, we have

$$B < \langle \mathcal{R}^p[v^i \wedge \frac{v^j + v^l}{\sqrt{2}} \wedge v^I], v^i \wedge \frac{v^j + v^l}{\sqrt{2}} \wedge v^I\rangle < A, \tag{3.9}$$

and similarly for $\langle \mathcal{R}^p[v^i \wedge \frac{v^j - v^l}{\sqrt{2}} \wedge v^I], v^i \wedge \frac{v^j - v^l}{\sqrt{2}} \wedge v^I\rangle$. Combining (3.8) and (3.9) gives (iii). ∎

We now begin the proof of the theorem. By (ii) of the Lemma, we may treat $\langle \mathcal{R}^p(v^i \wedge v^j \wedge v^I), v^k \wedge v^l \wedge v^I\rangle$ much like the curvature tensor. Fix p, I and set

$$\langle R^p(v^i \wedge v^j \wedge v^I), v^k \wedge v^l \wedge v^I\rangle = T_{ijkl}, \quad T_{ijij} = -S(i,j),$$

$$\begin{aligned}\langle R^p(v^i \wedge \frac{v^j + v^m}{\sqrt{2}} \wedge v^I), v^i \wedge \frac{v^j + v^m}{\sqrt{2}} \wedge v^I\rangle &= -S(i, j+m) \\ &= T_{i(j+m)i(j+m)},\end{aligned}$$

for distinct indices $i, j, k, l, m \notin I$. Note that $T_{ijkl} = 2R_{ijkl}$, but the S's are not sectional curvatures – in fact, $-S(i,j) = \sum_p$ for some basis $\{v^k\}$.

Fix $x \in M$ and assume that we have $B < \sum_p < A$ at x, so

$$B < -S(i, j+m) = T_{i(j+m)i(j+m)} < A. \tag{3.10}$$

Of course, T has the same Bianchi identity and symmetries as the curvature tensor R. Now the argument in (3.2)-(3.4) carries over to T, S. In particular, from

$$\begin{aligned}-S(i, l+j) &= \langle R^p(v^i \wedge \frac{v^l + v^j}{\sqrt{2}} \wedge v^I, \frac{v^l + v^j}{\sqrt{2}} \wedge v^I\rangle \\ &= \frac{1}{2}(T_{ilil} + T_{ijij} + T_{ilij} + T_{ijil}),\end{aligned}$$

we get

$$T_{ijil} = \begin{cases} -S(i,j), & j = l \\ \frac{1}{2}S(i,l) + \frac{1}{2}S(i,j) - S(i, l+j), & j \neq l. \end{cases}$$

We also know

$$2T_{(i-k)j(i-k)l} = T_{ijil} - T_{ijkl} - T_{kjil} + T_{kjkl}.$$

As with R_{ijkl}, we obtain

$$3T_{ijkl} = -2T_{(i-k)j(i-k)l} + 2T_{(j-k)i(j-k)l} + T_{ijil} + T_{kjkl} - T_{jijl} - T_{kikl}, \tag{3.11}$$

so (3.6) becomes

$$|T_{ijil}| \leq \frac{1}{2}(A - B). \tag{3.12}$$

Using the estimates (3.10), (3.12), we see that (3.11) implies

$$|T_{ijkl}| \leq \frac{4}{3}(A-B). \tag{3.13}$$

In summary, for $J \neq K$, we have by Lemma 3.1(i), (3.12) (3.13),

$$|\langle \mathcal{R}^p v^J, v^K \rangle| \leq \begin{cases} \frac{1}{2}(A-B), & |J \cap K| = p-1, \\ \frac{4}{3}(A-B), & |J \cap K| = p-2, \\ 0, & |J \cap K| \leq p-3. \end{cases}$$

Thus

$$\begin{aligned} \langle \mathcal{R}^p(a_I v^I), a_J v^J \rangle &= \sum_I a_I^2 \langle \mathcal{R}^p v^I, v^I \rangle + \sum_{I \neq J} a_I a_J \langle \mathcal{R}^p v^I, v^J \rangle \\ &> B \sum_I a_I^2 - \frac{1}{2}(A-B) \sum_{|I \cap J| = p-1} |a_I a_J| \\ &\quad + \frac{4}{3}(A-B) \sum_{|I \cap J| = p-2} |a_I a_J|. \end{aligned} \tag{3.14}$$

We now estimate the last two terms.

Lemma 3.2 *Let I, J be multi-indices of length p. Then for $1 \leq k \leq p$,*

$$\sum_I \sum_{\substack{J \\ |I \cap J| = k}} |a_I a_J| \leq \binom{p}{p-k} \binom{n-p}{p-k} \sum_I |a_I|^2.$$

Remark: This estimate is sharp, as can be seen by setting $a_I = \binom{n}{p}^{-1/2}$ for all I.

Proof: For fixed value of $\sum_I a_I^2$, the maximum of the left hand side of the inequality is attained at some vector $a = (a_I)$ with $a_I > 0$. Thus we may drop the absolute value signs on the left hand side. Set $N = \binom{n}{p}$ and consider the $N \times N$ matrix A given by

$$A_{IJ} = \begin{cases} 1, & |I \cap J| = k, \\ 0, & \text{otherwise.} \end{cases}$$

Here A acts on $\mathbf{R}^N$ with coordinates indexed by multi-indices of length p. Finding the maximum of the left hand side for fixed $\sum_I a_I^2$ is equivalent to finding the maximum eigenvalue of A. Observe that

$$\lambda_0 = \binom{p}{n-k} \binom{n-p}{p-k}$$

is an eigenvalue of A with eigenvector a having $a_I = \binom{n}{p}^{-1/2}$ for all I.

If $p = k$ the result is trivial. Otherwise we can apply Perron-Frobenius theory. Indeed, A has non-negative entries and is irreducible. (For if Q is any subset of multi-indices such that $I \in Q$ and $J \notin Q$ implies $A_{IJ} = 0$, then clearly $I \in Q$ implies $I' \in Q$ if I' is obtained from I by changing one index; iterating this argument shows that Q consists of all multi-indices.) Perron-Frobenius theory (see e.g. [14, Theorem 2, p. 53, and Remark 3, p. 62]) assures us that such A has a unique eigenvalue whose eigenvector has positive entries and that this is the maximum eigenvalue, as we require. ∎

Thus (3.14) becomes

$$\langle \mathcal{R}^p(a_I v^I), a_J v^J\rangle > B\|a_I v^I\|^2 - \frac{1}{2}(A-B)p(n-p)\sum_I a_I^2 - \frac{4}{3}(A-B)\binom{p}{2}\binom{n-p}{2}\sum_I a_I^2,$$

and this last term is positive provided

$$\left(1 + \frac{1}{2}p(n-p) + \frac{4}{3}\binom{p}{2}\binom{n-p}{2}\right)B - \left(\frac{1}{2}p(n-p) + \frac{4}{3}\binom{p}{2}\binom{n-p}{2}\right)A \geq 0.$$

This finishes the proof of Theorem 3.1.

Example: By Theorem 2.3, a compact n-manifold cannot admit both a metric of $[1-(n-2)^{-2}]$-pinched negative curvature and a $1-(2/(2+2(n-2)+\frac{4}{3}(n-p)(n-p-1)))$-pinched metric. Similar remarks apply to Proposition 2.2.

Corollary 3.1 *If for all $x \in M$, (3.1) holds for $C = C(x), A = A(x)$ with*

$$A(x)\left(\left[1 + \frac{1}{2}p(n-p) + \frac{4}{3}\binom{p}{2}\binom{n-p}{2}\right]C(x) - \left[\frac{1}{2}p(n-p) + \frac{4}{3}\binom{p}{2}\binom{n-p}{2}\right]\right) \tag{3.15}$$

strongly stochastically positive, then $\mathcal{R}^p$ is strongly stochastically positive. In particular, if M is compact and (3.15) holds, then $H^p(M;\mathbf{R}) = 0$.

Proof: The proof shows that $CA < \sum_{i=1}^p \sum_{j=p+1}^n K(v^i, v^j) < A$ implies

$$\mathcal{R}^p > CA - \frac{1}{2}p(n-p)(A - CA) - \frac{4}{3}\binom{p}{2}\binom{n-p}{2}(A - CA).$$

∎

We now construct a manifold M^6 which is $C(6,3)$-pinched, and so has $H^3(M;\mathbf{R}) = 0$, but which has planes of negative sectional curvature at each point. To our knowledge, this is the first explicit example of a Bochner vanishing theorem for the Laplacian on forms allowing such negative curvature. For example, the Gallot-Meyer result [13] for the curvature operator $\mathcal{C}$,

$$\mathcal{C} > 0 \ \Rightarrow \ R^k > 0,$$

implies positive sectional curvature, as

$$\langle \mathcal{C}(v^i \wedge v^j),\ v^i \wedge v^j\rangle = \langle -R_{ijkl}v^k \wedge v^l,\ v^i \wedge v^j\rangle = -2R_{ijij}$$

for an orthonormal frame $\{v^i\}$. Also, we know of no examples of manifolds with positive curvature operator on isotropic complex two-planes (and so $\pi_k(M) = 0$ for $1 < k \leq n/2$ by [20]) but allowing planes of negative sectional curvature.

We set $M = \Sigma_a \times S^4$, where Σ is a closed surface of genus $g > 1$ with a metric of constant negative curvature $-a$, and S^4 is the 4-sphere of constant positive curvature 1. We give M the product metric. Since $H^k(M;\mathbf{R}) \neq 0$ for $k \neq 3$, M cannot be $C(6,k)$-pinched for $k \neq 3$; this is easily verified by computing $\sum_{i=1}^k \sum_{j=k+1}^n K(v^i, v^j)$ for various permutations of an orthonormal frame $\{v^i\}$ with $\{v^1, v^2\} \in T^*\Sigma,\ \{v^3, v^4, v^5, v^6\} \in T^*S^4$.

We give two proofs that M has $\mathcal{R}^3 > 0$, one proof based on Theorem 3.1, and one by a direct calculation.

First, for the product metric on M, the curvature two-forms $\Omega_{ij} = R_{ijkl}v^k \wedge v^l$ vanish for $\{v^i\}$ as above, unless $i,j \in \{1,2\}$ or $i,j \in \{3,4,5,6\}$. Moreover, $R_{ijkl} = 0$ unless $i,j,k,l \in \{1,2\}$, or $i,j,k,l \in \{3,4,5,6\}$. Now $R_{1212} = a$, and for $s,t,r \in \{3,4,5,6\}$, (3.2) gives

$$R_{stsr} = \begin{cases} 0, & t \neq r \\ -1, & t = r. \end{cases}$$

Finally, if $i,j,k,l \in \{3,4,5,6\}$ are distinct indices, then (3.4) implies $R_{ijkl} = 0$. This determines the curvature tensor on M.

In particular, $R_{ijkl} = 0$ if there are at least three distinct indices among i,j,k,l. From (3.5), we see that $\langle R^p v^J, v^K\rangle = 0$ if $|J \cap K| \leq p-2$. A similar computation shows that if $|J \cap K| = p-1$, then $\langle R^p v^J, v^K\rangle$ is a sum of R_{ijkl} terms, each of which has three distinct indices. Thus in this case, $\langle R^p v^J, v^K\rangle = 0$ also. In summary, we have $\langle \mathcal{R}^3 v^I, v^J\rangle = 0$ unless $I = J$.

Let $\{w^i\}$ be an orthonormal frame on M. Writing $w^i = a^i_j v^j$ for an orthogonal matrix (a^i_j), we get by (3.7)

$$\sum_{i=1}^{3}\sum_{j=4}^{6} K(w^i, w^j) \tag{3.16}$$

$$
\begin{aligned}
&= \sum_{k_1,k_2,k_3,s_1,s_2,s_3} a^1_{k_1}a^2_{k_2}a^3_{k_3}a^1_{s_1}a^2_{s_2}a^3_{s_3}\langle \mathcal{R}^3(v^{k_1}\wedge v^{k_2}\wedge v^{k_3}), v^{s_1}\wedge v^{s_2}\wedge v^{s_3}\rangle \\
&= \sum_{k_1,k_2,k_3} \left((a^1_{k_1}a^2_{k_2}a^3_{k_3})^2\langle \mathcal{R}^3(v^{k_1}\wedge v^{k_2}\wedge v^{k_3}), v^{k_1}\wedge v^{k_2}\wedge v^{k_3}\rangle\right).
\end{aligned}
$$

Now

$$
\langle \mathcal{R}^3(v^{k_1}\wedge v^{k_2}\wedge v^{k_3}), v^{k_1}\wedge v^{k_2}\wedge v^{k_3}\rangle = \sum_{i=1}^{3}\sum_{j=4}^{6} K(\tilde{v}^i,\tilde{v}^j), \tag{3.17}
$$

for some rearrangement $\{\tilde{v}^i\}$ of $\{v^i\}$. Computing the cases where one, two or three of the first three of the $\tilde{v}^i$ are in TS^4, we see that the right hand side of (3.17) equals either $4-a$ or 3. Thus if we fix $a=1$, the right hand side of (3.16) becomes

$$
\begin{aligned}
\sum_{i=1}^{3}\sum_{j=4}^{6} K(w^i,w^j) &= 3\sum_{k_1,k_2,k_3}(a^1_{k_1}a^2_{k_2}a^3_{k_3})^2 \\
&= 3\left(\sum_{k_1}(a^1_{k_1})^2\right)\left(\sum_{k_2}(a^2_{k_2})^2\right)\left(\sum_{k_3}(a^3_{k_3})^2\right) = 3.
\end{aligned}
$$

Thus for $a=1$, M has $\sum_3$ constant, so by the theorem $\mathcal{R}^3>0$.

For the second proof, we compute R^3 directly, using our calculation that $\langle \mathcal{R}^3 v^I, v^J\rangle = 0$ unless $I=J$. Let $I=(i_1,i_2,i_3)$, and choose i_4,i_5,i_6 so that $\{i_q : q=1,...,6\} = \{1,...,6\}$. By (3.7), we see that

$$
\langle \mathcal{R}^3(a_I v^I), a_I v^I\rangle = (\sum_I a_I^2)\langle \mathcal{R}^3 v^I, v^I\rangle = \sum_{I,|I|=3} a_I^2 \sum_{q=1}^{3}\sum_{s=4}^{6} K(v^{i_q}, v^{i_s}). \tag{3.18}
$$

Since $\sum_3$ equals either $4-a$ or 3, taking $a<4$ makes the right hand side of (3.18) positive.

Remark: Let $\{w^i\}$ be an orthonormal frame. We can use [8,Thm. 5A] to conclude the stronger result $H_3(M;\mathbf{Z})=0$ once we check

$$
\sum_{i=1}^{3}\sum_{j=4}^{6} K(w^i,w^j) > \frac{\|\alpha\|^2}{2} - \frac{n|H|^2}{2} \tag{3.19}
$$

pointwise on M, where α is the second fundamental form, H is the mean curvature for an isometric immersion of M in some $\mathbf{R}^N$, and $n=6$. Here we first isometrically immerse Σ_a in some $\mathbf{R}^{N_1}$ for a fixed a and S^4 in $\mathbf{R}^5$ in the usual way. We then put M isometrically in $\mathbf{R}^{N_1+5}$.

For $a \le 1$, we know that the left hand side of (3.19) is at least 3. On the other hand, for a small, $\|\alpha\|^2$ is very close to $\|\alpha\|^2$ for S^4, and similarly for $n|H|^2$. On S^4, $\|\alpha\|^2 = 4$ and H is minus the radial vector, so on M, $(\|\alpha\|^2/2) - (n|H|^2/2)$ is approximately $2/3$. Thus (3.19) is satisfied.

4 Remarks on spectral gap estimates

The crucial Lemma 2.1 treats the L^∞ theory of harmonic forms, so it is natural to look for an L^2 version. For this, let $\lambda_1^q = \lambda_1^{h,q}$ be the spectral gap at zero for q-forms, i.e. $\lambda_1^q = \inf\left\{\mathrm{spec}(\Delta^{h,q}) - \{0\}\right\}$. Equivalently

$$\lambda_1^q = \inf_{\phi\in C_K^\infty, \phi\in H_q^\perp} \frac{\int_M \langle \Delta^h\phi, \phi\rangle\, \mu(dx)}{|\phi|_{L^2}^2}.$$

Here $H_q^\perp$ is the space of differential q-forms perpendicular to the the harmonic forms. We will show that the L^2 version of Lemma 2.1 corresponds to lower bound estimates for λ_1^q for differential forms.

Let $1 \le q \le n-1$. Suppose $\mathbf{R}^{q\pm 1}$ are bounded from below. For $\phi, \psi \in C_c^\infty\Lambda^q$, and in the notation of §2, we have

$$\begin{aligned}
&\left|\int_M \langle P_t\phi - \phi, \psi\rangle\, \mu(dx)\right| \\
= \;& \frac{1}{2}\left|\int_0^t \int_M \langle d\phi, P_s(d\psi)\rangle \mu(dx)\, ds + \int_0^t \int_M \langle \delta^h\phi, P_s(\delta^h\psi)\rangle \mu(dx)\, ds\right| \\
= \;& \frac{1}{2}\left|\int_M \langle d\phi, \int_0^t P_s(d\psi)\, ds\,\rangle \mu(dx)\right| + \frac{1}{2}\left|\int_M \langle \delta^h\phi, \int_0^t P_s(\delta^h\psi)\, ds\,\rangle \mu(dx)\right| \\
\le \;& \frac{1}{2}||d\phi||_{L^2}\cdot ||\int_0^t P_s(d\psi) ds||_{L^2} + \frac{1}{2}||\delta^h\phi||_{L^2}\cdot ||\int_0^t P_s(\delta^h\psi) ds||_{L^2}
\end{aligned}$$

Suppose $\lambda_1^{q+1} > 0$ and $\lambda_1^{q-1} > 0$. Then since $d\psi$ and $\delta^h\psi$ are orthogonal to the L^2 h-harmonic forms, we have the existence of $\int_0^\infty P_s(d\psi) ds$ and $\int_0^\infty P_s(\delta^h\psi) ds$ in L^2 with

$$||\int_0^\infty P_s(d\psi) ds|| \le \frac{2}{\lambda_1^{q+1}}\cdot ||d\psi||_{L^2}$$

and

$$||\int_0^\infty P_s(\delta^h\psi) ds|| \le \frac{2}{\lambda_1^{q-1}}\cdot ||\delta^h\psi||_{L^2}.$$

Thus letting $t \to \infty$ in the earlier inequality gives

$$\left|\int_M \langle H\phi - \phi, \psi\rangle \mu(dx)\right| \le \frac{1}{\lambda_1^{q+1}}\left(|d\phi|_{L^2}\right)\cdot\left(|d\psi|_{L^2}\right) + \frac{1}{\lambda_1^{q-1}}\left(|\delta^h\phi|_{L^2}\right)\cdot\left(|\delta^h\psi|_{L^2}\right).$$

Taking $\phi = \psi \in H^\perp$ and using $H\phi = 0$, we have

$$(||\phi||_{L^2})^2 \leq \frac{1}{\min\left\{\lambda_1^{q-1}, \lambda_1^{q+1}\right\}} \cdot \langle \Delta\phi, \phi\rangle$$

and so

$$\lambda_1^q \geq \min\left\{\lambda_1^{q-1}, \lambda_1^{q+1}\right\}.$$

This result also follows from the fact that $\lambda_1^q > 0$ if and only if d and δ on q-forms both have closed range [6]; see also [10] for the case $h \neq 0$. This shows that d and δ gives a conjugacy of the restriction of Δ^q to Ker $d^\perp$ with Δ^{q+1} restricted to image of d, and similarly for Δ^q on $(\text{Ker } \delta)^\perp$.

We can also estimate λ_1^q in terms of $\underline{\mathcal{R}}^{q\pm1}$, but these estimates are useful only when there are no L^2 harmonic $(q \pm 1)$-forms. If we denote by $\lambda_0(f)$ the bottom of the spectrum of the operator $\Delta + f$ on functions, then we have

$$\lambda_1^q \geq \min\left\{\lambda_0(\underline{R}^{q-1}), \lambda_0(\underline{R}^{q+1})\right\},$$

assuming $\mathcal{R}^{q+1}$ and $\mathcal{R}^{q-1}$ are bounded from below. This comes from [8, Theorem 3C] which states that the bottom of the spectrum of Δ^p is bigger or equal to $\lambda_0(\underline{R}^p)$ for each $p \in \{1, 2, \ldots, n-1\}$. Note that by [8, Proposition 4B], we have

$$\lambda_0(\underline{\mathcal{R}}^p) \geq -2\overline{\lim_{t\to\infty}} \frac{1}{t} \log \mathbf{E} e^{-\frac{1}{2}\int_o^t \underline{\mathcal{R}}^p(x_0)ds},$$

which gives an estimate for λ_1^q in terms of $\underline{\mathcal{R}}^{q-1}$ and $\underline{\mathcal{R}}^{q+1}$.

References

[1] M. F. Atiyah, *Elliptic operators, discrete groups, and von Neumann algebras*, Asterisque **32-33** (1976), 43–72.

[2] A. Avez, *Applications de la formule de Gauss-Bonnet-Chern aux variétés à quatre dimensions*, C. R. Acad. Sc. Paris **256** (1963), 5488–5490.

[3] R. Azencott, *Behavior of diffusion semigroups at infinity*, Bull. Math. Soc. France **102** (1974), 193–240.

[4] D. Bakry, *Un critére de non-explosion pour certaines diffusions sur une variété Riemannienne compléte*, C. R. Acad. Sc. Paris **303** (1986), 23–26.

[5] R. Brooks, *The fundamental group and the spectrum of the Laplacian*, Comment. Math. Helv. **56** (1981), 581–598.

[6] H. Donnelly, *The differential form spectrum of hyperbolic space*, Manuscripta Math. **33** (1981), 365–385.

[7] K. D. Elworthy and X.-M. Li, *Formulae for the derivatives of heat semigroups*, J. Functional Analysis **125** (1994), 252–286.

[8] K. D. Elworthy and S. Rosenberg, *Generalized Bochner theorems and the spectrum of complete manifolds*, Acta Appl. Math. **12** (1988), 1–33.

[9] ———, *Manifolds with wells of negative Ricci curvature*, Inventiones Math. **103** (1991), 471–495.

[10] ———, *The Witten Laplacian on negatively curved simply connected manifolds*, Tokyo J. Math. **16** (1993), no. 2, 513–524.

[11] ———, *Homotopy and homology vanishing theorems and the stability of stochastic flows*, Geometric and Functional Analysis **6** (1996), 51–78.

[12] M. P. Gaffney, *The conservation property of the heat equation on Riemannian manifolds*, Comm. Pure Appl. Math. **12** (1959), 1–11.

[13] S. Gallot and D. Meyer, *Opérateur de courbure et Laplacien des formes différentielles d'une vairété Riemannienne*, J. Math. Pures et Appl. **54** (1975), 259–284.

[14] F. R. Gantmacher, *The Theory of Matrices*, vol. 2, Chelsea Publishing Co., New York, 1960.

[15] H. B. Lawson, Jr. and J. Simons, *On stable currents and their applications to global problems in real and complex geometry*, Annals of Math. **98** (1973), 427–450.

[16] X.-M. Li, *Stochastic flows on noncompact manifolds*, Ph.D. thesis, University of Warwick, 1992.

[17] ———, *On extension of Myers' theorem*, Bull. London Math. Soc. **27** (1995), 392–396.

[18] T. Lyons and D. Sullivan, *Function theory, random paths and covering spaces*, J. Differential Geometry **19** (1984), 299–323.

[19] H. McKean, *An upper bound for the spectrum of Δ on a manifold of negative curvature*, J. Differential Geometry **4** (1970), 359–366.

[20] M. Micallef and J. D. Moore, *Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes*, Annals of Math. **87** (1988), 199–227.

[21] M. Micallef and J. Wolfson, *The second variation of area of minimal surfaces in four manifolds*, preprint.

[22] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Academic Press, Orlando, 1980.

[23] S. Rosenberg and D. Yang, *On the fundamental group of manifolds of almost positive Ricci curvature*, J. Math. Soc. Japan **46** (1994), 267–288.

[24] E. Witten, *Supersymmetry and Morse theory*, J. Differential Geometry **121** (1985), 169–186.

[25] S.-T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure and Applied Math. **38** (1975), 201–228.