Center for Mathematical Physics BU-CMP/03-01

Traces and Characteristic Classes on Loop Spaces

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Abstract

We construct Chern-Weil classes on infinite dimensional vector bundles with structure group $\mathcal{C}l_0^*(M,E)$, the group of zeroth order invertible classical pseudo-differential operators acting on a finite rank vector bundle E over a closed manifold M. $\mathcal{C}l_0^*(M,E)$ is the structure group of geometric bundles naturally associated to loop spaces of Riemannian manifolds. Mimicking the finite dimensional Chern-Weil construction, we replace the ordinary trace on matrices by different linear functionals on the Lie algebra of $\mathcal{C}l_0^*(M,E)$. We use (i) traces built from the leading symbol, and (ii) a linear map which considers all terms in the asymptotic expansion of a heat kernel regularized trace. For a specific bundle on loop spaces, the first approach yields non-vanishing Chern classes in all degrees. The second approach produces connection independent cohomology classes under stringent conditions. For the tangent bundle to a loop group, the first method gives a vanishing first Chern class, while the second method recovers the first Chern class investigated by Freed, and explains why this class is not connection independent.

1 Introduction

Infinite dimensional vector bundles with connections are frequently encountered in mathematical physics; basic examples include the tangent bundle of loop spaces [6] and infinite rank vector bundles associated to families of Dirac operators [4]. In this paper, we construct Chern forms and Chern classes for a class of vector bundles including the tangent bundle TLM to a loop space LM, and produce examples of non-vanishing classes. In light of Freed's curious example [6] of a connection dependent first Chern form on loop groups, an impossibility in finite dimensions, it seems worthwhile to examine extensions of Chern-Weil theory to infinite dimensions.

In contrast to finite dimensions, on infinite dimensional bundles one first has to choose the topology on the fiber and determine a structure group. One obvious choice, modeling the fiber on a Hilbert space H and the structure group on $GL(H)$, leads to a trivial theory by Kuiper's theorem [10]. Since a direct topological approach to characteristic classes seems difficult, we follow the geometric approach of Chern-Weil theory, which both historically preceded topological approaches and is perhaps more elementary. In our approach, we

[1]Partially supported by the NSF and the CNRS.
2000 Mathematical Subject Classification. Primary: 53C05. Secondary: 58J40

assume that our infinite dimensional bundle $\mathcal{E}$ has (i) fibers modeled on the space of sections of a finite rank bundle E over a closed manifold M, in either a Sobolev or C^∞ topology, and (ii) a connection whose connection one-form takes values in $\mathcal{C}l_{\leq 0} = \mathcal{C}l_{\leq 0}(M, E)$, the space of classical pseudo-differential operators (ΨDOs) of nonpositive order acting on the fibers. As in finite dimensions, $\mathcal{C}l_{\leq 0}$ should be the Lie algebra of the structure group. In our case, the structure group is therefore $\mathcal{C}l_0^*$, the space of zeroth order invertible (and hence elliptic) ΨDOs on sections of E. This framework includes the case of TLM [13].

Chern-Weil theory produces characteristic classes from invariant polynomials on the Lie algebra $\mathcal{C}l_{\leq 0}$. Avoiding the difficult question of determining all such invariants, we focus on those polynomials which produce the Chern classes in finite dimensions, namely $\mathrm{Tr}(\Lambda^k A)$, the trace of exterior powers of a matrix. However, powers of the curvature need not be trace class for our structure group. One main topic of the paper is the investigation in §3 of alternative traces on $\mathcal{C}l_{\leq 0}$. One of these traces, the leading symbol trace, produces nonvanishing Chern classes.

In general, the leading symbol trace picks out the leading term in the asymptotic expansion $\mathrm{Tr}(\Omega e^{-\varepsilon Q})$, where Ω is the curvature of the connection and Q is a generalized Laplacian on the fibre, while the weighted traces of e.g. [18] pick out the finite term. As a second main topic, in §4 we show that certain asymptotic coefficients are closed, and that the corresponding cohomology classes are independent of the connection under more stringent conditions. Thus, in contrast to finite dimensions, it is qualitatively harder to show that characteristic forms are connection independent. In fact, Freed's example occurs as an asymptotic coefficient which is closed but does not satisfy the stringent conditions. Thus both the leading symbol traces of §3 and the results of §4 give extensions of Chern-Weil theory that improve the weighted trace approach of [5, 14, 18].

In more detail, in §2 we review classical Chern-Weil theory, with an emphasis on traces as morphisms $\lambda : \mathrm{Ad}\, P \to \mathbb{C}$ from the adjoint bundle of a principal bundle P to the trivial $\mathbb{C}$ bundle. Here the structure group and the base may be infinite dimensional, and we are thinking of P as the principal bundle associated to $\mathcal{E}$. These morphisms λ produce characteristic forms and classes as in finite dimensions (Theorem 2.2).

In §3, we introduce two types of traces in infinite dimensions, each of which can be interpreted as generalizations of the ordinary trace on matrices. The first example is the Wodzicki residue, the unique trace on the space of classical ΨDOs. However, we show in §3.1 that the associated Chern forms vanish on loop groups, confirming results in [14]. We produce more interesting examples by noting that the Lie algebra of the structure group $\mathcal{C}l_0^*$ admits a family of "symbol traces" of the form $A \mapsto \Lambda(\sigma_0^A)$, where σ_0^A is the leading symbol of A and Λ is a distribution on the cosphere bundle of M. In the main section §3.2, we show that the associated Chern classes are non-zero in general for the structure group of loop spaces (Theorem 3.3). We also present evidence that, despite appearances, these classes are not given by integration over the fiber of Chern classes of a finite dimensional bundle.

In §3.3, we relate the symbol traces to regularization techniques familiar in statistical mechanics. In particular, in certain cases the symbol trace $\Lambda^Q(\sigma_0^A)$ equals the leading term in the asymptotic expansion $\mathrm{tr}(Ae^{-\varepsilon Q})$, where Λ^Q is a distribution associated to a generalized Laplacian Q (Propostion 3.4). This applies to loop spaces: $Q = D^*D$ at the loop γ, with D covariant differentiation in the direction $\dot\gamma$ along γ. We also build characteristic classes from symbol traces on the smaller algebra $\mathcal{C}l_{\leq p}$ of ΨDOs of order at most p for $p < 0$, and discuss their dependence on the choice of connection (Theorem 3.6). This is relevant to the loop group case, as the Levi-Civita connection one-form takes values in such an algebra.

In §4, we extend the discussion of §3.3 to relate symbol traces to other terms in the asymptotics of $\mathrm{tr}(Ae^{-\varepsilon Q})$, and in particular to the finite part. This finite part regularization of $\mathrm{tr}(A)$ is well known in the physics literature, but the corresponding Chern-Weil construction (i.e. replacing A by powers of the curvature form Ω) does not produce closed forms in general because of the Q dependence.

To analyze this difficulty, we consider the entire asymptotic series $\mathrm{tr}^Q_\varepsilon(\Omega^k) := \mathrm{tr}(\Omega^k e^{-\varepsilon Q})$ as a $2k$-form with values in the sum of (i) a formal Laurent series in $\varepsilon^{\frac{1}{q}}$ (for some $q \in \mathbb{N}$) and (ii) $\log\varepsilon$ times a power series in ε. We modify the given connection ∇ on $\mathcal{E}$ to a connection ∇^Q_ε with connection one-form taking values in the power series ring $\mathcal{C}l_{\leq 0}[[\varepsilon]]$. We use ∇^Q_ε to determine which coefficients of $\mathrm{tr}^Q_\varepsilon(\Omega^k)$ are closed (Theorem 4.4), and when their cohomology classes are independent of the connection (Theorem 4.6). Roughly speaking, the number of coefficients which are closed grows linearly in $-d := -\mathrm{ord}([\nabla, Q])$. Thus, the more (covariantly) constant Q is, the greater the number of closed forms. For example, when $d < \mathrm{ord}(Q)$ as for loop groups, the coefficient of the most divergent term is closed; for $k = 1$, this coefficient is precisely the first Chern form considered by Freed. The number of coefficients whose cohomology class is constant for a family of connections ∇_t also depends linearly on $-d$, provided the order of $\frac{d}{dt}\nabla_t$ is sufficiently smaller than d. From these theorems, we can see precisely why Freed's first Chern class is connection dependent.

The second author would like to thank Université Blaise Pascal for its hospitality during the preparation of this article. Conversations with Simon Scott on this subject are also gratefully acknowledged. We also thank the editor and a referee for helpful comments, and in particular for the referee's simplified proof of (4.4).

2 Chern-Weil calculus

In this section, we review finite dimensional Chern-Weil calculus as in e.g. [3] and check its extension to the infinite dimensional setting. We emphasize the role of linear functionals on the Lie algebra of the structure group, as the choice for functionals is the main topic of §3.

Let B be a finite dimensional manifold, G a Lie group and $P \to B$ a smooth principal G-bundle. A smooth representation $\rho : G \to GL(W)$ of G on a finite dimensional vector space W induces an associated smooth vector bundle $\mathcal{W} := P \times_\rho W \to B$. In particular, the adjoint representation $\rho : G \to \mathrm{Aut}(\mathrm{Lie}(G))$ determines a bundle Ad P.

This framework extends to Kriegl and Michor's "convenient setting" for global analysis [11], which includes principal bundles for regular Fréchet Lie groups G over Fréchet manifolds. We will work with the space $\mathcal{C}l_0^*$ of invertible zeroth order ΨDOs acting on smooth sections of a vector bundle over a closed manifold M. The Fréchet topology on $\mathcal{C}l_0^*$ is induced from the standard Fréchet topology on the coefficients of the homogeneous symbols σ_i of a ΨDO T and the C^k topology on the smoothing part $T - \sum_i \sigma_i$. (The σ_i and the smoothing part depend on the choice of a partition of unity on M and a cutoff function in the cotangent variables, which we make once and for all.) This puts a regular Fréchet Lie group structure on $\mathcal{C}l_0^*$.

We briefly recall the geometric constructions we need in the Banach and Fréchet setting, referring the reader to [11] for details. The finite dimensional constructions must be modified, as a representation $G \to GL(W)$ fails to be continuous in any reasonable sense once G and W are infinite dimensional. Indeed, $GL(W)$ cannot be equipped with an appropriate Lie group structure. If G is Banach and W is either Banach or Fréchet, one does not expect $GL(W)$ to be a Lie group for the topology in which the representation is expected to be continuous. If W is Frechet, $GL(W)$ is never a topological group unless W is Banach space, in which case it is a Banach Lie group in the operator norm topology [15].

To circumvent these difficulties, one works with the group action associated to a representation [11, §49.1]. In more detail, let G be a regular Fréchet, resp. Banach Lie group with Lie algebra A, and let $P \to B$ be a smooth principal bundle equipped with a connection given by a Lie algebra valued connection one-form $\omega \in \Omega^1(P, A)$. Let W be a Fréchet, resp. Banach vector space (and therefore a regular space in the sense of [11]). A representation ρ' of G on W which induces a jointly smooth map $\rho : G \times W \to W$, $\rho(g, w) = \rho'(g)(w)$, determines an associated vector bundle $\mathcal{W} := P \times_\rho G \to B$ [11, §37.12]. Note that the associated bundle is constructed just as in finite dimensions, but the smoothness requirement of the representation has been restated. The space $\Omega(B, \mathcal{W})$ of $\mathcal{W}$-valued forms on B can be identified via a canonical isomorphism with the space $(\Omega(P) \otimes W)_b$ of basic forms on P with values in the trivial bundle $P \times W$ [11, §37.31]. Recall that a form is basic if it is G-invariant and horizontal. Moreover, the connection one-form ω on P with curvature form $\Omega^P \in \Omega^2(P, A)$ induces a covariant derivative ∇ on smooth sections of $\mathcal{W}$ [11, §37.26], and its curvature $\Omega^{\mathcal{W}} \in \Omega^2(B, \mathrm{Hom}(\mathcal{W}))$ is related to Ω^P via the canonical isomorphism above [11, §37.32].

Let $W = A$ be the Fréchet, resp. Hilbert Lie algebra of G. Recall that the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(A)$ is the differential of conjugation in G: $\mathrm{Ad}_g a := (D_e C_g)a$, where $C_g : G \to G$ is $C_g(h) = ghg^{-1}$. The differential of Ad, $\mathrm{ad} = D\mathrm{Ad} : A \to \mathrm{End}(A)$, is given by $\mathrm{ad}_b(a) = [b, a]$. It is immediate that the adjoint representation satisfies the joint smoothness condition above. In particular, a connection one-form $\theta \in \Omega^1(P, A)$ yields a connection ∇^{ad} on $\mathrm{Ad}\,P$, with $\nabla^{\mathrm{ad}} = d + [\theta, \cdot]$. (For this reason, our $\mathrm{Ad}\,P$ is often denoted $\mathrm{ad}\,P$.)

A linear form on A:

$$\lambda : A \to \mathbb{C}$$

such that $\mathrm{Ad}^*\lambda := \lambda \circ \mathrm{Ad} = \lambda$ induces a bundle morphism

$$\lambda : \mathrm{Ad}\,P \to B \times \mathbb{C}$$

defined as follows. Given a local trivialization (U,Φ), where $U \subset B$ is open and $\Phi : \mathrm{Ad}\ P|_U \to U \times A$ is an isomorphism, and a local section $\sigma \in \Gamma(\mathrm{Ad}\ P|_U)$, we set

$$\lambda(\sigma) := \lambda(\Phi(\sigma)).$$

This definition is independent of the local trivialization. Indeed, given another local trivialization (V,Ψ), at $b \in U \cap V$ we have

$$\lambda(\Phi(\sigma)) = \lambda(\mathrm{Ad}_g \Psi(\sigma)) = \lambda(\Psi(\sigma)), \quad \text{for some } \ g = g_b \in G.$$

The connection ∇^{ad} on Ad P induced by a connection θ on P induces in turn a connection ∇^* on the dual bundle Ad P^* (i.e. $(\mathrm{Ad}\ P)^*$), which is locally described by $\nabla^{\mathrm{ad}^*} = d + \mathrm{ad}^*_\theta$. Since $\mathrm{Ad}^*\lambda = \lambda$ implies $\mathrm{ad}^*\lambda = 0$, we have $\nabla^{\mathrm{ad}^*}\lambda = d\lambda = 0$, since λ is locally constant. Summarizing, we have:

Lemma 2.1: *Let $\lambda : \mathrm{Ad}\ P \to B \times \mathbb{C}$ be the linear morphism induced by a linear form $\lambda : A \to \mathbb{C}$ with $\mathrm{Ad}^*\lambda = \lambda$. Let $\nabla^{\mathrm{ad}} = d + [\theta, \cdot] = d + \mathrm{ad}_\theta$ be a connection on Ad P induced by a connection θ on P. Then*

$$d \circ \lambda = \lambda \circ \nabla^{\mathrm{ad}}. \tag{2.1}$$

PROOF: Since $d\lambda = 0$, we have $d \circ \lambda = \lambda \circ d$ locally. However, $\mathrm{ad}^*\lambda = 0$ implies $\lambda \circ d = \lambda \circ (d + \mathrm{ad}_\theta) = \lambda \circ \nabla^{\mathrm{ad}}$, so $d \circ \lambda = \lambda \circ \nabla^{\mathrm{ad}}$ globally. □

Abusing notation, we will sometimes denote $\nabla^{\mathrm{ad}}\alpha$ by $[\nabla, \alpha]$, for α an Ad P-valued form, in analogy to the local description $\nabla^{\mathrm{ad}} = d + [\theta, \cdot]$, with the understanding that $[\nabla, \alpha]$ is a superbracket with respect to the $\mathbb{Z}_2$-grading on differential forms.

The lemma leads to the main result of this section. To set the notation, let $\mathcal{E} \to B$ be a vector bundle with structure group a Fréchet or Banach Lie group G and with fiber modeled on a vector space V. The associated principal G-bundle $P^{\mathcal{E}}$ is given by gluing copies of G over B via the transition maps of $\mathcal{E}$. Strictly speaking, if $\{U_\alpha\}$ is an open cover of B which trivializes $\mathcal{E}$ and with transition maps $g_{\alpha\beta}(x) \in G$, $x \in U_\alpha \cap U_\beta$, $g_{\alpha\beta}(x) : V \to V$, then $P^{\mathcal{E}} = \coprod_\alpha (U_\alpha \times G)/(x,g) \sim (x, g_{\alpha\beta}(x)g)$ with the quotient topology. As an example applicable to loop spaces, let M, X be smooth manifolds, $E \to X$ a finite rank vector bundle, $B := C^\infty(M,X)$, and let $\mathcal{E} \to B$ have fiber $\mathcal{E}_b = C^\infty(M, b^*E)$. Let $\{V_\beta\}$ be the path components of B, and pick $b_\beta \in V_\beta$. Then the structure group of $\mathcal{E}|_{V_\beta}$ is $G = G_\beta = C^\infty(M, \mathrm{Aut}(b_\beta^*E))$, and the associated G-bundle has fiber $P^{\mathcal{E}}|_b = C^\infty(M, \mathrm{Aut}(b^*E)) = C^\infty(M, b^*P^E)$, where P^E is the frame bundle of E.

A G-connection on $\mathcal{E}$ induces a connection one-form on $P^{\mathcal{E}}$, just as a connection induces a connection one-form on the G-frame bundle in finite dimensions. In particular, a connection ∇ on $\mathcal{E}$ induces a connection ∇^{ad} on Ad $P^{\mathcal{E}}$, and the curvature Ω of ∇ lies in $\Omega^2(B, \mathrm{Ad}\ P^{\mathcal{E}})$.

Theorem 2.2: *Let $P = P^{\mathcal{E}}$ be the principal bundle associated to a vector bundle with connection $(\mathcal{E}, \nabla) \to B$ with structure group G. Let Ω be the curvature of ∇. Let λ be as* *in Lemma 2.1. Then for any analytic function $f : \mathbb{C} \to \mathbb{C}$, the form $\lambda(f(\Omega))$ is closed, and its de Rham cohomology class in $H^*(B;\mathbb{C})$ is independent of the choice of ∇.*

As usual, we mean that the degree k piece of $\lambda(f(\Omega))$ is a closed $2k$-form, for all $k \in \mathbb{N}$.

Proof: The usual finite dimensional proof (see e.g. [3]) runs through, with ordinary traces replaced by λ.

In more detail, $\lambda(f(\Omega))$ is closed because $\lambda(\Omega^k)$ is closed for any $k \in \mathbb{N}$, which we check in a local trivialization of Ad P. We have

$$d\,\lambda(\Omega^k) = \lambda(\nabla^{\mathrm{ad}}\Omega^k) = \lambda(\sum_{j=1}^{k} \Omega^{j-1}(\nabla^{\mathrm{ad}}\Omega)\Omega^{k-j}) = 0,$$

where we have used the Bianchi identity $\nabla\Omega = 0$ in the last identity.

To check that the corresponding de Rham class is independent of the choice of connection, we consider a differentiable one-parameter family of connections $\{\nabla_t, t \in \mathbb{R}\}$ on $\mathcal{E}$. More precisely, connections are elements of the smooth one-forms $\Omega^1(\mathrm{Ad}\ P)$, i.e. smooth bundle maps $\alpha : TM \to \mathrm{Ad}\ P$ in the Fréchet topologies. Differentiable families of connections are defined similarly. ∇_t induces a family of connections ∇_t^{ad} on Ad P. Then

$$\begin{aligned} \frac{d}{dt}\lambda(\Omega_t^k) &= \lambda\left(\sum_{j=1}^{k} \Omega_t^{k-j}\left(\dot{\nabla}_t\nabla_t + \nabla_t\dot{\nabla}_t\right)\Omega_t^{j-1}\right) = \lambda\left(\sum_{j=1}^{k} \Omega_t^{k-j}(\nabla_t^{\mathrm{ad}}\dot{\nabla}_t)\Omega_t^{j-1}\right) \qquad (2.2) \\ &= \lambda\left(\sum_{j=1}^{k} \nabla_t^{\mathrm{ad}}\Omega_t^{k-j}\dot{\nabla}_t\Omega_t^{j-1}\right) = d\lambda\left(\sum_{j=1}^{k} \Omega_t^{k-j}\dot{\nabla}_t\Omega_t^{j-1}\right). \end{aligned}$$

In the first equality, we use $\nabla_t^2 = \Omega_t$, in the second we have extended the bracket connection to forms, and in the third we have used the Bianchi identity. (2.2) shows that the dependence on the connection is measured by an exact form and hence vanishes in cohomology.

□

This yields the usual Chern-Weil classes:

Corollary 2.3: *Let $G \subset GL(n,\mathbb{C})$ be a finite dimensional Lie group, and let $\mathcal{E} \to B$ be a vector bundle with structure group G. Let ∇ be a connection on $\mathcal{E}$ with curvature Ω. For any analytic function f, the forms $\mathrm{tr}(f(\Omega)) \in \Omega^*(B,\mathbb{C})$ are closed and their de Rham cohomology classes are independent of the choice of ∇.*

This follows from Theorem 2.2 by passing from $\mathcal{E}$ to $P^{\mathcal{E}}$ and using $\lambda = \mathrm{tr}$, the ordinary trace on matrices.

Remark: For $GL(n,\mathbb{C})$ and $U(n)$, all characteristic classes are generated by $\mathrm{tr}(\Omega^k), k \in \mathbb{N}$. However, we do not capture the Euler class for $SO(n,\mathbb{R})$ by this procedure, as this class is generated by the non-linear, but Ad-invariant function $\sqrt{\det}$. We can treat this case either by using the identity $\det(1+A) = \sum_k \mathrm{tr}(\Lambda^k A)$, or by noting that the proof of Theorem 2.2 does not use the linearity of λ.

Notation: Throughout the paper, "ΨDOs" means classical pseudo-differential operators, and $\mathcal{C}l(M,E)$ denotes the space of all classical ΨDOs acting on smooth sections of the finite dimensional Hermitian bundle E over a closed Riemannian manifold M. $\mathcal{C}l_k(M,E)$ denotes the subspace of ΨDOs of order $k \in \mathbb{R}$. $\mathcal{C}l_{\leq k}(M,E)$ denotes the space of ΨDOs of order at most k. $\mathcal{C}l^*_k(M,E)$ denotes the set of invertible operators in $\mathcal{C}l_k(M,E)$. $\mathcal{E}ll^+(M,E)$ denotes the space of positive order, elliptic operators in $\mathcal{C}l(M,E)$ with positive definite leading symbol.

A bundle $\mathcal{E}$ with fiber modeled on $C^\infty(M,E)$ or on $H^s(M,E)$ is a *ΨDO bundle* if the transition maps lie in the regular Fréchet Lie group $\mathcal{C}l^*_0(M,E)$. Here $C^\infty(M,E)$, $H^s(M,E)$ are the spaces of smooth and s-Sobolev class sections of E, respectively. Ad $P^{\mathcal{E}}$ is a bundle of algebras locally modeled on $\mathcal{C}l_{\leq 0}(M,E)$, and will be denoted $\mathcal{C}l_{\leq 0}(\mathcal{E})$. Note that Ad $P^{\mathcal{E}}$ equals the bundle $\mathcal{E} \times_{\mathcal{C}l^*_0(M,E)} \mathcal{C}l_{\leq 0}(M,E)$ associated to the adjoint representation. In §4, we also consider the larger bundle $\mathcal{C}l(\mathcal{E}) = \mathcal{E} \times_{\mathcal{C}l^*_0(M,E)} \mathcal{C}l(M,E)$, associated to the adjoint representation of $\mathcal{C}l^*_0(M,E)$ on the algebra $\mathcal{C}l(M,E)$; here $\mathcal{C}l(M,E)$ is given the inductive limit topology of the usual Fréchet topology on $\mathcal{C}l_{\leq k}(M,E)$.

A connection on a ΨDO bundle $\mathcal{E}$ is a *ΨDO connection* if its connection one-form takes values in $\mathcal{C}l(M,E)$ in any local trivialization.

Remark: If θ is the locally defined connection one-form of a ΨDO connection on a bundle $\mathcal{E}$ modeled on $C^\infty(M,E)$, then under a gauge change g, θ transforms to $g^{-1}\theta g + g^{-1}dg$. Since $g^{-1}dg$ is zeroth order if g is nonconstant, the connection one-form is usually of non-negative order. (For left-invariant connections on loop groups, $g^{-1}dg$ vanishes, and θ can be of any order.) When θ is of non-positive order, it is a bounded operator and hence extends to a connection on the extension of $\mathcal{E}$ to an $H^s(M,E)$-bundle. We call a connection with connection one-form taking values in $\mathcal{C}l_{\leq 0}(M,E)$ a $\mathcal{C}l_{\leq 0}$*-connection.* The curvature of such a connection is a $\mathcal{C}l_{\leq 0}$-valued two-form on the base.

3 Examples of traces and corresponding Chern classes in infinite dimensions

In this section we examine two examples of Theorem 2.2. The trace is furnished by the Wodzicki residue in the first example, and by various traces applied to the leading order symbol of a zeroth order ΨDO in the second. We also consider various traces applied to the leading order symbol of ΨDOs of negative order, for which an extension of Theorem 2.2 is needed.

In each case, we begin with a Fréchet or Hilbert vector bundle $\mathcal{E}$ over a base space B, with fiber modeled on $C^\infty(M,E)$ or $H^s(M,E)$, with structure group given by $\mathcal{C}l^*_0 = \mathcal{C}l^*_0(M,E)$. We will consider a connection on B with values in the corresponding Lie algebra $\mathcal{C}l_{\leq 0} = \mathcal{C}l_{\leq 0}(M,E)$. In the language of §2, we pass from $\mathcal{E}$ to the corresponding principal bundle $P = P^{\mathcal{E}}$ with fiber modeled on $\mathcal{C}l^*_0$. Then Ad P has fiber modeled on $\mathcal{C}l_{\leq 0}$, and we can apply the Chern-Weil machinery of §2, using either the Wodzicki residue or the leading symbol traces for the functional λ.

Note that in this section, we are treating the structure group $\mathcal{C}l_0^*$ as a generalization of $GL(n, \mathbb{C})$. As in finite dimensions, we focus only on invariant polynomials on the Lie algebras given by traces. We do not discuss the interesting question of whether all such polynomials on $\mathcal{C}l_{\leq 0}$ are generated by these traces.

Exactly how these examples generalize the finite dimensional situation is open to interpretation. When the manifold is reduced to a point, the leading symbol of an endomorphism in the fiber, a "zeroth order ΨDO," is just the endomorphism itself, and the only trace, up to normalization, is the ordinary trace on a vector space. In contrast, in finite dimensions the Wodzicki residue vanishes. So in this interpretation, the Wodzicki residue is a purely infinite dimensional phenomenon, while the symbol trace generalizes the finite dimensional theory.

On the other hand, both the Wodzicki residue and the symbol trace appear in the most divergent term in asymptotic expansions: the Wodzicki residue of an operator A is the residue of the pole of the zeta function regularization $\mathrm{Tr}(AQ^{-s})$ at $s = 0$ (for any positive elliptic operator Q), and the symbol trace is related to the coefficient of the most divergent term in the heat operator regularization $\mathrm{Tr}(Ae^{-\varepsilon Q})$ as $\varepsilon \to 0$. (The last statement is proved in Proposition 3.4.) Since the two corresponding "regularizations" in finite dimensions using a positive definite matrix Q simply reduce to $\mathrm{Tr}(A)$, we can alternatively view both examples as proper generalizations of the finite dimensional Chern-Weil theory.

Similarly, on the smaller algebra $\mathcal{C}l_{\leq p}(M, E)$ for $p < 0$, there are leading symbol traces that are also related to the coefficient of the leading term (usually the "most divergent" term) in the heat operator regularization. Because this is a proper subalgebra of the Lie algebra $\mathcal{C}l_{\leq 0}(M, E)$ of the structure group, we cannot expect a full Chern-Weil theory for $\mathcal{C}l_{\leq p}(M, E)$-connections. Nevertheless, in §3.3, we produce closed characteristic forms for these connections and show that the characteristic classes obtained this way are independent of the choice of connection, provided the two connections differ by a $\mathcal{C}l_{\leq p}(M, E)$-valued one-form. In §4, we improve this result and the results in [18] by formally keeping track of all terms in the relevant asymptotic expansions.

In summary, there seems to be no canonical generalization or regularization of finite dimensional Chern-Weil theory free from drawbacks: using the operator trace on trace class operators is too restrictive for zeroth order operators, the better adapted leading symbol traces vanish on trace class operators and operators of negative order, and the weighted traces of §4 are not true traces. Moreover, there is no canonical interpretation of whether a specific method is indeed a proper generalization, as the Wodzicki residue can be interpreted either as an extension of the finite dimensional theory or as a purely infinite dimensional pheonomenon. The particular choice of regularization depends on a combination of physical motivation, computability and nontriviality results.

3.1 The Wodzicki residue

Recall that the Wodzicki residue $\mathrm{res_w} A$ of a ΨDO A acting on sections of a bundle E over a closed manifold M is defined to be the residue of the pole term of $\mathrm{Tr}(AQ^{-s})$ at $s = 0$, for an elliptic operator Q with certain technical conditions. Alternatively, $\mathrm{res_w} A$ is proportional to the coefficient of $\log\varepsilon$ in the asymptotic expansion of $\mathrm{Tr}(Ae^{-\varepsilon Q})$ as $\varepsilon \to 0$, for $Q \in \mathcal{E}ll^+(M,E)$. The strengths of the Wodzicki residue are (i) its local nature:

$$\mathrm{res_w} A = \frac{1}{(2\pi)^n}\int_{S^*M} \mathrm{tr}\ \sigma^A_{-n}(x,\xi)\ d\xi\ dx,$$

where $n = \dim(M)$, S^*M is the unit cosphere bundle of M, and σ^A_{-n} is the $(-n)^{\text{th}}$ homogeneous piece of the symbol of A; and (ii) the fact that it is the unique trace on $\mathcal{C}l = \mathcal{C}l(M,E)$, up to normalization. Its drawback is its vanishing on all differential and multiplication operators, all trace class operators (and so all ΨDOs of order less than $-n$) and all operators of non-integral order.

Given an infinite dimensional bundle $\mathcal{E}$ over a base B with fibers modeled either on $H^s(M,E)$ (with $s \gg 0$) or on $C^\infty(M,E)$, and a connection on $\mathcal{E}$ with curvature $\Omega \in \Omega^2(B,\mathcal{C}l)$, we can form the k^{th} Wodzicki-Chern form by setting

$$c_k^{\mathrm{w}}(\Omega) = \mathrm{res_w}\Omega^k \in \Omega^{2k}(B).$$

By Theorem 2.2, $c_k^{\mathrm{w}}(\Omega)$ is closed and independent of the connection.

As an example, we show that the Wodzicki-Chern forms vanish for current groups $\mathcal{C} = H^{s+1}(M,G)$, the space of H^{s+1}-maps from a closed Riemannian manifold M to a Lie group G. (The same vanishing holds for Fréchet current groups.) The tangent space at any map f is the space of sections $H^s(M, f^*TG)$. Since TG is canonically trivial, so is $T\mathcal{C}$. For the trivial connection, we certainly have the vanishing of the Wodzicki-Chern forms. It follows that the Wodzicki-Chern classes vanish for any ΨDO connection.

We now check that the Levi-Civita connection on a current group is a ΨDO connection for a semi-simple Lie group G of compact type. (These assumptions ensure that the Killing form is nondegenerate and that the adjoint representation is antisymmetric for this form.) $\mathcal{C}$ is a Hilbert Lie group with Lie algebra $H^s(M,A)$, the space of H^s sections of the trivial bundle $M \times A$, where $A = \mathrm{Lie}(G)$. Thus the tangent bundle $T\mathcal{C}$ is a ΨDO bundle with fibers modeled on $H^s(M,A)$. For Δ the Laplacian on functions on M, we set $Q_0 := \Delta \otimes 1_A$, a second order elliptic operator acting densely on $H^s(M,A)$. Q_0 is non-negative for the scalar product $\langle\cdot,\cdot\rangle_0 := \int_M \mathrm{dvol}(x)(\cdot,\cdot)$, where $(\cdot,\cdot)$ is minus the Killing form. $T\mathcal{C}$ has a left-invariant weight $Q_\gamma = L_\gamma Q_0 L_\gamma^{-1}$ (i.e. a family of elliptic operators on the fibers), where L_γ is left translation by $\gamma \in \mathcal{C}$.

$\mathcal{C}$ has a left-invariant Sobolev s-metric defined by

$$\langle\cdot,\cdot\rangle^s := \langle {Q_0}^{\frac{s}{2}}\cdot, {Q_0}^{\frac{s}{2}}\cdot\rangle_0,$$

where Q_0 is really $Q_0 + P$, for P the orthogonal projection of Q_0 onto its kernel. The corresponding left-invariant Levi-Civita connection has the global expression $\nabla^s = d + \theta^s$, with θ^s a left-invariant $\mathrm{End}(T\mathcal{C})$-valued one-form on $\mathcal{C}$ induced by the $\mathrm{End}(H^s(M,A))$-valued one-form on $H^s(M,A)$

$$\theta_0^s(U) = \frac{1}{2}\left(\mathrm{ad}_U + {Q_0}^{-s}\mathrm{ad}_U\ {Q_0}^s - {Q_0}^{-s}\mathrm{ad}_{{Q_0}^s U}\right), \tag{3.1}$$

for $U \in H^s(M, A)$ [6, (1.9)]. By inspection, θ^s takes values in $\mathcal{C}l_{\leq 0}(M, M \times A)$.

The fact that Wodzicki-Chern classes vanish on current groups is not surprising, since the same argument works on any parallelizable manifold. It is more surprising that these classes vanish on any loop space, even when the target manifold is not parallelizable [14]. Thus the Wodzicki residue, the natural first choice for a trace functional, yields a Chern-Weil theory that is currently vacuous. As a result we look for other functionals with nontrivial Chern-Weil theory.

3.2 Leading symbol traces

The uniqueness of the trace on $\mathcal{C}l$ defined by the Wodzicki residue does not rule out the existence of other traces on subalgebras of $\mathcal{C}l$. Indeed, the ordinary operator trace on $\mathcal{C}l_{\leq -n/2}$ is an example. In this subsection, we will introduce a family of traces on $\mathcal{C}l_{\leq 0}$ and show that they produce non-vanishing Chern classes on the universal bundle associated to the gauge group for $\mathcal{E} = TLM$, the tangent bundle to the free loop space of a Riemannian manifold M. To our knowledge, this is the first example of non-vanishing Chern classes of infinite dimensional bundles above c_1.

We first produce a "trace" on $\mathcal{C}l_{\leq p}$ for fixed $p \leq 0$ with values in S^*M, and an associated family of true traces. A description of all traces on e.g. $\mathcal{C}l_{\leq p}$ for fixed $p \leq 0$ is an interesting question; we have preliminary results with J.-M. Lescure. Let $\mathcal{D}'(X)$ denote the space of complex valued distributions on a compact manifold X.

Lemma 3.1: *For $p \leq 0$, the map $\mathrm{Tr}_p : \mathcal{C}l_{\leq p}(M, E) \to C^\infty(S^*M)$ defined by $\mathrm{Tr}_p(A) = \mathrm{tr}_x(\sigma_p^A(x, \xi))$ has $\mathrm{Tr}_p(A + B) = \mathrm{Tr}_p(A) + \mathrm{Tr}_p(B)$, $\mathrm{Tr}_p(\lambda A) = \lambda \mathrm{Tr}_p(A)$, and $\mathrm{Tr}_p(AB) = \mathrm{Tr}_p(BA)$. For any $\Lambda \in \mathcal{D}'(S^*M)$, the map $\mathrm{Tr}_p^\Lambda : \mathcal{C}l_{\leq p} \to \mathbb{C}$ given by $\mathrm{Tr}_p^\Lambda(A) = \Lambda(\mathrm{Tr}_p(A))$ is a trace.*

Proof: Certainly taking the p^{th} order symbol is linear. When $p = 0$, since the leading order symbol is multiplicative, we have

$$\mathrm{tr}_x\ \sigma_0^{AB} = \mathrm{tr}_x(\sigma_0^A \cdot \sigma_0^B) = \mathrm{tr}_x(\sigma_0^B \cdot \sigma_0^A) = \mathrm{tr}_x\ \sigma_0^{BA}.$$

When $p < 0$, for $A, B \in \mathcal{C}l_{\leq p}(M, E)$, the products AB and BA lie in $\mathcal{C}l_{\leq 2p}(M, E)$ so that we have

$$\mathrm{tr}_x\ \sigma_p^{AB} = 0 = \mathrm{tr}_x\ \sigma_p^{BA}.$$

The proof of the second statement is immediate. □

In this subsection, we focus on the case $p = 0$, leaving the case $p < 0$ for the next subsection. For convenience we set $\mathrm{Tr} := \mathrm{Tr}_0$, $\mathrm{Tr}^\Lambda := \mathrm{Tr}_0^\Lambda$. When the distribution is given by $\Lambda(\phi) = \int_{S^*M} f(x, \xi)\phi(x, \xi)$ for all $\phi \in C^\infty(S^*M)$, we simply write Tr^f.

Remarks: (i) When $p < 0$, for $r \in [2p, p]$, $\mathrm{Tr}_r A = \mathrm{tr}_x(\sigma_r^A(x, \xi))$ is also a trace, as $\mathrm{Tr}_r(AB)$ trivially vanishes for $r > 2p$. The proof of the lemma covers the case $r = 2p$.

(ii) Let $Q \in \mathcal{E}ll^+(M,E)$ have scalar leading symbol $\sigma_L^Q(x,\xi) = f(x,\xi)\mathrm{Id}$. Define $\tilde{f} \in C^\infty(S^*M)$ by

$$\tilde{f} = \frac{(n-1)!\Gamma\left(\frac{n}{q}\right)\dim(E)}{q(2\pi)^n} f(x,\xi)^{-\frac{n}{q}},$$

where $n = \dim(M), q = \mathrm{ord}(Q)$. Then $\mathrm{Tr}^{\tilde{f}}(A)$ is the leading term in the asymptotics of $\mathrm{Tr}(Ae^{-\varepsilon Q})$ if $\mathrm{ord}(A) = 0$ (see Proposition 3.4).

Recall that the ring of characteristic classes for e.g. $U(n)$ bundles is generated by the Chern classes $c_k = [\mathrm{Tr}(\Lambda^k\Omega)]$, or equivalently by the components $\nu_k = [\mathrm{Tr}(\Omega^k)]$ of the Chern character. Note we are momentarily distinguishing between $\mathrm{Tr}(\Lambda^k A)$ and $\mathrm{Tr}(A^k)$ for a matrix A. We will concentrate on Chern forms, and abuse notation by writing $c_k = [\mathrm{Tr}(\Omega^k)]$.

Definition: Let $\mathcal{E}$ be a bundle over B modeled on $H^s(M,E)$ or $C^\infty(M,E)$, and let ∇ be a ΨDO-connection on $\mathcal{E}$. The k^{th} Chern class of ∇ with respect to $\Lambda \in \mathcal{D}'(S^*M)$ is defined to be the de Rham cohomology class

$$[c_k^\Lambda(\Omega)] = [\Lambda(\mathrm{tr}_x \sigma_0^{\Omega^k}(x,\xi))] \in H^{2k}(B;\mathbb{C}). \tag{3.2}$$

As before, when $\Lambda(\phi) = \int_{S^*M} f(x,\xi)\phi(x,\xi)$ we set $c_k^f = c_k^\Lambda$.

Remarks: (1) As an example, if $f = 1 \in C^\infty(S^*M)$ is the constant map with value 1 on S^*M, then

$$c_k^f(\Omega) = \int_{S^*M} \mathrm{tr}_x\ \sigma_0^{\Omega^k}(x,\xi).$$

At another extreme, if $\Lambda = \delta_{(x_0,\xi_0)}$ is a delta function, then

$$c_k^\Lambda(\Omega) = \mathrm{tr}_{x_0}\ \sigma_0^{\Omega^k}(x_0,\xi_0).$$

(2) As in the previous remark, we can define Chern classes $c_{r,k}^\Lambda$ for connections with curvature forms taking values in $\mathcal{C}l_{\leq p}$ for any $r \in [2p, p]$. Note that for $r < p$ and e.g. $\Lambda(\phi) = \int_{S^*M}\phi(x,\xi)$, these classes are defined only after a choice of coordinates on M, E and a partition of unity on M, since integrals of non-leading order symbols depend on such choices.

The following result justifies this definition:

Theorem 3.2: *Let ∇ be a $\mathcal{C}l_{\leq 0}$ connection on a* ΨDO *bundle $\mathcal{E}$. Then the differential forms $c_k^\Lambda(\Omega)$ are closed, and their cohomology classes are independent of the choice of $\mathcal{C}l_{\leq 0}$ connection.*

PROOF: By Lemma 3.1, Tr^Λ is a trace on $\mathcal{C}l_{\leq 0}(M,E)$, so we can apply Lemma 2.1 to the principal bundle $P = P^{\mathcal{E}}$ built from $\mathcal{E}$ to get the relation $d \circ \mathrm{Tr}^\Lambda = \mathrm{Tr}^\Lambda \circ \nabla^{\mathrm{ad}}$. We then apply Theorem 2.2 to get the corresponding Chern classes $[c_k^\Lambda(\Omega)]$. □

Remark: For the linear functionals Tr_p^{Λ}, the proof that the Chern forms are closed goes through. However, the proof of their independence of choice of connection breaks down, since the class of connections with curvature forms lying in $\mathcal{C}l_{\leq p}$ is not connected. In the next subsection, we nevertheless show that the independence holds on a restricted class of connections.

When the structure group reduces to a gauge group, we can construct an example of non-zero Chern classes $[c_k^f(\Omega)]$. Fix $n > k$ and consider the Grassmannian $BU(n) = Gr(n, \infty)$ with its universal vector bundle E_n. We consider the pullback bundle $E = \pi^* E_n$ over $S^1 \times BU(n)$, with π the projection onto $BU(n)$. We now "loopify" to form $B = L(S^1 \times BU(n))$, the free loop space of $S^1 \times BU(n)$, with bundle $\mathcal{E}$ whose fiber over a loop γ is the space of smooth sections of $\gamma^* E$ over S^1. ($\mathcal{E}_\gamma$ is the space of loops in E lying over γ, suitably interpreted at self-intersection points of γ, so we will write $\mathcal{E} = L\pi^* E_n$.) Since $\gamma^* E$ is (non-canonically) isomorphic to the trivial bundle $S^1 \times \mathbb{C}^n$ over S^1, it is easily checked that the structure group for $\mathcal{E}$ is the gauge group $\mathcal{G}$ of this trivial bundle. Indeed, as in the example before Theorem 2.2 with $M = S^1, X = S^1 \times BU(n)$, the structure group of $\mathcal{E}$ is $C^\infty(S^1, \mathrm{Aut}(\gamma^* E)) = C^\infty(S^1, Gl_n(\mathbb{C}))$.

Take a hermitian connection ∇ on E_n (e.g. the universal connection PdP, where P_x is the projection of $\mathbb{C}^\infty$ onto the n-plane x) and its pullback connection $\pi^*\nabla$ on $S^1 \times BU(n)$. As in the case of the tangent bundle to a loop space, we can take an L^2 or pointwise connection ∇^0 on $\mathcal{E}$ by setting

$$\nabla^0_X Y(\gamma)(\theta) = (\pi^*\nabla)_{X(\theta)} Y(\theta),$$

for X a vector field along γ (i.e. a tangent vector in B at γ) and Y a local section of $\mathcal{E}$. The curvature Ω^0 acts pointwise and hence is a multiplication operator: $(\Omega^0_\gamma u)(\theta) = (\pi^*\nabla^2)_{\gamma(\theta)} u(\theta)$. In particular, its symbol is independent of ξ.

Pick the distribution $\delta = (1, +)$ on $C^\infty(S^*S^1) = C^\infty(S^1 \times \{\pm\partial_\theta\})$: i.e.

$$\delta(f(\theta, \partial_\theta), g(\theta, -\partial_\theta)) = \frac{1}{2\pi}\int_{S^1} f(\theta, \partial_\theta)\, d\theta.$$

We claim that $[c_k^\delta(\Omega^0)]$ is nonzero in $H^{2k}(B; \mathbb{C})$. To see this, let $a = a_{2k} \in H_{2k}(BU(n), \mathbb{C})$ be such that $\langle c_k(E_n), a\rangle = 1$. Define $c \in H^{2k}(B)$ to be $c = \beta_* a$, where $\beta : BU(n) \to L(S^1 \times BU(n))$ is given by $\beta(x)(\theta) = (\theta, x)$. Now

$$\langle [c_k^\delta(\Omega^0)], c\rangle = \langle [c_k^\delta(\Omega^0)], \beta_* a\rangle = \langle [\beta^* c_k^\delta(\Omega^0)], a\rangle, \tag{3.3}$$

since β has degree one. For $\gamma \in L(S^1 \times BU(n))$, we have

$$c_k^\delta(\Omega^0)(\gamma) = \frac{1}{2\pi}\int_{S^1} \mathrm{tr}\left(\sigma_0^{(\Omega^0)^k}(\gamma(\theta), \partial_{\gamma(\theta)})\right)\, d\theta = \frac{1}{2\pi}\int_{S^1} \mathrm{tr}\left(\pi^*\Omega^k_{\gamma(\theta)}\right)\, d\theta. \tag{3.4}$$

For a tangent vector $X \in T_x BU(n)$, it is immediate that $\beta_*(X) \in T_{\beta(x)} B$ has $\beta_*(X)(\theta, x) = (0, X)$. Thus by (3.4),

$$\beta^* c_k^\delta(\Omega^0)(X_1, \ldots, X_{2k}) = \frac{1}{2\pi}\int_{S^1} \mathrm{tr}(\pi^*\Omega^k)(X_1, \ldots, X_{2k}) = \mathrm{tr}(\pi^*\Omega^k)(X_1, \ldots, X_{2k}). \tag{3.5}$$

Combining (3.3) and (3.5), we get

$$\langle [c_k^\delta(\Omega^0)], c\rangle = \langle [\mathrm{tr}(\Omega^k)], a\rangle = 1.$$

In particular, the class $[c_k^\delta(\Omega^s)]$ must be non-zero.

Theorem 3.3: *The cohomology classes $[c_k^\delta(\Omega)]$ are non-zero in general. In particular, the corresponding classes for the universal bundle $E\mathcal{G}$ are nonzero in the cohomology of the classifying space $B\mathcal{G}$, where $\mathcal{G}$ is the gauge group of the trivial bundle $S^1 \times \mathbb{C}^n$ over S^1.*

We have shown the first statement. To explain the second statement, note that although the structure group of $L\pi^*\gamma_n$ is the gauge group of the trivial bundle $S^1 \times \mathbb{C}^n$ over S^1, the curvature of the connection will take values in $\mathcal{C}l_{\leq 0}(S^1, S^1 \times \mathbb{C}^n)$ of this bundle. As a result, the classifying space is really $B\mathcal{C}l_0^*$. It can be shown [17] that the principal symbol map is the time one map of a deformation retraction of $\mathcal{C}l_0^*$ onto the gauge group of the trivial bundle over S^*S^1, which is just two copies of $\mathcal{G}$. Thus $B\mathcal{C}l_0^*$ is homotopy equivalent to $B\mathcal{G}\coprod B\mathcal{G}$, and each $[c_k^{(1,\pm)}]$ is non-zero in one copy of $B\mathcal{G}$. The proof of the second statement depends on the existence of a universal connection on $E\mathcal{G}$ over $B\mathcal{G}$ [17].

In fact, $B\mathcal{G}$ equals $L_0BU(n)$, the space of contractible loops on $BU(n)$ [2]. It is known that $H^*(B\mathcal{G}, \mathbb{C})$ is a super-polynomial (i.e. super-commutative) algebra with one generator in each degree $k \in \{1, \ldots, 2n\}$. In analogy with finite dimensions, we conjecture that $[c_k^\delta]$ is a nonzero multiple of the generator in degree $2k$. For $k = 1$, this is clear.

We now outline a conjectured construction of geometric representatives of the odd generators in $H^*(B\mathcal{G})$. The tangent bundle TLM of any loop space splits off a trivial line bundle, namely the span of $\dot\gamma$ at the loop γ. Note that for any connection on a bundle over LM,for any $\Lambda \in \mathcal{D}'(S^*S^1)$ we have

$$di_{\dot\gamma}c_k^\Lambda(\Omega) = di_{\dot\gamma}c_k^\Lambda(\Omega) + i_{\dot\gamma}dc_k^\Lambda(\Omega) = L_{\dot\gamma}c_k^\Lambda(\Omega), \tag{3.6}$$

where i is interior product and L is Lie derivative. We state without (the elementary) proof that (3.6) implies

$$di_{\dot\gamma}c_k^\Lambda(\Omega) = c_k^{\partial\Lambda}(\Omega), \tag{3.7}$$

where $\partial\Lambda$ is the derivative of Λ as a distribution on S^*S^1. In particular, we see that $i_{\dot\gamma}c_k^{(1,\pm)}(\Omega)$ are closed forms. It remains to be seen if $[i_{\dot\gamma}c_k^{(1,\pm)}(\Omega)] \in H^{2k-1}(B\mathcal{G}\coprod B\mathcal{G})$ are non-zero.

We can also use (3.7) to understand the dependence of $[c_k^\Lambda(\Omega)]$ on Λ. Since Λ is a zero current on S^*S^1 and hence is trivially closed, and since exact zero currents $\partial\Lambda$ produce vanishing Chern classes by (3.7), we see that the space of classes

$$\{[c_k^\Lambda(\Omega)] : f \in \mathcal{D}'(S^*S^1)\} \in H^{2k}(B\mathcal{G})$$

is isomorphic to the zeroth cohomology group of complex currents on S^*S^1. (Here we are extending the usual confusion of functions f and one-forms $f d\theta$ on S^1 to a confusion of zero- and one-currents.) This cohomology group is isomorphic to $H_0(S^*S^1)$, and is spanned by $(1,\pm)$. (The reader may wish to check directly from (3.7) that all δ-function currents on one copy of S^1 produce the same cohomology class. A more interesting exercise is to show that these delta functions produce the same cohomology class as one of $(1,\pm)$ using Fourier series.)

Remarks: (i) The general case, where the gauge group is associated to the bundle E over a closed manifold M, is more complicated. The cohomology of $B\mathcal{G}$ is known, and in general has odd dimensional cohomology [2]. We do not know at present which part of $H^*(B\mathcal{G})$ is spanned by $[c_k^\Lambda(\Omega^{E\mathcal{G}})]$, where we use the universal connection mentioned above. We also do not know how to produce geometric representatives of odd dimensional classes in $H^*(B\mathcal{G})$, nor do we know how the Chern classes depend on the distribution.

(ii) In the loop group case, Freed showed [6] that the curvature Ω^s of the H^s Levi-Civita connection is a ΨDO of order -1 for $s > 1/2$. For this connection, the Chern forms built from σ_0 trivially vanish. In the next subsection, we discuss the k-th Chern forms one can build using the symbol of order $-k$. For loop groups, the first Chern form requires the additional analysis in §4, while higher powers of the curvature are trace class operators, requiring no regularization.

(iii) For the distribution Λ given by integration over S^*M, the symbol trace looks like an integration over the fiber. Nevertheless, to the best of our knowledge, our Chern classes are not given by an integration of characteristic classes of an associated finite dimensional bundle.

In more detail, let $\mathcal{E}$ be a bundle over B with structure group $\mathcal{C}l_0^*$ and with a ΨDO connection ∇. Let $\mathcal{G}$ be the gauge group of π^*E over S^*M. As in [17], $\mathcal{E}$ reduces to a $\mathcal{G}$-bundle $\mathcal{F}'$ with connection ∇', where the connection one-form of ∇' is the zeroth order symbol of the connection one-form of ∇. The curvature of ∇' equals the zeroth order symbol of the curvature of ∇. The fiber of $\mathcal{F}'$ is still $C^\infty(M,E)$, and we can form a $\mathcal{G}$-bundle $\mathcal{F}$ over B with fiber $C^\infty(S^*M,\pi^*E)$ using the same gluing maps as for $\mathcal{F}'$. (While $\mathcal{G}$ acts on fibers of $\mathcal{F}'$ as zeroth order ΨDOs, it acts on fibers of $\mathcal{F}$ as multiplication operators.) The connection one-form for ∇' still transforms correctly on $\mathcal{F}$, and so defines a connection on $\mathcal{F}$, also denoted ∇'. The curvatures of the ∇' connections are equal.

$\mathcal{F}$ induces a finite dimensional bundle F over $B\times S^*M$, with fiber $\pi^*E|_{(x,\xi)}$ over (b,x,ξ). However, ∇' induces a connection ∇^F on F only after we specify how to differentiate in S^*M directions. Assume that we can specify these differentiations so that the curvature Ω^F is flat in S^*M directions. Ω^F will still agree with $\Omega^{\mathcal{F}}$ in B directions, and

$$\begin{aligned}[c_k^\Lambda(\Omega^{\mathcal{E}})] &= [c_k^\Lambda(\Omega^{\mathcal{F}'})] = [c_k^\Lambda(\Omega^{\mathcal{F}})] = \left[\int_{S^*M}\mathrm{tr}((\Omega^{\mathcal{F}})^k)\right]\\ &= \int_{S^*M}[\mathrm{tr}((\Omega^F)^k)] == \int_{S^*M} c_k(F),\end{aligned}$$

where $\int_{S^*M}$ outside the braces is the pushforward map from $H^*(B\times S^*M)$ to $H^*(B)$.

Thus under the assumption, the cohomology class $c_k^\Lambda(\mathcal{E})$ will indeed be the integration over the fiber of the Chern class of a finite dimensional bundle. However, the assumption is unreasonable: even if E is trivial, as for loop spaces, there is no canonical identification of the fibers of $\mathcal{E}$ with $H^s(M,E)$, so the trivial connection on E does not glue up to a connection on F which is trivial in S^*M directions.

3.3 Leading terms in heat-kernel asymptotic expansions

In this section, we consider traces Tr_p^Λ for $p \leq 0$. Theorem 2.2 no longer applies, as it did for $p=0$, since Tr_p^Λ defines a trace only on the subalgebra $\mathcal{C}l_{\leq p}$ of the Lie algebra $\mathcal{C}l_{\leq 0}$. We cannot expect these traces to produce a full Chern-Weil theory on ΨDO bundles. However, they do yield characteristic classes which are independent of the choice of the connection in some restricted class of connections.

We first relate leading symbol traces to leading terms in heat-kernel asymptotic expansions. We then use Tr_p^Λ to prove that the leading term in the asymptotic expansion of $\mathrm{tr}(\Omega e^{-\varepsilon Q})$ is closed, where Q is a generalized Laplacian and Ω the curvature on a ΨDO bundle. In fact, we show that if Q has positive leading symbol $\sigma_L(Q)(x,\xi) = f(x,\xi)\mathrm{Id}$ and if Ω has integer order $a > -\dim(M)$, then this leading term is given by the leading symbol trace $\mathrm{Tr}_a^f(\Omega)$.

The following folklore result follows from the analysis developed in [7], while the analysis in the following proof is hidden in the local nature of the Wodzicki residue.

Proposition 3.4: *Let $A \in \mathcal{C}l_{\leq 0}(M,E)$ have integral order $a > -n = -\dim(M)$, and let Q be an elliptic ΨDO of order q with positive scalar leading symbol $\sigma_L(Q)(x,\xi) = f(x,\xi)\mathrm{Id}$. Let $c = c(n,a,q)$ be*

$$c = \frac{\Gamma(\frac{n+a}{q})\dim(E)(n-1)!}{q(2\pi)^n}.$$

Then as $\varepsilon \to 0$,

$$\mathrm{tr}(Ae^{-\varepsilon Q}) = c\int_{S^*M} \mathrm{tr}\left(\sigma_a(A)(x,\xi)\right)\left(f(x,\xi)\right)^{-\frac{n+a}{q}} \cdot \varepsilon^{-\frac{n+a}{q}} + \mathrm{o}(\varepsilon^{-\frac{n+a}{q}}).$$

In particular, if $\sigma_L(Q)(x,\xi) = ||\xi||^k$ for some k, then

$$\mathrm{tr}(Ae^{-\varepsilon Q}) = c\int_{S^*M} \mathrm{tr}\left(\sigma_a(A)\right) \cdot \varepsilon^{-\frac{n+a}{q}} + \mathrm{o}(\varepsilon^{-\frac{n+a}{q}}).$$

Proof: We want to compute the coefficient $a_0(A,Q)$ in the known asymptotic expansion

$$\mathrm{tr}(Ae^{-\varepsilon Q}) = \sum_{j=0}^{a+n} a_j(A,Q)\varepsilon^{\frac{j-a-n}{q}} + b_0(A,Q)\log\varepsilon + \mathrm{O}(1), \tag{3.8}$$

for general $A \in \mathcal{C}l(M,E)$ of order a, and with $a_j(A,Q), b_0(A,Q) \in \mathbb{C}$. The coefficient $b_0(A,Q)$ satisfies $b_0(A,Q) = -\frac{1}{q}\mathrm{res_w}(A)$.

A Mellin transform yields

$$a_0(A,Q) = \mathrm{Res}_{z=\frac{n+a}{q}} \Gamma\left(\frac{n+a}{q}\right) \mathrm{tr}(AQ^{-z}),$$

(the case $A = 1$ considered in [10, (12)] easily extends to a general ΨDO A). Thus, for A as in the hypothesis,

$$a_0(A,Q) = \frac{\Gamma(\frac{n+a}{q})}{q\Gamma(n+a)} a_0(A, Q^{\frac{1}{q}}).$$

Thus it suffices to prove the formula for Q_1 of order one. Since $\mathrm{ord}(AQ_1^{-(n+a)}) = -n$, (3.8) becomes

$$\mathrm{tr}(AQ_1^{-(n+a)} e^{-\varepsilon Q_1}) = -\mathrm{res_w}(AQ_1^{-(n+a)}) \log \varepsilon + \mathrm{O}(1).$$

Differentiating this expansion $n+a$ times (recall that $n+a$ is a positive integer) with respect to ε, we get

$$\mathrm{tr}(Ae^{-\varepsilon Q_1}) \sim (n+a-1)! \, \mathrm{res_w}(AQ_1^{-(n+a)}) \varepsilon^{-(n+a)}.$$

The local formula for the Wodzicki residue yields:

$$\begin{aligned}
\mathrm{res_w}(AQ_1^{-(n+a)}) &= \frac{1}{(2\pi)^n} \int_{S^*M} \mathrm{tr}(\sigma_{-n}(AQ_1^{-(n+a)})) \\
&= \frac{1}{(2\pi)^n} \int_{S^*M} \mathrm{tr}\left(\sigma_a(A)\sigma_{-(n+a)}(Q_1^{-(n+a)})\right) \\
&= \frac{\dim(E)}{(2\pi)^n} \int_{S^*M} \mathrm{tr}(\sigma_a(A)) f(x,\xi)^{-(n+a)}.
\end{aligned}$$

Hence our original Q has

$$a_0(A,Q) = \frac{\Gamma(\frac{n+a}{q})}{q\Gamma(n+a)} a_0(A, Q^{\frac{1}{q}}) = c \int_{S^*M} \mathrm{tr}\left(\sigma_a(A)\right) \left(f(x,\xi)\right)^{-(n+a)/q}.$$

□

Lemma 3.5: *Let $\mathcal{E} \to B$ be a ΨDO bundle with a ΨDO connection ∇ whose connection one-form θ takes values in $\mathcal{C}l_{\leq 0}(M,E)$. Let $A \in \Omega^k(B, \mathcal{C}l(\mathcal{E}))$ be a $\mathcal{C}l_{\leq a}(\mathcal{E})$-valued form whose order a is independent of $b \in B$.*

(i) For any distribution $\Lambda \in \mathcal{D}'(S^*M)$, $d\mathrm{Tr}_a^\Lambda(A) = \mathrm{Tr}_a^\Lambda([\nabla, A])$. In particular, if $[\nabla, A] = 0$, then $\mathrm{Tr}_a^\Lambda(A) \in \Omega^k(B, \mathbb{C})$ is closed.

(ii) Let $Q = \{Q_b\} \in \Gamma(\mathcal{C}l(\mathcal{E}))$ be a smooth family of elliptic operators of constant order q and with positive scalar leading symbol independent of $b \in B$. Define $a_0(A,Q), b_0(A,Q) \in \Omega^k(B, \mathbb{C})$ as in (3.8). If $[\nabla, A] = 0$, then $b_0(A,Q)$ is closed, and $a_0(A,Q)$ is closed if $a \in \mathbb{Z}, a > -\dim(M)$.

Note that the condition on the leading symbol is independent of trivialization of $\mathcal{E}$.

PROOF:

(i) We have

$$\begin{aligned} d\mathrm{Tr}_a^\Lambda(A) &= \Lambda\left[d\mathrm{tr}_x(\sigma_a(A))\right] = \Lambda\left[\mathrm{tr}_x\left(\sigma_a(dA)\right)\right] = \Lambda\left[\mathrm{tr}_x\left(\sigma_a(dA+[\theta,A]\right)\right] \\ &= \Lambda\left[\mathrm{tr}_x\left(\sigma_a([\nabla,A])\right)\right]. \end{aligned}$$

Here we use the fact that θ has non-positive order, so that

$$\mathrm{tr}_x\left(\sigma_a([\theta,A])\right) = \mathrm{tr}_x\left([\sigma_0(\theta),\sigma_a(A)]\right) = 0,$$

if $[\theta,A]$ has expected order a. Finally, $\sigma_a([\theta,A]) = 0$ trivially if the order of $[\theta,A]$ is less than a.

(ii) $a_0(A,Q)$ is the leading term in the asymptotic expansion (3.8) and hence proportional to a leading symbol trace by the above proposition. It is therefore closed by (i). Since $b_0(A,Q) = -\frac{1}{q}\mathrm{res_w}(A)$, it is closed by §3.1. □

As a consequence, we can build "Chern-Weil type" closed forms from leading symbol traces Tr_p^Λ.

Theorem 3.6: *Let $\mathcal{E}\to B$ be a ΨDO bundle with a ΨDO connection ∇ whose connection one-form θ takes values in $\mathcal{C}l_{\leq 0}(\mathcal{E})$, and whose curvature two-form (which takes values in $\mathcal{C}l_{\leq 0}(\mathcal{E})$) has constant order a. Let $Q\in\Gamma\left(\mathcal{C}l(\mathcal{E})\right)$ be a smooth family of elliptic operators of constant order q and with positive scalar leading symbol independent of $b\in B$. In the notation of (3.8), the following elements of $\Omega^{2k}(B,\mathbb{C})$ are closed:*

(i) $\mathrm{Tr}_{ka}^\Lambda(\Omega^k)$, for any $\Lambda\in\mathcal{D}'(S^*M)$;

(ii) $a_0(\Omega^k,Q)$, for $ka\in\mathbb{Z}, a>-\frac{n}{k}$, where $n=\dim(M)$;

(iii) $b_0(\Omega^k,Q)$. Moreover, the cohomology class of $b_0(\Omega^k,Q)$ is independent of the choice of connection ∇.

Let $\{\nabla_t : t\in[0,1]\}$ be a smooth family of $\mathcal{C}l_{\leq 0}(\mathcal{E})$ connections such that $\dot{\nabla}_t\in\Omega^1(B,\mathcal{C}l_{\leq a}(\mathcal{E}))$ and $\Omega_t\in\Omega^2(B,\mathcal{C}l_{\leq a}(\mathcal{E}))$. The following de Rham cohomology classes are independent of t:

(iv) $\left[\mathrm{Tr}_{ka}^\Lambda(\Omega_t^k)\right]$;

(v) $\left[a_0(\Omega_t^k,Q)\right]$ for $ka\in\mathbb{Z}, a>-\frac{n}{k}$, where $n=\dim(M)$.

Note that when $a=0$, this gives back the results of Theorem 3.2.

PROOF: (i) – (iii) follow from Lemma 3.5. The fact that $[b_0(\Omega^k,Q)]$ is independent of the choice of connection follows from the results of §3.1, since $b_0(\Omega^k,Q)$ is proportional to $\mathrm{res_w}(\Omega^k)$. For (iv), we repeat (2.2), with λ replaced by Tr_{ka}^Λ. Note that we have to use Lemma 3.5 to swap the (covariant) differentiation and Tr_{ka}^Λ in this argument. Finally, since $a_0(\Omega_t^k,Q)$ is a leading order symbol by Proposition 3.4, we get (v). □

Remark: Theorem 3.6 does not apply to Freed's conditional first Chern form on loop groups. Even though, as we will see in §4, this Chern form corresponds to the finite part $a_0(\Omega,Q_0)$, the curvature of the Levi-Civita connection for the $H^{1/2}$ metric on LG has order $a=-1=-\dim(S^1)$, the borderline case for Theorem 3.6. Showing that Freed's conditional first Chern form on loop groups is closed [6] requires the more refined analysis of §4.

4 Characteristic classes and formal power series

In this section, we use heat kernel regularized traces to produce an asymptotic series of characteristic forms, provided the regularizing family of operators $\{Q_b\}$ is "fairly covariantly constant." This improves the weighted trace approach of [18], and is based on regularization techniques common in quantum field theory.

We begin with some calculations leading to Lemma 4.1, which measures the effect of trying to push a connection ∇ on a bundle $\mathcal{E}$ past a heat operator or a weighted trace. For $\{A_0, A_1, \cdots, A_n\} \subset \mathcal{C}l(M,E)$ and $(\sigma_0, \cdots, \sigma_n) \in (\mathbb{R}^+)^{n+1}$, the operator $A_0 e^{-\sigma_0 Q} A_1 e^{-\sigma_1 Q} \cdots A_n e^{-\sigma_n Q}$ is smoothing and hence trace class. We define *trace forms*

$$\langle A_0, A_1, \cdots, A_n \rangle_{\varepsilon,n,Q} := \int_{\Delta_n} \operatorname{tr}\left(A_0 e^{-\varepsilon\sigma_0 Q} A_1 e^{-\varepsilon\sigma_1 Q} \cdots A_n e^{-\varepsilon\sigma_n Q}\right), \tag{4.1}$$

where Δ_n is the standard n-simplex, in agreement with [9] (although there the A_i are bounded). In particular, we call the Q-weighted trace of A (with ε-cut-off) the linear functional $\langle A_0 \rangle_{\varepsilon,0,Q} = \operatorname{tr}_\varepsilon^Q(A_0)$.

The concept of trace form and hence of weighted trace extends to sections of a ΨDO bundle. Recall that a ΨDO bundle $\mathcal{E}$ with structure group $\mathcal{C}l_0^*(M,E)$ has an associated bundle of algebras $\mathcal{C}l_{\leq 0}(\mathcal{E}) = \operatorname{Ad} P^{\mathcal{E}}$ with fibers modeled on $\mathcal{C}l_{\leq 0}(M,E)$. A weight is a section $Q \in \Gamma(\mathcal{C}l(\mathcal{E}))$ with Q elliptic with positive definite leading symbol and of constant order. These conditions are independent of local chart, since the transition maps are ΨDOs. In particular, if $\{g_b\}$ is the transition map between two trivializations of $\mathcal{E}$ over b, then Q_b transforms into $g_b^{-1} Q_b g_b$; the same holds for sections of $\mathcal{C}l(\mathcal{E})$. For $A \in \Gamma(\mathcal{C}l(\mathcal{E}))$, $\operatorname{tr}_\varepsilon^Q(A)$ is well-defined, since

$$\operatorname{tr}_\varepsilon^{g^{-1}Qg}(g^{-1}Ag) = \operatorname{Tr}(g^{-1}Age^{-\varepsilon g^{-1}Qg}) = \operatorname{Tr}(g^{-1}Agg^{-1}e^{-\varepsilon Q}g) = \operatorname{tr}_\varepsilon^Q(A). \tag{4.2}$$

In the same way, for $\{A_0, A_1, \cdots, A_n\} \subset \Gamma(\mathcal{C}l(\mathcal{E}))$, the trace form $\langle A_0, A_1, \cdots, A_n \rangle_{\varepsilon,n,Q}$ is well-defined.

We set $\operatorname{tr}^Q(A)$ to be the finite part of $\operatorname{tr}_\varepsilon^Q(A)$ as $\varepsilon \to 0$. In other words, $\operatorname{tr}^Q(A)$ is the coefficient of ε^0 in the asymptotic expansion (3.8). This is equivalent to taking the zeta function regularization $\operatorname{Tr}(AQ^{-z})|_{z=0}$, provided Q is invertible and (3.8) contains no log terms.

If $Q = Q_0 + Q_1$, with Q_0 elliptic of order $q_0 > 0$ and Q_1 of order $q_1 < q_0$, the Volterra formula [1, 3, 8] (the first and third references treat the Banach algebra setting) states

$$e^{-\varepsilon(Q_0+Q_1)} = \sum_{k=0}^{\infty} (-\varepsilon)^k \int_{\Delta^k} e^{-\sigma_0 \varepsilon Q_0} Q_1 e^{-\sigma_1 \varepsilon Q_0} Q_1 \cdots Q_1 e^{-\sigma_k \varepsilon Q_0} d\sigma_0 d\sigma_1 \cdots d\sigma_k.$$

The convergence holds in the trace operator norm topology, and so

$$\operatorname{tr}\left(e^{-\varepsilon(Q_0+Q_1)}\right) = \sum_{k=0}^{\infty} (-\varepsilon)^k \langle 1, Q_1, \cdots, Q_1 \rangle_{\varepsilon,k,Q_0}.$$

For the moment, let $\mathcal{E}$ be a trivial vector bundle over B modeled on $C^\infty(M,E)$ or $H^s(M,E)$, with structure group $\mathcal{C}l_0^* = \mathcal{C}l_0^*(M,E)$, and with the trivial connection d. Let Q be a weight on $\mathcal{E}$. For $h \in T_{b_0}B$, writing $Q_b = Q_{b_0} + dQ(b_0)\cdot h + \mathrm{o}(h)$ and substituting $Q_0 = Q_{b_0}, Q_1 = dQ(b_0)\cdot h + \mathrm{o}(h)$ in (4.1) yields

$$e^{-\varepsilon Q_b} - e^{-\varepsilon Q_{b_0}} = -\varepsilon \int_0^1 e^{-\varepsilon t Q_{b_0}} \left(dQ(b_0)\cdot h\right) e^{(1-t)\varepsilon Q_{b_0}} dt + \mathrm{o}(h)$$

in the trace operator norm topology. From this we derive Duhamel's formula:

$$de^{-\varepsilon Q} = -\varepsilon \int_0^1 e^{-\varepsilon t Q} dQ e^{-(1-t)\varepsilon Q} dt = -\varepsilon \int_0^1 e^{-(1-t)\varepsilon Q} dQ e^{-\varepsilon t Q} dt. \tag{4.3}$$

Remark: In this derivation we implicitly restrict attention to a compact subset K of B, so that the $\mathrm{o}(h)$ term is uniform on K (see [3]). This applies throughout this section. In particular, we check that a form ω is closed on B by evaluating $d\omega$ over every closed cycle in B. Since the image of a cycle is compact, $d\omega$ is well defined, and formulas like (4.3) are valid. Moreover, we can use Duhamel's formula to justify differentiating asymptotic expansions of the form $\mathrm{Tr}(A_b e^{-tQ_b})$ term by term, provided the asymptotic expansions contain $\varepsilon^{\pm k/q}$ terms (possibly with zero coefficients) with the k ranging over a subset of $\mathbb{Z}$ independent of b. This is certainly the case if the order of A is constant in b.

For $A, Q = Q_{b_0}$ as above, we also have

$$[e^{-\varepsilon Q}, A] = -\varepsilon \int_0^1 e^{-(1-t)\varepsilon Q} [Q,A] e^{-\varepsilon t Q} dt. \tag{4.4}$$

Indeed, differentiating the map $t \mapsto [e^{-tQ}, A]$ (which is differentiable as a bounded linear map from $H^{a+q}(\mathcal{E})$ to $H^0(\mathcal{E})$ for $a = \mathrm{ord}(A)$, $q = \mathrm{ord}(Q)$), we get $\left(\frac{d}{dt} + Q\right)[e^{-tQ}, A] = [A,Q]e^{-tQ}$. Solving this equation by (the other) Duhamel's formula for first order inhomogeneous linear differential equations gives $[e^{-\varepsilon Q}, A] = \int_0^1 e^{-sQ}[Q,A]e^{-(\varepsilon - s)Q}dt$, and substituting $t = \varepsilon s$ yields (4.4). This identity, which holds *a priori* in the space of bounded linear maps from $H^{a+q}(\mathcal{E})$ to $H^0(\mathcal{E})$, persists as long as both sides of the equation make sense.

Replacing A by $[A,Q]$ and Q by σQ in (4.4) yields

$$\begin{aligned}
e^{-(1-\sigma)\varepsilon Q}[Q,A]e^{-\varepsilon\sigma Q} &= e^{-\varepsilon Q}[Q,A] + \varepsilon\sigma \int_0^1 e^{(-(1-\sigma_1)\sigma - (1-\sigma))\varepsilon Q} \left[Q,[Q,A]\right] e^{-\varepsilon\sigma\sigma_1 Q} d\sigma_1 \\
&= e^{-\varepsilon Q}[Q,A] + \varepsilon\sigma \int_0^1 e^{-(1-\sigma_1\sigma)\varepsilon Q} \left[Q,[Q,A]\right] e^{-\varepsilon\sigma\sigma_1 Q} d\sigma_1 \\
&= e^{-\varepsilon Q}[Q,A] + \varepsilon \int_0^\sigma e^{-(1-\sigma_1)\varepsilon Q} \left[Q,[Q,A]\right] e^{-\varepsilon\sigma_1 Q} d\sigma_1,
\end{aligned}$$

and so

$$[e^{-\varepsilon Q}, A] = -\varepsilon e^{-\varepsilon Q}[Q,A] - \varepsilon^2 \int_{\Delta_2} e^{-(1-\sigma_1)\varepsilon Q} \left[Q,[Q,A]\right] e^{-\sigma_1 \varepsilon Q} d\sigma_1 d\sigma_0. \tag{4.5}$$

For a ΨDO A, define $[A]^j_Q, j \in \mathbb{N} \cup \{0\}$, by

$$[A]^0_Q = A, \quad [A]^{j+1}_Q = [Q, [A]^j_Q] = (\text{ad } Q)^{j+1}(A).$$

We now make the important assumption that Q have scalar symbol. Iterating (4.5) gives

$$[e^{-\varepsilon Q}, A] = -\sum_{j=1}^{N-1} \frac{\varepsilon^j}{j!}[A]^j_Q e^{-\varepsilon Q} + R_{A,N}(\varepsilon), \tag{4.6}$$

for $N \in \mathbb{N}$, with

$$R_{A,N}(\varepsilon) = \varepsilon^N \int_{\Delta_N} e^{-\varepsilon(1-\sigma_1)Q}[A]^N_Q e^{-\sigma_1 \varepsilon Q} d\sigma_1 \dots d\sigma_N.$$

One can check that for $k > 0$, $R_{A,N}(\varepsilon) = \mathrm{O}(\varepsilon^k)$ for $N = N(k) \gg 0$ (cf. [12, Lemma 4.2]). In particular, we have

$$\mathrm{tr}^Q_\varepsilon([A,B]) = \mathrm{tr}(e^{-\varepsilon Q}[A,B]) = \mathrm{tr}([e^{-\varepsilon Q}, A]B) = \sum_{j=1}^{N-1} \frac{\varepsilon^j}{j!}\mathrm{tr}^Q_\varepsilon\left(A[B]^j_Q\right) + \mathrm{O}(\varepsilon^k) \tag{4.7}$$

for $N \gg 0$. Letting $Q = Q_b$ vary again, and using (4.3), (4.6), we obtain

$$d\, e^{-\varepsilon Q} = -\sum_{j=1}^{N} \frac{\varepsilon^j}{j!}[d\,Q]^{j-1}_Q e^{-\varepsilon Q} + \tilde{R}_{dQ,N+1},$$

where $\tilde{R}_{dQ,N+1}(\varepsilon) := -\varepsilon \int_0^1 \sigma^{N+1} e^{-\varepsilon Q} R_{dQ,N+1}(\varepsilon\sigma) d\sigma$. Taking traces yields

$$\mathrm{tr}\left((de^{-\varepsilon Q})A\right) = -\sum_{j=1}^{N} \frac{\varepsilon^j}{j!}\mathrm{tr}^Q_\varepsilon\left([d\,Q]^{j-1}_Q A\right) + \mathrm{O}(\varepsilon^k). \tag{4.8}$$

We now pass to the general setting by dropping the assumption that $\mathcal{E}$ is trivial. We assume that $\mathcal{E}$ has a ΨDO connection ∇ .

Lemma 4.1: *(i) For $\varepsilon > 0$,*

$$[\nabla, e^{-\varepsilon Q}] = -\sum_{j=1}^{N} \frac{\varepsilon^j}{j!}\,[[\nabla, Q]]^{j-1}_Q\, e^{-\varepsilon Q} + \tilde{R}_{[\nabla,Q],N+1}(\varepsilon)$$

where $\tilde{R}_{[\nabla,Q],N+1}(\varepsilon) = -\varepsilon \int_0^1 \sigma^{N+1} e^{-\varepsilon Q} R_{[\nabla,Q],N+1}(\varepsilon\sigma) d\sigma$.

(ii) For $\alpha \in \Omega^*(B, \mathcal{C}l(\mathcal{E}))$ and $k > 0$, there exists $N \gg 0$ such that

$$[\nabla, \mathrm{tr}^Q_\varepsilon](\alpha) := \left(\nabla\, \mathrm{tr}^Q_\varepsilon - \mathrm{tr}^Q_\varepsilon\, \nabla\right)(\alpha) = -\sum_{j=1}^{N} \frac{\varepsilon^j}{j!}\mathrm{tr}^Q_\varepsilon\left([[\nabla, Q]]^{j-1}_Q\, \alpha\right) + \mathrm{O}(\varepsilon^k).$$

Proof: Locally, we have $\nabla = d+\theta$ where θ is a local $\mathcal{C}l(M,E)$-valued one-form on B. We can apply (4.6), (4.8) to obtain

$$\begin{aligned}[\nabla, e^{-\varepsilon Q}] &= d\, e^{-\varepsilon Q} + [\theta, e^{-\varepsilon Q}] \\ &= -\sum_{j=1}^{N} \frac{\varepsilon^j}{j!}[d\,Q]_Q^{j-1} e^{-\varepsilon Q} + e^{-\varepsilon Q}\sum_{j=1}^{N}\frac{\varepsilon^j}{j!}\,[[Q,\theta]]_Q^{j-1} + \tilde{R}_{[\nabla,Q],N+1}(\varepsilon) \\ &= -e^{-\varepsilon Q}\sum_{j=1}^{N}\frac{(-\varepsilon)^j}{j!}\,[[\nabla,Q]]_Q^{j-1} + \tilde{R}_{[\nabla,Q],N+1}(\varepsilon),\end{aligned}$$

where $\tilde{R}_{[\nabla,Q],N+1}(\varepsilon) := \tilde{R}_{dQ,N+1}(\varepsilon) + \tilde{R}_{[\theta,Q],N+1}(\varepsilon)$. For α a $\mathcal{C}l(M,E)$-valued form on B, we get by (4.7)

$$\begin{aligned}[\nabla, \mathrm{tr}_\varepsilon^Q](\alpha) &= d\,\mathrm{tr}(e^{-\varepsilon Q}\alpha) - \mathrm{tr}\left(e^{-\varepsilon Q}[\nabla,\alpha]\right) \\ &= \mathrm{tr}\left((de^{-\varepsilon Q})\alpha\right) - \mathrm{tr}(e^{-\varepsilon Q}[\theta,\alpha]) \\ &= -\sum_{j=1}^{N}\frac{\varepsilon^j}{j!}\mathrm{tr}_\varepsilon^Q\left([[\nabla,Q]]_Q^{j-1}\,\alpha\right) + \mathrm{O}(\varepsilon^k)\end{aligned}$$

provided N is chosen so large that $\mathrm{tr}_\varepsilon^Q(\alpha\tilde{R}_{[\nabla,Q],N+1}(\varepsilon)) = \mathrm{O}(\varepsilon)$. □

We now extend a familiar construction for ordinary algebras [8, 16] to bundles of algebras by considering $\mathcal{C}l(\mathcal{E})[[\varepsilon]]$, the space of formal power series in the variable ε with coefficients in $\mathcal{C}l = \mathcal{C}l(\mathcal{E})$. Thus an element $A(\varepsilon)$ of $\mathcal{C}l[[\varepsilon]]$ has the form $A(\varepsilon) = \sum_{j=0}^{\infty} A_j\varepsilon^j,\quad A_j \in \Gamma(\mathcal{C}l)$. The reader unhappy with these formal sums can just work with finite sums with error estimates, as in Theorems 4.4, 4.6 below.

Recall that the algebra $\mathcal{C}l(M,E)$ of classical (polyhomogeneous) ΨDOs is given by finite sums $\sum_{i=1}^{n} A_i$, where each A_i is polyhomogeneous in the sense that the symbol of A_i has an asymptotic expansion $\sigma(A_i) \sim \sum_{j=0}^{\infty}\sigma_{o_i-j}$, $o_i = \mathrm{ord}(A_i)$ with σ_{o_i-j} having the standard homogeneity and growth conditions for the symbol class S^{o_i-j} [7]. The order of A is then the maximum of the o_i. It is standard that each A_i, has an asymptotic expansion as $\varepsilon\to 0$ of the form

$$\mathrm{tr}_\varepsilon^Q(A_i) \sim \sum_{j=0}^{\infty} a_j(A_i,Q)\varepsilon^{\frac{j-o_i-n}{q}} + \sum_{k=0}^{\infty} b_k(A_i,Q)(\log\varepsilon)\varepsilon^k + \sum_{\ell=0}^{\infty} c_\ell(A_i,Q)\varepsilon^\ell, \tag{4.9}$$

where $a_j(A,Q), b_k(A,Q), c_\ell(A,Q)\in\mathbb{C}$ and $n = \dim(M)$. Thus A has a similar asymptotic expansion. We set

$$\begin{aligned}\left(\mathrm{tr}_\varepsilon^Q(A_i)\right)_{\mathrm{asy}} &:= \sum_{j=0}^{\infty} a_j(A_i,Q)\varepsilon^{\frac{j-o_i-n}{q}} + \sum_{k=0}^{\infty} b_k(A_i,Q)(\log\varepsilon)\varepsilon^k \\ &\qquad + \sum_{\ell=0}^{\infty} c_\ell(A_i,Q)\varepsilon^\ell \in \mathbb{C}[\log\varepsilon][\varepsilon^{-\frac{1}{q}},\varepsilon^{\frac{1}{q}}]],\end{aligned}$$

and define $\left(\mathrm{tr}_\varepsilon^Q(A)\right)_{\mathrm{asy}}$ by linearity. Given a weight $Q \in \Gamma(\mathcal{C}l(\mathcal{E}))$ as above, the $\mathbb{C}$-linear morphism $\mathrm{tr}_\varepsilon^Q$ defined for fixed ε partially extends to a $\mathbb{C}[[\varepsilon]]$-morphism

$$\mathrm{tr}_\varepsilon^Q : \Gamma\left(C\ell(\mathcal{E})[[\varepsilon]]\right) \to \mathbb{C}[\log\varepsilon][\varepsilon^{-\frac{1}{q}}, \varepsilon^{\frac{1}{q}}]], \quad A = \sum_{k=0}^{\infty} A_k \varepsilon^k \mapsto \sum_{k=0}^{\infty} \left(\mathrm{tr}_\varepsilon^Q(A_k)\right)_{\mathrm{asy}} \varepsilon^k. \tag{4.10}$$

In the last term in (4.10), we formally rearrange the sum to produce an element of $\mathbb{C}[\log\varepsilon][\varepsilon^{-\frac{1}{q}}, \varepsilon^{\frac{1}{q}}]]$, provided the number of terms contributing to each ε^ℓ and $(\log\varepsilon)\varepsilon^\ell$ is finite.

It is not hard to give conditions that guarantee that $\mathrm{tr}_\varepsilon^Q(\sum_{k=0}^\infty A_k\varepsilon^k)$ exists in this formally rearranged sense:

Lemma 4.2: *If the* $a_i := \mathrm{ord}(A_i)$ *satisfy* $\lim_{i\to\infty} qi - a_i = \infty$, *then* $\mathrm{tr}_\varepsilon^Q(\sum_{i=0}^\infty A_i\varepsilon^i)$ *exists as a rearranged sum.*

Proof: We may assume that each A_k is classical polyhomogeneous, as replacing A_k by a finite sum of such operators does not affect the proof. For fixed i, j, the term $a_j(A_i, Q)$ in (4.9) appears in $\mathrm{tr}_\varepsilon^Q(\sum_{i=0}^\infty A_i\varepsilon^i)$ as a coefficient of $\varepsilon^{\frac{j-a_i-n+qi}{q}}$. The hypothesis guarantees that only a finite number of i, j can contribute to the coefficient of a fixed e^{k_0}. Similar arguments apply to $b_k(A_i, Q), c_\ell(A_i, Q)$. □

Motivated by Lemma 4.1, we introduce a $C\ell(\mathcal{E})[[\varepsilon]]$-valued connection ∇_ε^Q defined in terms of the connection $\nabla^{\mathcal{C}l(\mathcal{E})} = [\nabla, \cdot] = \nabla^{\mathrm{Ad}\ P^{\mathcal{E}}}$ on $\mathcal{C}l(\mathcal{E})$ and the weight Q:

$$\nabla_\varepsilon^Q \alpha := \nabla^{\mathcal{C}l(\mathcal{E})}\alpha - \sum_{j=1}^{\infty} \frac{\varepsilon^j}{j!} \left[[\nabla, Q]\right]_Q^{j-1} \alpha, \ \alpha \in \Omega^*(B, \mathcal{C}l(\mathcal{E})). \tag{4.11}$$

We now show that ∇_ε^Q has the key property of commuting with the weighted trace $\mathrm{tr}_\varepsilon^Q$:

Lemma 4.3: *Let* ∇ *be a* $\mathcal{C}l_{\leq 0}$*-connection on* $\mathcal{E}$, *and let* Q *be a weight on* $\mathcal{E}$ *with scalar leading symbol. For* $\alpha \in \Omega^*(B, C\ell(\mathcal{E}))$, *we have*

$$d \ \circ \mathrm{tr}_\varepsilon^Q \alpha = \mathrm{tr}_\varepsilon^Q \circ \nabla_\varepsilon^Q \alpha.$$

Proof: By Lemma 4.1, we have

$$d \circ \mathrm{tr}_\varepsilon^Q \alpha - \mathrm{tr}_\varepsilon^Q \circ \nabla_\varepsilon^Q \alpha = [\nabla_\varepsilon^Q, \mathrm{tr}_\varepsilon^Q](\alpha) = [\nabla, \mathrm{tr}_\varepsilon^Q](\alpha) - \sum_{j=1}^{\infty} \frac{(-\varepsilon)^j}{j!} \mathrm{tr}_\varepsilon^Q \left(\left[[\nabla, Q]\right]_Q^{j-1} \alpha\right) = 0,$$

provided we show that $\mathrm{tr}_\varepsilon^Q$ can be applied to $\nabla_\varepsilon^Q\alpha$. In fact, if $d := \mathrm{ord}[\nabla, Q] \leq q$, then the order of $[[\nabla, Q]]_Q^{j-1}$ is $a_j \leq d + (j-1)(q-1)$, since Q has scalar leading symbol. Thus the hypothesis of Lemma 4.2 is satisfied. □

Remarks: (i) The preceeding proof assumes that Lemma 4.1 extends to formal power series of operators. This justification, while not difficult, is somewhat lengthy and is omitted.

(ii) If ∇_ε^Q were induced from a connection on $\mathcal{E}$, Lemma 4.3 would guarantee a Chern-Weil theory for the curvature Ω_ε^Q: each coefficient in $\mathrm{tr}_\varepsilon^Q(\Omega_\varepsilon^Q)$ would be a closed from independent of the connection. However, we will see in Corollary 4.7 that for loop groups, the leading order coefficient is the Kähler form $\mathrm{Tr}^Q(\Omega)$ for the $H^{1/2}$ Levi-Civita connection. The corresponding non-zero Kähler class is certainly not independent of connecction, since TLG is trivial. Theorem 4.6 gives a more refined analysis of this example.

Despite the last remark, we can use Lemma 4.3 to produce a Chern-Weil theory under additional hypotheses.

Theorem 4.4: *Let ∇ be a $\mathcal{C}l_{\leq 0}$-connection and Q a weight on $\mathcal{E}$ of order q and with scalar leading symbol. Let d be the order of the $\mathcal{C}l(\mathcal{E})$-valued form $[\nabla, Q]$, and set $r := q - d$.*

(i) For $\alpha \in \Omega^*(B, \mathcal{C}l(\mathcal{E}))$ of constant order a, we have

$$d\mathrm{tr}_\varepsilon^Q(\alpha) = \mathrm{tr}_\varepsilon^Q(\nabla^{\mathcal{C}l(\mathcal{E})}\alpha) + \mathrm{o}(\varepsilon^{\frac{-a-n+r-\eta}{q}}),$$

for all $\eta > 0$.

(ii) Let Ω, the curvature of ∇, have constant order a. Then the coefficient of $\varepsilon^{\frac{\gamma}{q}}$ in the asymptotic expansion of $\mathrm{tr}_\varepsilon^Q(\Omega^k)$ is closed, for all $\gamma < -ka - n + r$. In particular, if $r > 0$, the coefficient of the leading order term $\varepsilon^{\frac{-ka-n}{q}}$ is closed. The coefficients of $\log\varepsilon \cdot \varepsilon^\ell$ are closed for all $\ell < \frac{-ka-n+r}{q}$.

(iii) Let Ω have constant order a. The coefficient of $\log\varepsilon$ in the asymptotic expansion of $\mathrm{tr}_\varepsilon^Q(\Omega^k)$ is closed.

Note that part (ii) of the theorem only applies if $r > 0$, which occurs e.g. if the leading order symbol of Q is independent of $b \in B$. We need $r > ka + n$ to obtain information about the $\log\varepsilon \cdot \varepsilon^\ell$ terms with $\ell > 0$.

PROOF: (i) By Lemma 4.3, we have

$$d\mathrm{tr}_\varepsilon^Q(\alpha) = \mathrm{tr}_\varepsilon^Q(\nabla_\varepsilon^Q\alpha) = \mathrm{tr}_\varepsilon^Q(\nabla^{\mathcal{C}l(\mathcal{E})}\alpha) - \sum_{j=1}^\infty \frac{\varepsilon^j}{j!}\mathrm{tr}_\varepsilon^Q\left([[\nabla, Q]]_Q^{j-1}\alpha\right).$$

We want to show that for $\eta > 0$,

$$\lim_{\varepsilon\to 0}\varepsilon^{\frac{n+a-r+\eta}{q}}\left(\sum_{j=1}^\infty \frac{\varepsilon^j}{j!}\mathrm{tr}_\varepsilon^Q\left([[\nabla, Q]]_Q^{j-1}\alpha\right)\right) = 0.$$

Since the infinite sum is a rearrangable formal power series, we mean that each exponent k_0 of the rearranged series satisfies $\frac{n+a-r+\eta}{q} + k_0 > 0$. Since the leading asymptotic term $a_j\varepsilon^{\gamma_j}$ of $\mathrm{tr}_\varepsilon^Q\left([[\nabla, Q]]_Q^{j-1}\alpha\right)$ contributes the exponent $j + \gamma_j$, it suffices to show that

$$\frac{n+a-r+\eta}{q} + j + \gamma_j > 0, \tag{4.12}$$

for all $j \in \mathbb{N}$. (A similar argument treats the case where the leading asymptotic term contains $\log \varepsilon$.) As in Lemma 4.3, $d_j := \mathrm{ord}([[\nabla, Q]]_Q^{j-1})$ satisfies $d_j \leq d + (j-1)(q-1)$ for $d := \mathrm{ord}[\nabla, Q]$. Thus

$$\gamma_j \geq \frac{-d_j - a - n}{q} \geq \frac{r + j - 1 - a - n - jq}{q},$$

which implies (4.12).

(ii) Since $\nabla^{\mathcal{C}l(\mathcal{E})} \Omega^k = 0$ and $\mathrm{ord}(\Omega^k) = ka$, it follows from (i) that

$$d\mathrm{tr}_\varepsilon^Q\left(\Omega^k\right) = -\sum_{j=1}^{\infty} \frac{\varepsilon^j}{j!} \mathrm{tr}_\varepsilon^Q\left([[\nabla, Q]]_Q^{j-1} \Omega^k\right) = \mathrm{o}\left(\varepsilon^{\frac{-ka-n+r-\eta}{q}}\right), \tag{4.13}$$

for all $\eta > 0$. Thus all coefficients of powers ε^γ with $\gamma < -ka - n + r$ are closed. A similar argument handles the $\log \varepsilon$ terms.

Finally, since the coefficient of the $\log \varepsilon$ term is proportional to the Wodzicki residue $\mathrm{res}_{\mathrm{w}}(\Omega)$, (iii) follows from the discussion in §3.1. □

This clarifies the non-closed weighted traces of [18].

Corollary 4.5: *Let ∇ be a $\mathcal{C}l_{\leq 0}$ connection, Q a weight on $\mathcal{E}$ with scalar leading symbol, $\Omega = \nabla^2$ the curvature of ∇, and $\mathrm{tr}^Q(\Omega^k)$ its Q-weighted trace, i.e. $\mathrm{tr}^Q(\Omega^k)$ is the finite part of $\mathrm{tr}_\varepsilon^Q(\Omega^k)$ as $\varepsilon \to 0$. Then $d\mathrm{tr}^Q(\Omega^k)$ is an explicit finite linear combination of coefficients in the asymptotic expansion of $\mathrm{tr}_\varepsilon^Q\left([[\nabla, Q]]_Q^{j-1} \Omega^k\right)$, $j \in \mathbb{N}$.*

Proof: This follows from (4.13) and the fact that $\sum_{j=1}^\infty \frac{\varepsilon^j}{j!} \mathrm{tr}_\varepsilon^Q\left([[\nabla, Q]]_Q^{j-1} \Omega^k\right)$ is rearrangable. In particular, the coefficient of ε^0 is constructed as stated. □

We now discuss the independence of the closed forms in Theorem 4.4 on the choice of connection. Note that the hypotheses are more stringent than in Theorem 4.4.

Theorem 4.6: *(i) Let Q be a weight on $\mathcal{E}$ with scalar leading symbol, and let $\{\nabla_t : t \in [0,1]\}$ be a smooth family of $\mathcal{C}l_{\leq 0}(\mathcal{E})$ connections such that $\dot{\nabla}_t \in \Omega^1(B, \mathcal{C}l_{\leq -s}(\mathcal{E}))$, for some $s \geq 0$, for all t. Then*

$$\frac{d}{dt} \mathrm{tr}_\varepsilon^Q(\Omega_t^k) = d\mathrm{tr}_\varepsilon^Q\left(\sum_{j=1}^{k} \Omega_t^{k-j} \dot{\nabla}_t \Omega_t^{j-1}\right) + \mathrm{o}\left(\varepsilon^{\frac{-n+s-(k-1)a-\eta}{q}}\right),$$

for all $k \in \mathbb{N}$ and for all $\eta > 0$.

(ii) Let $a = \mathrm{ord}(\Omega_t)$ be independent of t. Let $q = \mathrm{ord}(Q)$, and set $r := q - d$, where we assume that $d := \mathrm{ord}([\nabla_t, Q])$ is independent of t. If $s > r - a$, then the cohomology class of the coefficient of $\varepsilon^{\frac{\gamma}{q}}$ in the asymptotic expansion of $\mathrm{tr}_\varepsilon^Q(\Omega_t^k)$ is independent of t for all $\gamma < -ka - n + r$. The cohomology class of the coefficient of $\log \varepsilon \cdot \varepsilon^\ell$ is independent of t for all $0 < \ell < \frac{-ka-n+r}{q}$.

(iii) Let $a = \mathrm{ord}(\Omega_t)$ be independent of t. The cohomology class of the coefficient of $\log \varepsilon$ in the asymptotic expansion of $\mathrm{tr}_\varepsilon^Q(\Omega_t^k)$ is independent of t.

As with Theorem 4.4, this theorem is only meaningful if $s > 0$.

PROOF: Mimicking the finite dimensional proof, we have

$$\begin{aligned}
\frac{d}{dt}\mathrm{tr}_\varepsilon^Q(\Omega_t^k) &= \mathrm{tr}_\varepsilon^Q\left(\sum_{j=1}^k \Omega_t^{k-j}[\nabla_t, \dot\nabla_t]\Omega_t^{j-1}\right) = \mathrm{tr}_\varepsilon^Q\left(\sum_{j=1}^k \Omega_t^{k-j}(\nabla_t^{\mathcal{C}l(\mathcal{E})}\dot\nabla_t)\Omega_t^{j-1}\right)\\
&= \mathrm{tr}_\varepsilon^Q\left(\nabla_t^{\mathcal{C}l(\mathcal{E})}\sum_{j=1}^k \Omega_t^{k-j}\dot\nabla_t\Omega_t^{j-1}\right)\\
&= \mathrm{tr}_\varepsilon^Q\left(\nabla_{\varepsilon,t}^Q\sum_{j=1}^k \Omega_t^{k-j}\dot\nabla_t\Omega_t^{j-1}\right) + \sum_{j=1}^\infty \frac{\varepsilon^j}{j!}\mathrm{tr}_\varepsilon^Q\left([[\nabla_t, Q]]_Q^{j-1}\sum_{j=1}^k \Omega_t^{k-j}\dot\nabla\Omega_t^{j-1}\right)\\
&= d\mathrm{tr}_\varepsilon^Q\left(\sum_{j=1}^k \Omega_t^{k-j}\dot\nabla_t\Omega_t^{j-1}\right) + \sum_{j=1}^\infty \frac{\varepsilon^j}{j!}\mathrm{tr}_\varepsilon^Q\left([[\nabla_t, Q]]_Q^{j-1}\sum_{j=1}^k \Omega_t^{k-j}\dot\nabla\Omega_t^{j-1}\right).
\end{aligned}$$

The leading term in the asymptotics of $\frac{\varepsilon^j}{j!}\mathrm{tr}_\varepsilon^Q\left([[\nabla_t, Q]]_Q^{j-1}\sum_{j=1}^k \Omega_t^{k-j}\dot\nabla\Omega_t^{j-1}\right)$ is of the form $a_j\varepsilon^{\gamma_j}$, with

$$\gamma_j > \frac{-n + jq - d - (j-1)(q-1) + s - (k-1)a}{q} \geq \frac{-n + s - (k-1)a}{q}.$$

This proves (i). For (ii), we note that this last fraction will be greater than $(-n - ka + r)/q$ provided $s > r - a$. As in the previous theorem, the proof of (iii) follows from properties of the Wodzicki-Chern class of §3.1. □

With the previous two theorems, we have developed a theory of characteristic forms that explains why Freed's conditional first Chern form is closed and why its cohomology class cannot be connection independent.

Corollary 4.7: *Let $\Omega = \Omega^{(\frac{1}{2})}$ be the curvature of the Levi-Civita connection for the $H^{\frac{1}{2}}$ metric on the loop group LG. Then Freed's conditional first Chern form coincides with the weighted first Chern form $\mathrm{tr}^Q(\Omega)$ for any left invariant scalar weight Q on LG, and hence is closed.*

PROOF: The conditional trace of the Levi-Civita curvature in [6] is $\mathrm{tr}\,(\mathrm{tr}_{\mathrm{Lie}}(\Omega))$, where $\mathrm{tr}_{\mathrm{Lie}}$ denotes the trace with respect to the Killing form in the Lie algebra of G, and the outer trace is the ordinary operator trace. In particular, $\mathrm{tr}_{\mathrm{Lie}}(\Omega)$ is a trace class ΨDO on the trivial $\mathbb{C}$ bundle over S^1. As in [5], for any left invariant scalar weight Q we have

$$\mathrm{tr}\,(\mathrm{tr}_{\mathrm{Lie}}(\Omega)) = \lim_{\varepsilon\to 0}\mathrm{tr}\left(\mathrm{tr}_{\mathrm{Lie}}(\Omega)e^{-\varepsilon Q}\right) = \lim_{\varepsilon\to 0}\mathrm{tr}_\varepsilon^Q(\Omega) = \mathrm{tr}^Q(\Omega),$$

so Freed's conditional first Chern form is the weighted trace $\mathrm{tr}^Q(\Omega)$. Recall that the curvature two-form is a ΨDO of order $a = -1$. Since Q is left invariant and scalar, $[\nabla, Q] = dQ + [\theta, Q] = [\theta, Q]$ has order $r = q - 1 = 1$. Theorem 4.4 with $n = 1, q = 2, r = 1, a = -1$

shows that the constant term $a_0(\Omega, Q)$ in the asymptotic expansion of $\mathrm{tr}^Q_\varepsilon(\Omega)$ is closed. Since this constant term equals $\lim_{\varepsilon\to 0}\mathrm{tr}^Q_\varepsilon(\Omega) = \mathrm{tr}^Q(\Omega)$, it follows that $\mathrm{tr}^Q(\Omega)$ is closed for loop groups. □

Remark: Theorem 4.6 does not apply to Freed's conditional first Chern form. In particular, if we shrink the connection one-form θ to zero using the family $t\theta$, we cannot apply Theorem 4.6, since for this family we have $s = 0$.

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