Intersection Theory on Arithmetic Surfaces

Main Reference: "Advanced Topics in the Arithmetic of Elliptic Curves" by J. Silverman

1 Arithmetic Surfaces

Arithmetic surfaces are special types of fibred surfaces. Let R be a Dedekind domain with fraction field K, and set $S = \mathrm{Spec}(R)$.

Definition. An **arithmetic surface** over R is a *nice* R-scheme $\mathcal{C}$ whose generic fiber is a non-singular connected projective curve C/K and who special fibers are unions of curves over the appropriate residue fields. (The word **nice** is an abbreviation for integral, normal excellent scheme which is flat and of finite type over R.)

Example.

$$\mathcal{C} : y^2 = x^3 + 2x^2 + 6, \quad \text{over } \mathrm{Spec}(\mathbb{Z}).$$

Note that $\Delta = -2^6 \cdot 3 \cdot 97$, so for all primes $p \neq 2, 3, 97$ the fiber $\mathcal{C}_p$ is a non-singular elliptic curve over $\mathbb{F}_p$.

Example. Let R denote the localization of $\mathbb{Z}$ at a prime ideal (p). Thus

$$R = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, p \not| b \right\}, \quad K = \mathbb{Q}.$$

Let $\mathcal{C} = \mathrm{Spec}(R[x,y]/(xy - p))$. Since R is a DVR, the underlying space of $\mathrm{Spec}\, R$ has two points corresponding to the ideals (0) and (p). Thus we get a generic fiber $\mathrm{Spec}(\mathbb{Q}[x,y]/(xy-p))$ and a special fiber $\mathrm{Spec}(\mathbb{F}_p[x,y]/(xy))$

2 Weil Divisors

Now we specialize to R a DVR with maximal ideal $\mathfrak{p}$ and residue field $k = R/\mathfrak{p}$, and let $\mathcal{C}$ be an arithmetic surface over R.

Definition. A **prime divisor** (or **irreducible divisor**) on $\mathcal{C}$ is a closed integral subscheme of codimension 1. Let $\mathrm{Div}(\mathcal{C})$ denote the free abelian group generated by the prime divisors. Then $\mathrm{Div}(\mathcal{C})$ is the (Weil) divisor group of $\mathcal{C}$, and elements of $\mathrm{Div}(\mathcal{C})$ are called (Weil) divisors. A divisor is **effective** if it can be expressed in the form

$$D = \sum_i n_i \Gamma_i, \quad n_i \geq 0, \quad \textbf{prime divisors } \Gamma_i.$$

Furthermore, each non-zero function $f \in K(\mathcal{C})$ defines a **principal divisor**

$$(f) = \sum_{\text{prime } \Gamma \in \mathrm{Div}(\mathcal{C})} \mathrm{ord}_\Gamma(f)\Gamma \in \mathrm{Div}(\mathcal{C}).$$

3 Local Intersection Multiplicity

Definition. Let $\Gamma \in \mathrm{Div}(\mathcal{C})$ be an irreducible divisor and let $x \in \mathcal{C}_{\mathfrak{p}}$ be a point on the special fiber of $\mathcal{C}$. A **uniformizer for** Γ **at** x is a function $f \in \mathcal{O}_{\mathcal{C},x}$ in the local ring of $\mathcal{C}$ at x with the property that

$$\mathrm{ord}_{\Gamma}(f) = 1, \quad \mathrm{ord}_{\Gamma'}(f) = 0, \forall \text{ irreducible } \Gamma' \neq \Gamma \text{ with } x \in \Gamma'.$$

Definition. Let $\Gamma_1, \Gamma_2 \in \mathrm{Div}(\mathcal{C})$ be distinct irreducible divisors and let $x \in \mathcal{C}$ be a closed point on the special fiber $\mathcal{C}_{\mathfrak{p}}$ of $\mathcal{C}$. Choose uniformizers $f_1, f_2 \in \mathcal{O}_{\mathcal{C},x}$ for Γ_1, Γ_2 respectively. The **(local) intersection index of** Γ_1 and Γ_2 at x is

$$(\Gamma_1 \cdot \Gamma_2)_x = \dim_k \mathcal{O}_{\mathcal{C},x}/(f_1, f_2).$$

Note that $x \notin \Gamma_1 \cap \Gamma_2$ then $(\Gamma_1 \cdot \Gamma_2)_x = 0$.

We would like this to be invariant under linear equivalence, so we need to restrict the set of allowable divisors.

Definition. An irreducible divisor which is a component of the special fiber is called a **fibral divisor**, and an irreducible divisor which maps onto $\mathrm{Spec}(R)$ is called a **horizontal divisor**.

Example. Let $R = \mathbb{Z}_{(p)}$ and $\mathcal{C} = \mathrm{Spec}(R[x])$. Then

$$\mathrm{Spec}(R[x]/(p)) \cong \mathrm{Spec}(\mathbb{F}_p[x])$$

is a fibral divisor, while

$$\mathrm{Spec}(R[x]/(x)) \cong \mathrm{Spec}(R)$$

is a horizontal divisor.

Definition. A divisor $D = \sum n_i \Gamma_i$ is called **fibral** if every component Γ_i of D is a fibral divisor. The **group of fibral divisors on** $\mathcal{C}$ is denoted $\mathrm{Div}_{\mathfrak{p}}(\mathcal{C})$. Notice that this is a subgroup of $\mathrm{Div}(\mathcal{C})$.

4 Global Intersection Multiplicity

Theorem. *Let R be a DVR, and let $\mathcal{C}/R$ be a regular arithmetic surface which is proper over R. There is a unique bilinear pairing*

$$\mathrm{Div}(\mathcal{C}) \times \mathrm{Div}_{\mathfrak{p}}(\mathcal{C}) \to \mathbb{Z}, \quad (D, F) \mapsto D \cdot F,$$

with the following properties:

(i) If $\Gamma \in \mathrm{Div}(\mathcal{C})$ and $F \in \mathrm{Div}_{\mathfrak{p}}(\mathcal{C})$ are distinct irreducible divisors, then

$$\Gamma \cdot F = \sum_{x \in \Gamma \cap F} (\Gamma \cdot F)_x.$$

(ii) If $D_1, D_2 \in \mathrm{Div}(\mathcal{C})$ and $F \in \mathrm{Div}_{\mathfrak{p}}(\mathcal{C})$ are divisors with D_1 linearly equivalent to D_2, then $D_1 \cdot F = D_2 \cdot F$. In particular,

$$(f) \cdot F = 0, \quad \forall f \in K(\mathcal{C})^*, \forall F \in \mathrm{Div}_{\mathfrak{p}}(\mathcal{C}).$$

(iii) If $F_1, F_2 \in \mathrm{Div}_{\mathfrak{p}}(\mathcal{C})$ are fibral divisors, then $F_1 \cdot F_2 = F_2 \cdot F_1$.

Example. Consider $R = \mathbb{Z}_{(p)}$ and $\mathcal{C} = \mathrm{Spec}(R[x])$. Let

$$\Gamma_1 = \mathrm{Spec}(R[x])/(p)) \cong \mathrm{Spec}(F_p[x]),$$

$$\Gamma_2 = \mathrm{Spec}(\mathrm{Spec}(R[x])/(x)).$$

Note that p is a local uniformizer for Γ_1, while x is a local uniformizer for Γ_2. Then $k = R/(p) \cong \mathbb{F}_p$ and

$$(\Gamma_1 \cdot \Gamma_2)_{(p,x)} = \dim_{\mathbb{F}_p} \mathcal{O}_{\mathcal{C},(p,x)}/(p,x).$$

$$\mathcal{O}_{\mathcal{C},(p,x)}/(p,x) = R[x]_{(p,x)}/(p,x) \cong \mathbb{Z}[x]_{(p,x)}/(p,x).$$

The equality $\mathbb{Z}[x]_{(p,x)}/(p,x) \cong \mathbb{F}_p$ follows from the fact that quotients and localizations commute.
Since $\Gamma_1 \cap \Gamma_2 = \{(p,x)\}$, it follows that

$$\Gamma_1 \cdot \Gamma_2 = (\Gamma_1 \cdot \Gamma_2)_{(p,x)} = \dim_{\mathbb{F}_p} \mathbb{F}_p = 1.$$

But how could one compute self-intersections?

Example. Let p be an odd prime and let $R = \mathbb{Z}_{(p)}$. Consider

$$\mathcal{C} = \mathrm{Proj}(R[x.y,z]/(xy - pz^2)).$$

The special fiber is the reducible curve

$$\mathcal{C}_p = \mathrm{Proj}(\mathbb{F}_p[x.y,z]/(xy))$$

which is the union of the x-axis (call it Γ_1) and the y-axis (call it Γ_2). One can compute that $(\Gamma_1 \cdot \Gamma_2) = 1$. On the other hand, the sum $\Gamma_1 + \Gamma_2$ is the principal divisor defined by the function p, since $\Gamma_1 + \Gamma_2$ is the whole special fiber. It follows that

$$0 = (\Gamma_1 \cdot (\Gamma_1 + \Gamma_2)) = (\Gamma_1 \cdot \Gamma_1) + (\Gamma_1 \cdot \Gamma_2),$$

so $(\Gamma_1 \cdot \Gamma_1) = -1$.