Aash

Part 1: Overview of GZ

Thm 1 $g_A(z) = \sum_{m \geq 1} \langle c, T_m c^\sigma \rangle e^{2\pi i m z}$

is a cusp form of wt $2u$ which satisfies

$$(f, g_A) = \frac{u^2 |D|^{1/2} L'_A(f,1)}{8\pi^2}$$ for all f in the

space of wt 2 newforms in $\Gamma_0(N)$.

where:

$$\begin{array}{ccc} Cl_K & \text{Artin} & Gal(H/K) \\ A & \longleftrightarrow & G \end{array}$$

$c = (x) - (\infty)$ and x = Heegner Point

$\langle \cdot, \cdot \rangle$ height pairing

J is Jacobian of $X_0(N)$

$K = \mathbb{Q}(\sqrt{D})$ class number h

H/K Hilbert class field

$2u$ roots of unity in K.

$D = D_K$ discriminant of Heegner pt

Goal: $\langle , \rangle$ breaks up into local height pairings

talk about arch

Thm 2 $$L'(f, \chi, 1) = \frac{8\pi^2 (f,f) \hat{h}(c_{\chi,f})}{h u |D|^{1/2}}$$ canonical height

. .

where c_χ is $\sum_{\sigma \in Gal(H/K)} \chi^{-1}(\sigma)\, c^\sigma$

χ a character of $Gal(H/K)$

$G_{\chi,f}$ is the projection to the f-isotypical elt

ie the f-Eigenspace of $J(H) \otimes \mathbb{C}$ under the action of $\mathbb{T}$ (f a newform so it's an eigenvector for $\mathbb{T}$)

Thm 2 $\Rightarrow$ if $L' \neq 0$ then E contains an elt of int order (combined w/ waldspruger's formula) assoc to f

Part 2: Height Pairings

Let v be a place of $H/K \leftarrow$ quad im, only one arch

$$|\,|_v : H_v^\times \longrightarrow \mathbb{R}_+^\times$$

$|\alpha|_v = \alpha\bar{\alpha}$ if $H_v \simeq \mathbb{C}$ (arch case)

$= q^{-v(\alpha)}$ for non-archimedean v

Then Néron's theory gives a unique symbol on relatively prime divisors (divisors whose supports are disjoint)

This pairing when defined splits up as

$$\langle a,b\rangle = \sum_v \langle a,b\rangle_v .$$

In GZ formula want height pairing

$$\langle c, T_m c^\sigma \rangle$$

$$c = (x) - (\infty)$$ $\boxed{\text{Problem!}}$

T_m sends cusp $\to$ cusp, won't have disjoint support.

Instead, take $d = (x) - (0)$

$(0) - (\infty)$ is finite order in $J(\mathbb{Q})$ so height is 0,

↳ cusps "Manin-Drinfeld theorem"

Then $\langle c, T_m c^\sigma \rangle = \langle c, T_m d^\sigma \rangle$

Rmk $r_{C_A}(m) = 0$, and $N > 1$

\# integral ideals in class of A w/ norm m

level of the modular curve.

then c, $T_m d^\sigma$ are realatively prime.

Generalities on Local Height Pairings

S compact Rieman surface There exists a partially defined

$$\langle -, - \rangle : \mathrm{Div}^0(S) \times \mathrm{Div}^0(S) \to \mathbb{R}$$

which satisfies

(1) $\langle a,b \rangle$ defined when a,b have disjoint support

(2) $\langle -,- \rangle$ bi-additive when defined

(3) f meromorphic on S, $a = \sum n_i x_i$

$$\langle \operatorname{div} f, a \rangle = \sum n_i \log |f(x_i)|^2$$

(4) $\langle a, \sum m_j y_j \rangle$ is continuous on $S \setminus |a|$ wrt each y_j, $|a|$ = support of a.

Uniqueness of height: we will show it descends to the Jacobian (differences descend)

consider the differences of 2 symbols

$$\langle -,- \rangle - \langle -,- \rangle = [\langle -,- \rangle]$$

Fix $a \in \operatorname{Div}^\circ(S)$

The map $b \mapsto \langle b, a \rangle$ descends.

Can always choose reps w/ disjoint support.

ie $[\langle b + (f), a \rangle] = [\langle b, a \rangle] + [\langle f, a \rangle]$

$$= [\langle b, a \rangle] + \underbrace{\langle f, a \rangle_1 - \langle f, a \rangle_2}_{\text{these will cancel} = 0}$$

So $J \to \mathbb{R}$ is
$b \mapsto [\langle b, a \rangle]$ is a continous homomorphism

therefore the image is 0, as 0 is the only compact subgroup of $\mathbb{R}$.

Fix $x_0, y_0 \in S$, Green's Function (to prove existence of height pairing)

$G(x, y) = \langle (x) - (x_0), (y) - (y_0) \rangle$

$x \neq y, \quad y \neq x_0, \quad x \neq y_0$

Biadditivity $\Rightarrow \langle a, b \rangle \overset{(*)}{=} \sum_{i,j} n_i m_j G(x_i, y_j)$

$a = \sum_i n_i (x_i) \qquad b = \sum_j m_j (b_j)$

$y_0 \notin |a|, \quad x_0 \notin |b|$

Conversely, $G(x,y)$ will define a symbol $\langle -, - \rangle$ if for fixed $x \neq y_0$, $y \mapsto G(x,y)$ on $S \setminus \{x, x_0\}$

(1) is continuous

Re log $\to$ (2) is harmonic, ie $\nabla_y^2 G(x,y) = 0$.
on complex plane is harmonic

(3) logarithmic singularities of residue $+1, -1$ at $y = x, y = x_0$

[log sing. at z_0 if $f(z) - \alpha \log|\rho(z)|^2$ is cts near z_0, ρ holo near z_0 + vanishing to order 1 near z_0, α is called the residue at z_0, ρ is uniformizing parameter]

Same symmetric conditions on x.

(*) defines well-defined, cts, bi-additive if

$$\left[|a| \cup \{x_0\} \right] \cap \left[|b| \cup \{y_0\} \right] = \emptyset.$$

Want to extend to

$$|a| \cap |b| = \emptyset$$

Sufficient to show

$G(x_1, y) - G(x_2, y)$ makes sense

as $y \to y_0$. $x_1, x_2 \notin |b| \cup \{y_0\}$

$$G(x_i, y) = -\log|\rho(y)|^2 + c_i + O(\rho(y))$$

Therefore $G(x_1, y) - G(x_2, y) \to c_1 - c_2$

as $y \to x_0$

Therefore well-defined and continuous by hypothesis (3) on $G(x,y)$

Thm

$\langle -, - \rangle$ defined + cts + bi-additive

Proof Consider $(f) = \sum_{j=1}^{k} m_j (y_j)$ principal divisor, $x_0 \notin |f|$ and Define

$$\delta : x \mapsto \langle (x) - (x_0), f \rangle$$
$$- \left(\log |f(x)|^2 - \log |f(x_0)|^2 \right)$$
$$= \sum_j m_j G(x, y_j) - \left[\log|f(x)|^2 - \log|f(x_0)|^2 \right]$$

is harmonic for $x \in S \setminus \{y_0, \dots, y_k\}$ + cts everywhere therefore the difference is constant.

$$\langle \sum n_i (x_i), (f) \rangle - \sum n_i \log |f(x_i)|^2$$
$$= \sum' n_i \delta(x_i) = \sum' n_i C = 0.$$

So G w/ given hypothesis gives us $\langle -, - \rangle$.

Where do we find G?

Now we set $S = X_0(N)(\mathbb{C})$.

set $x_0 = \infty$, $y_0 = 0$

Conditions on G needed:

G1: G real valued cts harmonic fn on

$E = \{(z,z') \in \mathcal{H}^2 \mid z \notin \Gamma_0(N) z'\}$

such that $G(\gamma z, \gamma' z') = G(z,z')$

$\forall (z,z') \in E$, $\gamma, \gamma' \in \Gamma_0(N)$

G2: Fix $z \in \mathcal{H}$

$$G(z,z') = e_z \log|z-z'|^2 + O(1)$$

as $z' \to z$ where e_z is the order of the stabilizer in $\Gamma_0(N)$

G3: For fixed $z \in \mathcal{H}$, $G(z,z') = 4\pi y' + O(1)$

as $z' = x' + iy' \to \infty$ and $G(z,z') = O(1)$

at other cusps.

G4: $z' \in \mathcal{H}$ fixed, $G(z,z') = \frac{4\pi y}{N|z|^2} + O(1)$

a 0 $z = x + i+y \to 0$

and $G(z,z') = O(1)$ as $z \to$ any other cusp.

G_2, G_3, G_4 come from uniformizing parameters for log singularities: $\infty \sim e^{2\pi i z}$, $0 \sim e^{-\frac{2\pi i}{Nz}}$, $|z'-z|$ other non cusp

The Green's function we eventually get is

$$G(z,z') = \lim_{s\to 1}\Big[G_{N,s}(z,z') + 4\pi E_N(\omega_N z, s)$$

$$+ 4\pi E_N(z',s) + \frac{K_N}{s-1}\Big] + C$$

idea: sum something PSL_2-invariant over orbits, there are good candidates

Problem: diverges!

have to do this instead, sum this other fn / orbits, not harmonic, introduce this s-parmeter

Notation: E_N eisenstein series

$$K_N = \frac{-12}{[SL_2(\mathbb{Z}) : \Gamma_0(N)]}$$