Alex: Serre-Tate Local Moduli

References:
- Woods Hole conference proceedings (not easy to read, but first appeared here)
- Katz, LNM 868 §1.2 based on Donfeld (appendix)
- Hida, "Geometric Modular forms & Elliptic Curves 2nd Ed." "modern" v. of Katz
- Michigan Seminar

Motivation: Recall, trying to argue
\[
\left( x, \gamma_m(x) \right) = \sum_{n=1}^{\infty} \# \text{Hom}_{\text{CM}} \left( \mathbb{F}_q, \mathbb{F}_q \left[ \mathbb{C}_m \left( x \right) \right] \right)
\]

Last time: Lifted curves w/ an endomorphism CM curves.

This time: Discuss the "space" of all $W$ of same $E_0/F_q$.

1) We will work more generally w/ abelian schemes $E$ e.g. Elliptic curve, products, well restrictions, Jacobians of higher genus curves, $R$ a ring, $\text{AbSch}(R)$ is a category of abelian schemes over $R$.

Reduction: $A/\mathbb{Z}_p$ which reduce to $A_0$

\[
\downarrow
\]

$A_0/\mathbb{F}_p$

Need to remember extra data about $A_0$ to distinguish $A/\mathbb{Z}_p$

Fix $W$ complete DVR residue field $\overline{\mathbb{F}_p}$, e.g. Witt vectors of $\overline{\mathbb{F}_p}$ or the completion of the ring of integers of the maximal unramified ext of $\mathbb{Q}_p$.

\[
W = W_\infty = \lim_{\leftarrow n} W/p^n
\]

Let $R$ be a complete local $W_\infty$-algebra. Given $A/R$ an abelian scheme, $A[p^\infty] = \bigcup A[p^n]$ is a $p$-divisible group scheme. (More generally, a $p$-divisible group $G/R$ of height $h$ is an inductive system $(G_v, iv)$ such that $G_v$ are group schemes and each $G_v$ is order $p^v$ and $0 \to G_v \xrightarrow{iv} G_{v+1} \xrightarrow{p^v} G_{v+1}$ is exact.)

E.g., for constant $p$-adic sheaf, $G_v = \left( \mathbb{Z}/p^v \right)^n$, $\lim G_v = \left( \mathbb{Q}_p/\mathbb{Z}_p \right)^n$. 

\(\Phi\)}
2. Serre-Tate Theorem

Given a map of rings $R \rightarrow R_0$ we define $\text{Def}(R, R_0) = \{(A_0, G, \epsilon) \mid A_0 / R_0 \text{ abelian sch, } G/R \text{ p-divisible gp, } \epsilon : G_0 \otimes A_0 \rightarrow G \otimes R_0\}$.

If $R$ is a ring with $p$ nilpotent, $I \subseteq R$ nilpotent ideal, $R_0 = R / I$, we have a map $\text{AbSch}(R) \rightarrow \text{Def}(R, R_0)$

Functor $A \mapsto (A_0, A[p^\infty], \epsilon)$

"id" meaning forget for the theorem (Serre-Tate). This functor is an equivalence of categories.

The deformation $G$ of a fixed $R_0$ are in correspondence to deformations $\hat{G}$ of its p-divisible gp.

3. Drinfeld's Proof

Drinfeld extracts a key lemma about lifting functors that is applied many times.

Set-up: $R$ local $\text{W}_\text{m-alg}$, $I^{m+1} = 0$, $R_0 = R / I$, $N = \text{power of } p$

$NI = 0$

Given $A$ an $R$-algebra, consider $A / IA = A \otimes R_0$, $A / mA$ maximal.

If we have a functor $G : R$-algebras $\rightarrow \text{C}$ an abelian category, we can define $G^\wedge(A) = \ker(G(A) \rightarrow G(A/I))$

And

$$G^\wedge(A) = \ker(G(A) \rightarrow G(A / mA))$$

$G^\wedge \subset \hat{G}$. $G$ could be formal group

a collection of power series in $R(\{1, x_1, \ldots, x_n, y_1, \ldots, y_m\})$ satisfying certain conditions.
Extended e.g. The formal gp of an elliptic curve
we can write down explicit eqns for \(x(t), y(t), F(x(t), y(t))\) with sage, for use Newton iteration to solve
\(x(t), y(t)\) after knowing pole order, replace formula for gp law.

Given a complete local ring \(R\) and comm. local \(R\)-alg. \(A\) can ‘eval’ a formal gp at \(\text{mA}^n\) to get a group structure on \(\text{mA}^n\).

gives functor \(G:\) complete local \(R\)-alg \(\rightarrow\) Ab Groups
\[ G(A) \rightarrow (\text{mA}^n, F) \]

**Lemma** Let \(G\) be a commutative formal group \(R\), \(G_I, G\) subgroup
functors. Then \(G_I\) is \(N^\infty\) torsion.

**Proof**
\[ \text{(NIA)}^2 = (IA)^2 \] inductively...
\[ (N^v)^2 = 0 \text{ since } N = 0 \]

**Defn** Have a covariant functor \(G\)
complete local \(R\)-alg \(\rightarrow\) AbGps \(\text{st for any } A \rightarrow C\)

faithfully flat finite type, we have \(G(A) \rightarrow G(C)\)

"The sheaf condition on \(A \rightarrow C\)"
is called an \(\text{fppf abelian sheaf}\).

Eg. \(G(A) = E(A)\) for \(E/R\) elliptic curve, or an abelian scheme is an \(\text{fppf}\)
abelian sheaf.

**Important Lemma** Let \(G, H\) be \(\text{fppf}\) abelian sheaves. \(G_0 = (\text{restiction})\)
\(P_{\text{R-alg}} \rightarrow \text{AbGps}\)

(1) \(G\) is \(p\)-divisible
(2) \(H\) is formal
\(H = \text{same}\) such that or maybe \(R??\)

(3) \(H(A) \rightarrow H(A/J)\) surj for any nilpotent \(J\) "formal smoothness."


Then:

1. $\text{Hom}(G,H)$ and $\text{Hom}(G_0,H_0)$ are $p$-torsion free.
2. $\text{Hom}(G,H) \rightarrow \text{Hom}(G_0,H_0)$ is injective.
3. For $f \in \text{Hom}(G_0,H_0)$, $\exists! \phi \in \text{Hom}(G,H)$ with $\phi \equiv f \mod I$.

Write $\phi = \tilde{N}^G(\psi) \in \text{Hom}(G,H) \otimes Q$.

We get $f = \frac{\tilde{N}^G(\psi)}{N} \in \text{Hom}(G,H)$.

\[ \Rightarrow \tilde{N}^G(N^G[Q]) = 0 \]

Proof

Assume $pf = 0$ i.e. $pf(x) = 0 \quad \forall x \in G$.

By $p$-divisibility,

\[ \exists y \quad py = x \quad \Rightarrow 0 = f(py) = f(x) \]

8. Use $H$ is formal, use $p$-divisibility, $\tilde{N}^G$ kills everything.

Proof of Serre Tate

Can apply Dainfeld's lemma with all the various functors $A, A', A[\varphi], A'[\varphi], A_0[\varphi], A'[\varphi]$.