Alex: Serre-Tate Local Moduli

References:

- Woods Hole conference proceedings (not easy to read, but first appeared here)
- Katz, LNM 868 §1,2 based on Drinfeld (appendix)
- Hida "Geometric Modular forms & Elliptic Curves 2nd Ed!" "modern" v. of Katz
- Michigan Seminar

Motivation: Recall, trying to argue

$$(\underline{X}, T_m \underline{X}^\sigma) = \frac{1}{2}\sum_{n=1}^{\infty} \# \mathrm{Hom}_{W/\mathbb{F}}(\underline{X}^\sigma, \underline{X})_{\deg M}$$

Last time: Lifted curves w/ an endo to CM curves.

This time: discuss the "space" of all lifts of same $E_0/\mathbb{F}_q$

1) We will work more generally w/ ab sch e.g. Elliptic curve, products, weil restrictions, Jacobians of higher genus curves, R a ring, $AbSch(R)$ is a category of Abelian schemes $/R$.

Reduction: $A/\mathbb{Z}_p$ which reduce to A_0

$$\begin{array}{c} A/\mathbb{Z}_p \\ \downarrow \\ A_0/\mathbb{F}_p \end{array}$$

need to remember extra data abt A_0 to distinguish $A/\mathbb{Z}_p$

Fix W complete DVR residue field $\overline{\mathbb{F}}_p$, e.g. Witt vectors of $\overline{\mathbb{F}}_p$ or the completion of the ring of integers of the maximal ur ext of $\mathbb{Q}_p$.

$$W = W_\infty = \varprojlim_n \underbrace{W/p^n}_{W_n}$$

let R be a complete local W_m-algebra. Given A/R an abelian sch, $A[p^\infty] = \bigcup_n A[p^n]$ is the associated p-divisible group scheme. (More generally, a p-divisible group G/R of height h is an inductive system (G_v, i_v) such that G_v are finite group schemes and ~~[illegible]~~ each G_v is order p^{vh} [Do you also need flat?]

and $0 \to G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{p^v} G_{v+1}$ is exact.

E.g. for constant gp sch, $G_v \cong (\mathbb{Z}/p^v)^h$, $\varinjlim G_v = (\mathbb{Q}_p/\mathbb{Z}_p)^h$.

eg. $A[p^\infty]$ has height $2 \dim A$. [Why 2?]

2 Serre-Tate theorem

Given a map of rings $R \to R_0$ we define $Def(R, R_0) = \{(A_0, G, \varepsilon)$

: A_0/R_0 abelian sch, G/R p-divisible gp, $\varepsilon : G_0 \xrightarrow{\sim} A_0[p^\infty]\}$

$G_0 = G \otimes R_0$

If R is a ring with p nilpotent $I \subseteq R$ nilpotent ideal, $R_0 = R/I$

have a ~~map~~ Functor $AbSch(R) \to Def(R, R_0)$

$A \mapsto (A_0, A[p^\infty], \varepsilon)$

$A_0 = A \otimes R_0$, ε = "id" the thing you get for free

<u>Theorem</u> (Serre-Tate) This functor is an equivalence of categories.

ie deformations of a fixed R_0 are in correspondence to deformations of its p-divisible gp.

3 Drinfeld's Proof

Drinfeld extracts a key lemma about lifting functors that is applied many times.

Set-up: R local W_m-alg, $I^{\nu+1} = 0$, $R_0 = R/I$, N = power of p, $NI = 0$

given A a local R-algebra, consider $A/IA = A \otimes R_0$, A/m max'l ideal

If we have a functor G: R algebras $\to \underset{\sim}{C}$ abelian category (can take kernels)

can define $G_I(A) = \ker(G(A) \to G(A/I))$

$$\hat{G}(A) = \ker(G(A) \to G(A/m_A))$$

$G_I \subset \hat{G}$. G could be formal group

ie an (n-dim'l formal gp over R) is a collection of n power series in $R[[x_1, \dots, x_n, y_1, \dots, y_n]]$ ~~[illegible]~~ satisfying certain conditions.

Extended e.g: The formal gp of an ell curve

we can write down explicit eqns for $x(t)$, $y(t)$,
$F(x(t), y(t))$ with sage. [or: use Newton iteration to solve $x(t)$, $y(t)$ after knowing pole order, replace formula for gp law]

Given a complete local ring R and comp. local R-alg. A can 'eval' a comm formal gp at $\mathfrak{m}_A^n$ to get a group structure on $\mathfrak{m}_A^n$.
gives functor

$$G: \begin{matrix}\text{complete} \\ \text{local } R\text{-alg}\end{matrix} \longrightarrow \text{Ab Groups}$$

$$G(A) \longmapsto (\mathfrak{m}_A^n, F)$$

<u>Lemma</u> Let G be a commutative formal group $/R$, G_I, $\vec{G}$ subgroup functors. Then G_I is N^ν torsion.

Pf ~~First want to show~~ Let $\underline{a} \in G_I(A)$, $a_i \in I$ coords

$[N]\underline{a} = N\underline{a} + \text{h.o.t.}$

$$\Rightarrow [N]a_i = \underbrace{\overset{\in}{N(IA)}}_{\substack{=0 \\ \text{since } NI=0}} + \overset{\in}{(IA)^2} \qquad \text{inductively} \ldots [N^\nu]a_i \in IA^{2\nu} = 0$$

<u>Defn</u> Have a covariant functor G

$$\begin{matrix}\text{complete local} \\ R\text{-alg}\end{matrix} \longrightarrow \text{Ab Grps}$$

st for any $A \hookrightarrow C$ faithfully flat finite type we have $G(A) \hookrightarrow G(C)$ & $\underbrace{\text{"The sheaf condition on } A \hookrightarrow C\text{"}}_{\text{mysterious}}$ is called an ~~fppf abelian sheaf~~.

Eg. $G(A) = E(A)$ for E/R elliptic curve. or an abelian sch is an fppf abelian sheaf.

$\boxed{\text{Important}}$ <u>Lemma</u> Let G, H be fppf abelian sheaves. $G_0 = $ (restriction) $R_0\text{-alg} \to \text{Ab gps}$
$H_0 = $ same such that (1) G is p-divisible, (2) $\hat{H}$ is formal
(3) $H(A) \to H(A/J)$ surj for any nilpotent J "formal smoothness" or maybe R??

Then:

Ⓐ $\mathrm{Hom}(G,H)$ and $\mathrm{Hom}(G_0,H_0)$ are p-torsion free

Ⓑ $\mathrm{Hom}(G,H) \to \mathrm{Hom}(G_0,H_0)$ is injective

Ⓒ For $f_0 \in \mathrm{Hom}(G_0,H_0)$, $\exists!\ \varphi \in \mathrm{Hom}(G,H)$ with $\varphi \equiv N^\vee f_0 \bmod I$.

write $\varphi = \underbrace{\tilde{N}^\vee f}_{\phi} \in \mathrm{Hom}(G,H) \otimes \mathbb{Q}$

Ⓓ We get $f = \dfrac{\tilde{N}^\vee f}{N} \in \mathrm{Hom}(G,H) \iff \tilde{N}^\vee(G[N^\vee]) = 0$

<u>Proof</u> Ⓐ Assume $pf = 0$ ie $pf(x) = 0$ $\forall \alpha \in G$. ~~...~~ By p-divisibility ~~...~~ $\exists y$ ~~...~~ $py = x$, then $0 = \underset{pf(y)}{f(py)} = f(x)$.

Ⓑ use H is formal, use prev lemma that mult by $N^\vee$ kills everything

<u>Proof of Serre Tate</u> Can apply Drinfeld's lemma with all the various functors, $A, A', A[p^\infty], A'[p^\infty], A_0[p^\infty], A'[p^\infty]$