**John Sim: Modular Forms Background**

References: Darmon "Lecture Notes", Weinstein "Invitation to Modular Forms"

**Def:** \( \Gamma(N) = \{ (a \ b) \in \text{SL}_2 \mathbb{Z} \mid (a \ b) \equiv (\ 0 \ ) \mod N^2 \} \)

\( \Gamma \leq \text{SL}_2 \mathbb{Z} \) is a **congruence subgroup** if it contains \( \Gamma(N) \) for some \( N \).

**Key subgps:**
- \( \Gamma_1(N) = \{ (a \ b) \equiv (\ast \ast) \mod N \} \)
- \( \Gamma_0(N) = \{ (a \ b) \equiv (0 \ 1) \mod N \} \)

**Def:** \( f : \mathcal{H} \to \mathbb{C} \) is a **modular form** of wt \( 2k \) of \( \Gamma \) (wrt \& char) if:
  - \( f \) is holomorphic on \( \mathcal{H} \)
  - \( f \) is holomorphic at \( \infty \)
  - \( f|_{2k} \gamma (z) = f(z) \quad \forall \gamma \in \Gamma \)

\( \gamma = c_\mathcal{H} + d \), \( c_\mathcal{H} \neq 0 \)

**Example:**
\( \Gamma = \text{SL}_2 \mathbb{Z} = \langle (1 \ 1), (0 \ 1) \rangle \sim \text{SL}_2 \mathbb{Z} \)

- \( f(z+1) = f(z) \)
- \( f\left(\frac{1}{z}\right) \quad (cz+d)^{-2k} = f(z) \)
- \( f(z) = f(q) \), where \( q = e^{2\pi i z} \)

\( \mathbb{H} = \{ z \in \mathbb{C} : \text{Re}(z) > \frac{1}{2} \} \)

\( \sum_{n=0}^{\infty} a_n q^n \quad q = e^{2\pi i z} \)

\( q \) is punctured open at \( \infty \) using Fourier expansion:

Check holomorphicity disk.
Def: $f$ is a **cusp form** if $a_0 = 0$ and $f$ is a modular form.

Def: $M_k(\Gamma)$ is the space of $wt \ k$ mod forms.

$S_k(\Gamma)$ is the space of $wt \ k$ cusp forms.

E.g. $G_{2k}(\mathbb{H}) = \sum_{m,n \in \mathbb{Z}} \frac{1}{(mz+n)^{2k}}$

\[
\Delta(z) = \frac{q^3 - q^2}{1728}, \quad g_2 = \frac{G_4(z)}{25(4)} \quad \text{cusp form of weight 12}
\]

\[
g_3 = \frac{G_6(z)}{25(6)}
\]

**Thm:** $D: M_k \to S_{k+12}$

$f \mapsto \Delta f$ is an isomorphism and we have the decompositions:

$\forall k < 0$, $M_k = 0$, $k = 2$, $M_k = 0$,

$k$ odd $M_k = 0$, otherwise for all $k \in \mathbb{Z^+}$ $M_k = S_{k+G_{k,2}}$.

**Proof:** $f \cdot \Delta \left( \frac{-1}{z} \right) = (cz+d)^{-k} f(z) (cz+d)^{-2} \Delta(z)$

$= (cz+d)^{(k+12)} \left( f \cdot \Delta \right)(z)$.

If $k < 0$, $f \in M_k$, $f^{12} \Delta f \in S_0 = 0$

If $k = 0$, $M_0$ holomorphic on $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \cong \mathbb{C}$.

If $k$ odd, $(-1, 0) \in \text{SL}_2(\mathbb{Z})$ so $f(\overline{z}) = (-1)^{\frac{k}{2}} f(z) = -f(z)$. 

For every positive:
\[ M_k \rightarrow \mathbb{C} \]
\[ f(q) = a_0 + a_1 q + \ldots \rightarrow a_0 \]
\[ \dim \left( \frac{M_k}{\text{ker}=S_k} \right) \leq 1 \]
\[ M_k = S_k + G_k \cdot \mathbb{C} \]

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**Hecke Operators**

**Def:** $\Lambda$ is a lattice if it is a rk 2 $\mathbb{Z}$-module in $\mathbb{C}$ s.t. $\mathbb{C}/\Lambda$ is compact, write $\Lambda = \tau, \mathbb{Z} + \tau_2 \mathbb{Z}$, $\dim R \tau_1 \mathbb{R} + \tau_2 \mathbb{R} = 2$ and fix $\tau_1, \tau_2$ so that $\text{Im}(\tau_2/\tau_1) > 0$.

$F$ is a homogeneous lattice function of weight $k$ if $F: R \rightarrow \mathbb{C}$ where $R$ is a set of lattices

\[ F(\lambda \Lambda) = \lambda^{-k} F(\Lambda). \]

$F$ is holomorphic if $F: \mathbb{H} \rightarrow \mathbb{C}$, $f(\tau) = F(\mathbb{Z} + \tau \mathbb{Z})$
is holomorphic on $\mathcal{H}$.

$$\begin{align*}
\text{holomorphic} & \quad \longleftrightarrow \quad \text{weight } k \\
\text{homogeneous} & \quad \text{functions} & \quad \text{modular} \\
\text{wt } k & \quad \text{forms}
\end{align*}$$

$$F \quad \longleftrightarrow \quad \begin{cases} fF \quad \text{if } \ f \in \mathcal{M}(\mathbb{C}) \\
F(\tau, z + \tau z) = f(\tau \alpha_0)\quad \text{for } \alpha_0 \in \mathcal{SL}_2(\mathbb{Z})
\end{cases}$$

Def: $F$ holomorphic homogeneous holo weight $k$ lattice function then

$$T_{n,k} F(\lambda) = \lambda^{k-1} \sum_{\lambda' | \lambda} F(\lambda')$$

$$[\lambda : \lambda'] = n$$

$$T_{n,k} f = \mathcal{F}_{T_{n,k}}(F_{\lambda}) .$$

$$T_{n,k} f(z) = \lambda^{k-1} \sum_{\lambda' | \lambda} f(\lambda') (\lambda z + d)^{-k}$$

$$\lambda' \in \mathcal{SL}_2(\mathbb{Z}) \setminus \mathcal{M}$$

where $\mathcal{M} \subset \mathcal{SL}_2(\mathbb{Z})$ $2 \times 2$ matrices of integer entries det $n$.

Fourier expansion of Hecke operators:

Let $f(q) = \sum_{n=0}^{\infty} a_n q^n$. Then

$$T_{n,k} (f(q)) = \sum_{m=0}^{\infty} \sum_{d | \text{lcm}(m,n)} a_{m,n} q^m$$

If $(a,b) = 1$, fixing $k$,

$$T_a T_b = T_{ab}$$

$p$ prime

$$T_p T_p^{e+1} = T_{p^{e+1}} + p^{k-1} T_{p^{e-1}}$$
The Hecke operator $T_{n,k}$ operates on $M_k(SL_2 \mathbb{Z})$

Def: We say $f \in M_k(SL_2 \mathbb{Z})$ is an eigenform if it is a simultaneous eigenvector for $\sum_{n=1}^{\infty} T_n$.

There is an inner product $\langle \cdot, \cdot \rangle$ on $S_k \times S_k \to \mathbb{C}$

\[ \langle f, g \rangle = \int f(z) \overline{g(z)} y^k \frac{dz dy}{y^2} \quad (\text{Petersson inner product}) \]

The fundamental domain of $S_k$.

\[ \text{Fundamental Domain} \]

Thus we can decompose $S_k$ into a direct sum

\[ S_k = \bigoplus_{i} \mathbb{C} f_i \]

where the $f_i$ are eigenforms such that $\langle f_i, f_j \rangle = 0$.

Also, $G_k$ is an eigenform

\[ M_k = G_k \mathbb{C} + \bigoplus_{i} \mathbb{C} f_i \]

Def: $f$ is a normalized eigenform if $a_1(f) = 1$, $a_n(f) = a_1(T_n(f)) = a_n(\lambda_n f) = \lambda_n a_1(f) = \lambda_n$.

Prop: If $f \in S_k$ is a normalized eigenform then...
\( \mathbb{Q}(a_1(t), a_2(t), \ldots) \) is a finite totally real extension of \( \mathbb{Q} \) with degree less than or equal to 
\( \dim S_k := d. \)

\( \text{Pf} \quad \frac{G_k}{\mathfrak{S}(k)} \) and \( \Delta \) have rational coefficients and so we can view \( M_k \) as a \( \mathbb{Q} \)-v.s. 
\( \mathfrak{S}(k) \) := \( f_1 \mathbb{Q} + \ldots + f_d \mathbb{Q} \) 

Then \( T_n \) operates on this \( \mathbb{Q} \)-v.s. choice of basis for so \( T_n \to \text{Mat}_d(\mathbb{Q}) \), and furthermore 
\[
T_{k, \mathbb{Q}} := \mathbb{Q} \langle T_1, T_2, \ldots \rangle \to \text{Mat}_d(\mathbb{Q})
\]
\( \mathbb{Q} \)-algebra generated by \( T_2 \)

Therefore for all \( q \in \text{Hom}(T_k, \mathbb{Q}, \mathbb{Q}) \), \( q \) lies in a 
\( \text{deg} \leq d \) extension. Totally real because \( T_n \) are self-adjoint wrt a Hermitian inner product. \( \square \)

Generalization

\( \Gamma \) as any congruence subgroup
\( M_k(\Gamma) = \text{Eisenstein} + S_k(\Gamma) \)
\( M_k(\text{SL}_2 \mathbb{Z}) \subset M_k(\Gamma) \)
\( M_k(\Gamma(N)) \)
\( d | N, f \in M_k(\Gamma(d)) \)
\( g(z) := f(\frac{N}{d} \cdot z) \in M_k(\Gamma(N)) \) "dilation"

For large enough \( k \), we can find a basis of
\[ M_k(\Gamma(N)) \text{ by taking } M_k(\Gamma) \text{ where } \Gamma \supset \Gamma(N) \]
+ dilating
+ product of \( M_a(\Gamma(N)) \) where \( 1 < a < k \)
& Hecke operators. 

\( \text{Moral(?) : To find basis: use dilations from other } M_k(\Gamma). \) (Damon)

For \( (u, N) = 1, \ f \in M_k(\Gamma(N)) \)

\[ T_{n,k} f = n^{k-1} \sum_{\Gamma \backslash M_n} f(\gamma)(z) \]

\( M_n \) is upper triangular w/ \( \det n \).

For primes \( l, \ l \nmid N, \ T_l(f(q)) = \sum a_n l^n q^n + l^{k-1} \sum a_n <l> f(q^n) \)

"good operators"

\( \text{l|N, } T_l(f(q)) = \sum q^{nl} \text{ for } <l> f(z) = \left( f\left| k\begin{pmatrix} l & 0 \\ 0 & l \end{pmatrix}\right. \right)(z). \)

"bad operators"

Def. Let \( g \in S_k(\Gamma(N)) \) such that \( g(z) = f(dz) \) for some \( f \in S_k(\Gamma(N/d)). \) Then \( g \) is an oldform.

A newform is an \( f \in S_k(\Gamma(N)) \) such that \( \langle f, g \rangle = 0 \) for all oldform \( g \in S_k(\Gamma(N))^{\text{old}}. \)