John Sim: Modular Forms Background I

References: Darmon "Lecture Notes", Weinstein "Invitation to Modular Forms"

Def: $\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$

$\Gamma \subseteq SL_2\mathbb{Z}$ is a congruence subgroup if it contains $\Gamma(N)$ for some N.

Key subgrps: $\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\}$

$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}$

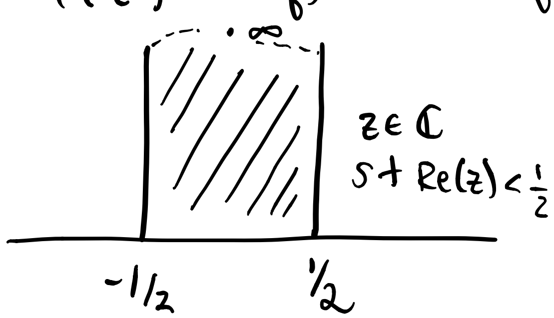
Def: $f: \mathcal{H} \to \mathbb{C}$ is a modular form of wt $2k$ of Γ (wrt ε char)

if
- f is holomorphic on $\mathcal{H}$
- f is holomorphic at ∞
- $f|_{2k}\gamma(z) = f(z) \quad \forall \gamma \in \Gamma$

where $f|_{2k}\gamma := (cz+d)^{-2k} f(\gamma z)\, \varepsilon(d)$

E.g. $\Gamma = SL_2\mathbb{Z} = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle SL_2\mathbb{Z}$

- $f(z+1) = f(z)$
- $f\left(\frac{-1}{z}\right)(cz+d)^{-2k} = f(z)$
- $f(z) = f(q)$ where $q = e^{2\pi i z}$



$z \in \mathbb{C}$ st $Re(z) < \frac{1}{2}$

$f(q) = \sum_{n=0}^{\infty} a_n q^n$

$q \in$ punctured open disc

check holomorphicity at ∞ using Fourier expansion in

Def f is a cusp form if $a_0 = 0$ and f is a modular form.

Def $M_k(\Gamma)$ is the space of wt k mod forms
$S_k(\Gamma)$ is the space of wt k cusp forms

E.g. $G_{2k}(z) = \sum'_{m,n \in \mathbb{Z}} \frac{1}{(mz+n)^{2k}}$

$$\Delta(z) = \frac{g_2^3 - g_3^2}{1728}$$

cusp form of weight 12

$$g_2 = \frac{G_4(z)}{2\zeta(4)}$$

$$g_3 = \frac{G_6(z)}{2\zeta(6)}$$

Thm: $D: M_k \to S_{k+12}$, $f \mapsto \Delta f$ is an isomorphism

and we have the decompositions:
$\forall k < 0$, $M_k = 0$, $k = 2$, $M_k = 0$,

k odd $M_k = 0$, otherwise for all $k \in 2\mathbb{Z}_+$
$M_k = S_k + G_k \mathbb{C}$.

Proof: $f \cdot \Delta(-\frac{1}{z}) = (cz+d)^{-k} f(z) (cz+d)^{-12} \Delta(z)$
$= (cz+d)^{-(k+12)} (f \cdot \Delta)(z)$.

If $k < 0$, $f \in M_k$, $f^{12} \Delta^k \in S_0 = 0$
If $k = 0$ M_0 holo fns on $\overline{SL_2\mathbb{Z} \backslash \mathcal{H}} \cong \mathbb{C}$.
If k odd, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL_2(\mathbb{Z})$ so $f(z) = (-1)^k f(z)$
$= -f(z)$.

For even positive:

$$M_k \to \mathbb{C}$$

$$f(q) = a_0 + a_1 q + \cdots \mapsto a_0$$

$$\dim\left(M_k / \ker = S_k\right) \leq 1$$

$$M_k = S_k + G_k \cdot \mathbb{C}.$$

Table:

k	$\dim M_k$	$\dim S_k$
<0	0	0
0	1	0
2	0	0
4	1	0
6	1	0
8	1	0
10	1	0
12	2	1
14	1	0
16	2	1
18	2	1
⋮	⋮	⋮

Hecke Operators

Def: Λ is a lattice if it is a rk 2 $\mathbb{Z}$-module in $\mathbb{C}$ st $\mathbb{C}/\Lambda$ is compact, write $\Lambda = \tau_1 \mathbb{Z} + \tau_2 \mathbb{Z}$, $\dim_{\mathbb{R}} \tau_1 \mathbb{R} + \tau_2 \mathbb{R} = 2$ and fix τ_1, τ_2 so that $\operatorname{Im}(\tau_2/\tau_1) > 0$.

F is a homogeneous lattice function of weight k $F: R \to \mathbb{C}$ where R is a set of lattices

$$F(\lambda \Lambda) = \lambda^{-k} F(\Lambda).$$

F is holomorphic if $f: \mathcal{H} \to \mathbb{C}$, $f(\tau) = F(\mathbb{Z} + \tau\mathbb{Z})$

is holomorphic on $\mathcal{H}$.

$$\left\{\begin{array}{l}\text{holomorphic} \\ \text{homogeneous} \\ \text{wt } k \text{ lattice} \\ \text{functions}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{weight } k \\ \text{modular} \\ \text{forms}\end{array}\right\}$$

$$F \longmapsto f_F$$

$$f_F(\tau) = F(\mathbb{Z} + \tau\mathbb{Z})$$

$$\overset{F_f}{F(\tau_1\mathbb{Z} + \tau_2\mathbb{Z})} = f(\tau_2/\tau_1) \longleftarrow\!\shortmid\ f_k$$

<u>Def</u>: F homogeneous holo weight k lattice function then

$$T_{n,k}F(\Lambda) = n^{k-1} \sum_{\substack{\Lambda' \subseteq \Lambda \\ [\Lambda:\Lambda'] = n}} F(\Lambda')$$

$$T_{n,k} f = f_{T_{n,k}(F_f)}.$$

$$T_{n,k} f(z) = n^{k-1} \sum_{\gamma \in SL_2\mathbb{Z} \backslash M_n} f(\gamma z)(cz+d)^{-k}$$

where $M_n \subset M_2\mathbb{Z}$ 2x2 matrices of integer entries det n.

Fourier expansion of Hecke operators:

Let $f(q) = \sum_{n=0}^{\infty} a_n q$. Then

$$T_{n,k}(f(q)) = \sum_{m=0}^{\infty} \sum_{d \mid (m,n)} d^{k-1} a_{\frac{m \cdot n}{d^2}} q^m$$

If $(a,b) = 1$, fixing k,

$$T_a T_b = T_{ab}$$

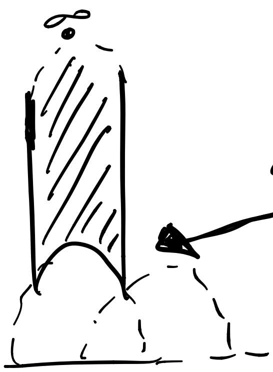
p prime $\quad T_p T_{p^t} = T_{p^{t+1}} + p^{k-1} T_{p^{t-1}}$

The Hecke operator $T_{n,k}$ operates on $M_k(SL_2\mathbb{Z})$

Def We say $f \in M_k(SL_2\mathbb{Z})$ is an eigenform if it is a simultaneous eigenvector for $\{T_n\}_{n=1}^{\infty}$.

There is an inner product $\langle \cdot , \cdot \rangle$ on $S_k \times S_k \to \mathbb{C}$

st $\langle f, g \rangle = \int_{\mathcal{F}} f(z)\overline{g(z)}\, y^k \frac{dz\,dy}{y^2}$ (Peterson inner product)



"Fundamental Domain"

e.g.

Key property T_n is self adjoint in S_k

ie $\langle T_n f, g \rangle = \langle f, T_n g \rangle$.

Thus we can decompose S_k into a direct sum

$$S_k = \bigoplus_i \underset{\text{eigenforms}}{f_i}\, \mathbb{C}$$

eigenforms st $\langle f_i, f_j \rangle = 0$

also G_k is an eigenform

$$M_k = G_k\mathbb{C} + \underbrace{\bigoplus f_i\, \mathbb{C}}_{S_k}$$

Def f is a normalized eigenform if $a_1(f) = 1$

$a_n(f) = a_1(T_n(f)) = a_n(\lambda_n f) = \lambda_n a_1(f) = \lambda_n$.

Prop If $f \in S_k$ is a normalized eigenform then

$\mathbb{Q}(a_1(f), a_2(f), \dots)$ is a finite totally real extension of $\mathbb{Q}$ w/ deg less than or equal to $\dim S_k := d$.

<u>Pf</u> $\frac{G_k}{\zeta(k)}$ and Δ have rational coefficients and so we can view M_k as a $\mathbb{Q}$ v.s. $:= \underbrace{f_1\mathbb{Q} + \dots + f_d\mathbb{Q}}_{\text{choice of basis for } \mathbb{Q} \text{ v.s.}}$

Then T_n operates on this $\mathbb{Q}$ v.s.

so $T_n \hookrightarrow \mathrm{Mat}_d(\mathbb{Q})$, and furthermore

$$\mathbb{T}_{k,\mathbb{Q}} := \underbrace{\mathbb{Q}[T_1, T_2, \dots]}_{\substack{\mathbb{Q}\text{-algebra} \\ \text{generated by } T_i}} \hookrightarrow \mathrm{Mat}_d(\mathbb{Q})$$

Therefore for all $\varphi \in \mathrm{Hom}(\mathbb{T}_{k,\mathbb{Q}}, \overline{\mathbb{Q}})$, φ lies in a deg $\leq d$ extension. Totally real because T_n are self-adjoint wrt a Hermitian inner product. $\square$

Generalization

Γ as any congruence subgroup

$M_k(\Gamma) =$ Eisenstein $+ \ S_k(\Gamma)$

$M_k(SL_2\mathbb{Z}) \subset M_k(\Gamma)$

$M_k(\Gamma(N))$

$d \mid N$, $f \in M_k(\Gamma(d))$

$g(z) := f(\frac{N}{d} \cdot z) \in M_k(\Gamma(N))$ "dilation"

For large enough k, we can find a basis of

$M_k(\Gamma(N))$ by taking $M_k(\Gamma)$ where $\Gamma \supset \Gamma(N)$

+ dilating

certain specific choices (?)

+ product of $M_a(\Gamma(N))$ $a<k$

& Hecke operators.

Moral(?): To find basis: use dilations from other $M_k(\Gamma)$. (Darmon)

For $(n,N)=1$, $f \in M_k(\Gamma(N))$

$$T_{n,k} f = n^{k-1} \sum_{\Gamma \backslash M_n} f(\gamma)(z)$$

M_n is upper triangular w/ det n.

For primes ℓ, $\ell \nmid N$, $T_\ell(f(q)) = \sum a_{n\ell} q^n + \ell^{k-1} \sum a_n \langle \ell \rangle f q^{n\ell}$

"good operators"

$\ell \mid N$, $T_\ell(f(q)) = \sum a_{n\ell} q^n$

$$\langle \ell \rangle f(z) = \left(f\Big|_k \begin{pmatrix} \ell & 0 \\ 0 & \ell^{-1} \end{pmatrix}\right)(z).$$

"bad operators"

Def. Let $g \in S_k(\Gamma(N))$ such that $g(z) = f(dz)$ for some $f \in S_k(\Gamma(\frac{N}{d}))$. Then g is an oldform.

A newform is an $f \in S_k(\Gamma(N))$ such that $\langle f, g \rangle = 0$ for all $g \in S_k(\Gamma(N))^{old}$.