

# CHIRAL DE RHAM COMPLEX AND ORBIFOLDS

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ABSTRACT. Suppose that a finite group  $G$  acts on a smooth complex variety  $X$ . Then this action lifts to the Chiral de Rham Complex  $\Omega_X^{\text{ch}}$  of  $X$  and to its cohomology by automorphisms of the vertex algebra structure. We define twisted sectors for  $\Omega_X^{\text{ch}}$  (and their cohomologies) as sheaves of twisted vertex algebra modules supported on the components of the fixed-point sets  $X^g, g \in G$ . Each twisted sector sheaf carries a BRST differential and is quasi-isomorphic to the de Rham complex of  $X^g$ . Putting the twisted sectors together with the vacuum sector and taking  $G$ -invariants, we recover the additive and graded structures of Chen-Ruan orbifold cohomology. Finally, we show that the orbifold elliptic genus is the partition function of the direct sum of the cohomologies of the twisted sectors.

## 1. INTRODUCTION

In [MSV] the authors constructed a sheaf of vertex superalgebras on any smooth complex variety  $X$ , called the chiral de Rham complex, and denoted by  $\Omega_X^{\text{ch}}$  (a chiral algebra version of this construction has been given in [BD]). The cohomology of  $\Omega_X^{\text{ch}}$ ,  $H^*(X, \Omega_X^{\text{ch}})$ , is also a vertex superalgebra. Thus, one attaches a vertex superalgebra to an arbitrary smooth algebraic manifold, and when  $X$  is Calabi-Yau, it is believed to be related to the chiral algebra of the topological twist of the type  $A$  sigma-model with target  $X$  (at least in the large radius limit). This explains the recent interest in the chiral de Rham complex: using it one may hopefully obtain a rigorous mathematical construction of a conformal field theory associated to  $X$  (such as the sigma model of  $X$ ) without resorting to non-rigorous methods like the path integral. The chiral de Rham complex also has a number of interesting applications in various fields of geometry and representation theory, such as mirror symmetry [Bo], elliptic genera [BL1], [BL2], as well as others [GMS], [GMS2], [GMS3], [S]. We hope that in the future it may be used to access other aspects of “stringy” geometry.

The aim of this paper is to extend the construction of  $\Omega_X^{\text{ch}}$  to orbifolds. Suppose that  $X$  carries the action of a finite group  $G$ . In physics terms, we would then like to construct an orbifold conformal field theory (such as the sigma model) corresponding to the stack  $[X/G]$ . In the language of vertex algebras, we must first take the  $G$ -invariant part of  $H^*(X, \Omega_X^{\text{ch}})$  and then include the so-called twisted sectors, one for each conjugacy class in  $G$ . Let  $T$  denote the set of conjugacy classes of  $G$ , and  $[g]$  the conjugacy class of an element  $g \in G$ . In this paper we give a geometric construction of these twisted sectors as follows. For each  $g \in G$ , let  $X^g$  denote the fixed-point set of  $g$  on  $X$ . We construct a twisted version of  $\Omega_X^{\text{ch}}$ , denoted  $\Omega_X^{\text{ch},g}$ , as a sheaf supported

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on  $X^g$ . We show that  $\Omega_X^{\text{ch},g}$  is a sheaf of  $g$ -twisted  $\Omega_X^{\text{ch}}$ -modules, and that furthermore,  $H^*(X, \Omega_X^{\text{ch},g})$  is a  $g$ -twisted module for  $H^*(X, \Omega_X^{\text{ch}})$ . Taking  $G$ -invariants, we show that  $H^*(X, \Omega_X^{\text{ch},g})^{C(g)}$  is an ordinary vertex algebra module for  $H^*(X, \Omega_X^{\text{ch}})^G$ , where  $C(g)$  denotes the centralizer of  $g$ . Furthermore, if  $[g] = [g']$ , then we have a canonical isomorphism

$$H^*(X, \Omega_X^{\text{ch},g'})^{C(g')} \simeq H^*(X, \Omega_X^{\text{ch},g})^{C(g)}$$

of  $H^*(X, \Omega_X^{\text{ch}})^G$ -modules. In particular,  $\Omega_X^{\text{ch},1} = \Omega_X^{\text{ch}}$ . The space

$$\mathcal{H} = \bigoplus_{[g] \in T} H^*(X, \Omega_X^{\text{ch},g})^{C(g)}$$

is then the conjectural space of states of the orbifold model. If  $X$  is Calabi-Yau and  $G$  acts preserving the volume form, then the untwisted sector of  $\mathcal{H}$  has the structure of an  $N = 2$  topological vertex algebra.

As shown in [MSV],  $\Omega_X^{\text{ch}}$  carries a BRST differential, denoted  $d^{\text{ch}}$ , and there is an inclusion of the usual de Rham complex

$$(\Omega_X, d_{DR}) \hookrightarrow (\Omega_X^{\text{ch}}, d^{\text{ch}})$$

which is a quasi-isomorphism. In the orbifold setting, each sheaf  $\Omega_X^{\text{ch},g}$  carries a BRST differential  $d^{\text{ch},g}$ , and there is a quasi-isomorphism

$$(\Omega_{X^g}, d_{DR}) \hookrightarrow (\Omega_X^{\text{ch},g}, d^{\text{ch},g}).$$

It follows that the hypercohomology of  $\Omega_X^{\text{ch},g}$  is isomorphic to

$$\mathbb{H}^*(X, \Omega_X^{\text{ch},g})^{C(g)} \cong H^*(X^g, \mathbb{C})^{C(g)} \cong H^*(X^g/C(g), \mathbb{C})$$

Combining all of the twisted sectors together, we see that

$$(1.1) \quad \bigoplus_{[g] \in T} \mathbb{H}^*(X, \Omega_X^{\text{ch},g})^{C(g)} \cong \bigoplus_{[g] \in T} H^*(X^g/C(g), \mathbb{C})$$

is thus isomorphic to the Chen-Ruan orbifold cohomology of  $[X/G]$  (see [CR]). The important point of our construction is that  $\mathcal{H}$ , and hence the right hand side of (1.1) acquires a natural gradation from the vertex algebra point of view. We show that this gradation coincides with the somewhat mysterious gradation (shifted by “age” or “fermionic shift”) on the orbifold cohomology introduced by Chen and Ruan. Thus, we obtain an isomorphism between the left hand side of (1.1) and the orbifold cohomology at the level of graded vector spaces. We expect that the product structure on the orbifold cohomology may also be constructed by the vertex algebra methods, along the lines of [MS] (where the question of constructing of the product structure on the ordinary de Rham cohomology in terms of the chiral de Rham complex was considered). We hope to return to this question in a future publication.

In [GMS],[GMS2], a purely even version of the chiral de Rham complex, the sheaf of chiral differential operators, was defined for those varieties  $X$  on which the second Chern character vanishes (see [BD] for a chiral algebra version). If a finite group  $G$  acts on such a variety, then we can define in a similar way sheaves of twisted modules over the sheaf of chiral differential operators.

It has been shown in [BL] that the elliptic genus of  $X$  may be expressed as the trace of the operator  $y^{J_0} q^{L_0}$  on  $H^*(X, \Omega_X^{\text{ch}})$ . Here we prove an analogue of this statement in the orbifold setting. Namely, we show that the orbifold elliptic genus of  $[X/G]$  may be expressed as the trace of  $y^{J_0} q^{L_0}$  on  $\mathcal{H}$ . The latter may be viewed as an invariant of the corresponding orbifold conformal field theory.

Our results can be generalized to more general orbifolds, which are not global quotients.

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## 2. VERTEX ALGEBRAS AND TWISTED MODULES

In this paper we will use the language of vertex superalgebras, their modules, and twisted modules. For an introduction to vertex algebras and their modules [FLM, K, FB], and for background on twisted modules, see [FFR, D, DLM, FS].

We recall that a conformal vertex superalgebra is a  $\mathbb{Z}_+$ -graded super vector space

$$V = \bigoplus_{n=0}^{\infty} V_n,$$

$$V_n = V_n^{\bar{0}} \oplus V_n^{\bar{1}}$$

together with a vacuum vector  $|0\rangle \in V_0^{\bar{0}}$ , an even translation operator  $T$  of degree 1, a conformal vector  $\omega \in V_2^{\bar{0}}$  and an even linear map

$$Y : V \rightarrow \text{End } V[[z^{\pm 1}]],$$

$$A \mapsto Y(A, z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}.$$

These data must satisfy certain axioms (see [FLM, K, FB]). In what follows we will denote the collection of such data simply by  $V$ , and the parity of an element  $A \in V$  homogeneous with respect to the  $\mathbb{Z}/2\mathbb{Z}$  grading by  $p(A)$ .

A vector superspace  $M$  is called a  $V$ -module if it is equipped with an even linear map

$$Y^M : V \rightarrow \text{End } M[[z^{\pm 1}]],$$

$$A \mapsto Y^M(A, z) = \sum_{n \in \mathbb{Z}} A_{(n)}^M z^{-n-1}$$

such that for any  $v \in M$  we have  $A_{(n)}^M v = 0$  for large enough  $n$ . This operation must satisfy the following axioms:

- $Y^M(|0\rangle, z) = \text{Id}_M$ ;
- For any  $v \in M$  and homogeneous  $A, B \in V$  there exists an element

$$f_v \in M[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$$

such that the formal power series

$$Y^M(A, z)Y^M(B, w)v, \quad (-1)^{p(A)p(B)}Y^M(B, w)Y^M(A, z)v, \quad \text{and}$$

$$Y_M(Y(A, z-w)B, w)v$$

are expansions of  $f_v$  in  $M((z))((w))$ ,  $M((w))((z))$  and  $M((w))((z-w))$ , respectively.

The power series  $Y^M(A, z)$  are called vertex operators. We write the vertex operator corresponding to  $\omega$  as

$$Y^M(\omega, z) = \sum_{n \in \mathbb{Z}} L_n^M z^{-n-2},$$

where  $L_n^M$  are linear operators on  $V$  generating the Virasoro algebra. Following [D], we call  $M$  *admissible* if  $L_0^M$  acts semi-simply with integral eigenvalues.

Now let  $\sigma_V$  be a conformal automorphism of  $V$ , i.e., an even automorphism of the underlying vector superspace preserving all of the above structures (so in particular  $\sigma_V(\omega) = \omega$ ). We will assume that  $\sigma_V$  has finite order  $m > 1$ . A vector space  $M^\sigma$  is called a  $\sigma_V$ -*twisted*  $V$ -module (or simply twisted module) if it is equipped with an even linear map

$$\begin{aligned} Y^{M^\sigma} : V &\rightarrow \text{End } M^\sigma[[z^{\pm \frac{1}{m}}]], \\ A &\mapsto Y^{M^\sigma}(A, z^{\frac{1}{m}}) = \sum_{n \in \frac{1}{m}\mathbb{Z}} A_{(n)}^{M^\sigma} z^{-n-1} \end{aligned}$$

such that for any  $v \in M^\sigma$  we have  $A_{(n)}^{M^\sigma} v = 0$  for large enough  $n$ . Please note that we use the notation  $Y^{M^\sigma}(A, z^{\frac{1}{m}})$  rather than  $Y^{M^\sigma}(A, z)$  in the twisted setting. This operation must satisfy the following axioms (see [FFR, D, DLM, Li, FS]):

- $Y^{M^\sigma}(|0\rangle, z^{\frac{1}{m}}) = \text{Id}_{M^\sigma}$ ;
- For any  $v \in M^\sigma$  and homogeneous  $A, B \in V$ , there exists an element

$$f_v \in M^\sigma[[z^{\frac{1}{m}}, w^{\frac{1}{m}}]][z^{-\frac{1}{m}}, w^{-\frac{1}{m}}, (z-w)^{-1}]$$

such that the formal power series

$$Y^{M^\sigma}(A, z^{\frac{1}{m}})Y^{M^\sigma}(B, w^{\frac{1}{m}})v, \quad (-1)^{p(A)p(B)}Y^{M^\sigma}(B, w^{\frac{1}{m}})Y^{M^\sigma}(A, z^{\frac{1}{m}})v, \quad \text{and}$$

$$Y^{M^\sigma}(Y(A, z-w)B, w^{\frac{1}{m}})v$$

are expansions of  $f_v$  in  $M^\sigma((z^{\frac{1}{m}}))((w^{\frac{1}{m}}))$ ,  $M^\sigma((w^{\frac{1}{m}}))((z^{\frac{1}{m}}))$  and  $M^\sigma((w^{\frac{1}{m}}))((z-w))$ , respectively.

- If  $A \in V$  is such that  $\sigma_V(A) = e^{\frac{2\pi ik}{m}} A$ , then  $A_{(n)}^{M^\sigma} = 0$  unless  $n \in \frac{k}{m} + \mathbb{Z}$ .

The series  $Y^{M^\sigma}(A, z)$  are called twisted vertex operators. In particular, the Fourier coefficients of the twisted vertex operator

$$Y^{M^\sigma}(\omega, z^{\frac{1}{m}}) = \sum_{n \in \mathbb{Z}} L_n^{M^\sigma} z^{-n-2},$$

generate an action of the Virasoro algebra on  $M^\sigma$ .

### 3. CHIRAL DE RHAM COMPLEX

In this section we review the construction of the chiral deRham complex due to [MSV]. We follow the presentation of [FB], Sect. 17.4.

**3.1. The vertex superalgebra  $\widehat{\Omega}_N$ .** Let  $\mathcal{H}_N$  denote the infinite-dimensional Heisenberg algebra with generators  $a_n^i, b_m^j, \quad i, j = 1, \dots, N, n, m \in \mathbb{Z}$ , and relations

$$[a_n^i, b_m^j] = \delta_{i,j} \delta_{n,-m}.$$

Let  $\mathcal{H}_N^+$  denote the abelian Lie subalgebra generated by  $a_n^i, n \geq 0, b_m^j, m > 0$ . Let

$$V_N = \text{Ind}_{\mathcal{H}_N^+}^{\mathcal{H}_N} \mathbb{C},$$

where  $\mathbb{C}$  denotes the trivial representation of  $\mathcal{H}_N^+$ . Then  $V_N$  has the structure of a vertex algebra (see [FB]). It is shown in [MSV] that the vertex algebra structure on  $V_N$  can be naturally extended to its completion

$$\widehat{V}_N = \mathbb{C}[[b_0^i]] \otimes_{\mathbb{C}[b_0^i]} V_N.$$

Let  $Cl_N$  be the infinite-dimensional Clifford algebra with generators  $\psi_n^i, \phi_m^j, \quad i, j = 1, \dots, N, n, m \in \mathbb{Z}$ , and relations

$$[\psi_n^i, \phi_m^j]_+ = \delta_{i,j} \delta_{n,-m}$$

Let  $Cl_N^+$  denote the abelian Lie sub-superalgebra spanned by  $\psi_n^i, n \geq 0, \phi_m^j, m > 0$ . Let

$$\Lambda_N = \text{Ind}_{Cl_N^+}^{Cl_N} \mathbb{C},$$

where  $\mathbb{C}$  denotes the trivial representation of  $Cl_N^+$ . Then  $\Lambda_N$  has the structure of a vertex superalgebra (see, e.g., [FB]).

The tensor product of vertex (super) algebras has the structure of a vertex (super) algebra in a natural way. Let

$$\widehat{\Omega}_N = \widehat{V}_N \otimes \Lambda_N$$

Let us denote by  $Y : \widehat{\Omega}_N \mapsto \text{End}(\widehat{\Omega}_N)[[z, z^{-1}]]$  the corresponding vertex operator.

**3.2. Superconformal structure.** Define the vectors  $L, J, Q, G \in \widehat{\Omega}_N$  as follows:

$$L = \sum_{i=1}^N (b_{-1}^i a_{-1}^i + \phi_{-1}^i \psi_{-1}^i),$$

$$J = \sum_{i=1}^N \phi_0^i \psi_{-1}^i, \quad Q = \sum_{i=1}^N a_{-1}^i \phi_0^i, \quad G = \sum_{i=1}^N \psi_{-1}^i b_{-1}^i$$

The corresponding fields,  $L(z) = Y(L, z), J(z) = Y(J, z), Q(z) = Y(Q, z), G(z) = Y(G, z)$  are:

$$L(z) = \sum_{i=1}^N (: \partial_z b^i(z) a^i(z) : + : \partial_z \phi^i(z) \psi^i(z) :),$$

$$J(z) = \sum_{i=1}^N : \phi^i(z) \psi^i(z) :, \quad Q(z) = \sum_{i=1}^N : a^i(z) \phi^i(z) :, \quad G(z) = \sum_{i=1}^N : \psi^i(z) \partial_z b^i(z) :$$

It is known from [FMS] (see also [MSV]) that the fields  $L(z), J(z), Q(z), G(z)$  have the OPE's of an  $N = 2$  superconformal algebra of central charge 0. These OPE's are:

$$L(z)L(w) = \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w}$$

$$J(z)J(w) = \frac{N}{(z-w)^2}; \quad L(z)J(w) = -\frac{N}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w}$$

$$G(z)G(w) = 0; \quad L(z)G(w) = \frac{2G(w)}{(z-w)^2} + \frac{\partial_w G(w)}{z-w}; \quad J(z)G(w) = -\frac{G(w)}{z-w}$$

$$Q(z)Q(w) = 0; \quad L(z)Q(w) = \frac{Q(w)}{(z-w)^2} + \frac{\partial_w Q(w)}{z-w}; \quad J(z)Q(w) = \frac{Q(w)}{z-w}$$

$$Q(z)G(w) = \frac{N}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{L(w)}{z-w}$$

**3.3. Charges and gradings.** The operator

$$L_0 = \sum_{i=1}^N \sum_{n \in \mathbb{Z}} (: a_n^i b_{-n}^i : + : \phi_n^i \psi_{-n}^i :)$$

defines a  $\mathbb{Z}_{\geq 0}$  grading on  $\widehat{\Omega}_N$  by conformal weights. Let

$$\widehat{\Omega}_{N,k} = \{v \in \widehat{\Omega}_N | L_0 v = kv\}$$

The operator

$$F = J_0 = \sum_{i=1}^N \sum_{n \in \mathbb{Z}} : \phi_n^i \psi_{-n}^i :$$

defines a  $\mathbb{Z}$  grading on  $\widehat{\Omega}_N$  by fermionic charge. Let

$$\widehat{\Omega}_N^p = \{v \in \widehat{\Omega}_N | Fv = pv\}$$

$L_0$  and  $J_0$  commute, and thus define a  $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$  bigrading on  $\widehat{\Omega}_N$ . We use the notation  $\widehat{\Omega}_{N,k}^p$  for the bigraded components, and call it the component of bidegree  $(k, p)$ . Let

$$d^{\text{ch}} = -Q_0 = - \sum_{i=1}^N \sum_{n \in \mathbb{Z}} a_n^i \phi_{-n}^i$$

be the BRST or deRham differential of  $\widehat{\Omega}_N$ . It has bidegree  $(0, 1)$  and  $d^{\text{ch}} \circ d^{\text{ch}} = 0$ .

**3.4. Construction of  $\Omega_X^{\text{ch}}$ .** Let  $D_N$  be the formal  $N$ -dimensional disk

$$D_N = \text{Spec } \mathbb{C}[[t_1, \dots, t_N]].$$

Let  $W_N$  denote the topological Lie algebra of vector fields on  $D_N$ ,  $W_N^+$  the Lie subalgebra of vector fields vanishing at the origin, and  $\text{Aut } \mathcal{O}_N$  the unipotent proalgebraic group of continuous algebra automorphisms of  $\mathbb{C}[[t_1, \dots, t_N]]$ . We have that  $\text{Lie}(\text{Aut } \mathcal{O}_N) = W_N^+$ , and  $(W_N, \text{Aut } \mathcal{O}_N)$  forms a Harish-Chandra pair.

Let  $U(\widehat{\Omega}_N)$  denote the Lie algebra of Fourier coefficients of vertex operators on  $\widehat{\Omega}_N$ . It follows from the results in [MSV] that there exists a Lie algebra homomorphism

$$\nu : W_N \mapsto U(\widehat{\Omega}_N)$$

defined by the formula

$$(3.1) \quad f(t_i) \partial_{t_j} \mapsto \text{Res}_z \left( : f(b^i(z)) a^j(z) : + \sum_{k=1}^N : (\partial_{t_k} f)(b^i(z)) \phi^k(z) \psi^j(z) : \right)$$

Since  $U(\widehat{\Omega}_N)$  acts on  $\widehat{\Omega}_N$ , we obtain a representation of  $W_N$  on  $\widehat{\Omega}_N$  by residues of vertex operators. Over  $W_N^+$  this action can be exponentiated to an action of  $\text{Aut } \mathcal{O}_N$  by vertex algebra automorphisms.

The infinitesimal action of  $\text{Aut } \mathcal{O}_N$  on the generating fields of  $\widehat{\Omega}_N$  is given by

$$(3.2) \quad [\nu(f(t_i) \partial_{t_j}), a^k(w)] = : (\partial_{t_k} f)(b^i(w)) a^j(w) : + \sum_{l=1}^N : (\partial_{t_l} \partial_{t_k} f)(b^i(w)) \phi^l(w) \psi^j(w) :$$

$$(3.3) \quad [\nu(f(t_i) \partial_{t_j}), b^k(w)] = \delta_{j,k} f(b^i(w))$$

$$(3.4) \quad [\nu(f(t_i) \partial_{t_j}), \phi^k(w)] = -\delta_{j,k} \sum_{l=1}^N (\partial_{t_l} f)(b^i(w)) \phi^l(w)$$

$$(3.5) \quad [\nu(f(t_i) \partial_{t_j}), \psi^k(w)] = (\partial_{t_k} f)(b^i(w)) \psi^j(w)$$

Let  $X$  be a smooth complex variety of dimension  $N$ . Then  $X$  carries a canonical Harish-Chandra structure  $\widehat{X}$

$$\widehat{X} = \{(x, (t_{i,x})_{i=1 \dots N})\}$$

where  $x \in X$  and  $(t_{i,x})_{i=1, \dots, N}$  is a set of topological generators of the maximal ideal of  $\widehat{\mathcal{O}}_x$ . It is clear that  $\widehat{X}$  is an  $\text{Aut } \mathcal{O}_N$ -principal bundle over  $X$  via the map  $(x, (t_{i,x})) \mapsto x$  (see [FB], where it is denoted by  $\text{Aut}_X$ ). Moreover, the Lie algebra  $W_N$  acts on  $\widehat{X}$  simply

transitively, extending the fiberwise action of  $W_N^+$  induced by the action of  $\text{Aut } \mathcal{O}_N$ . Let

$$\tilde{\Omega}_X^{\text{ch}} = \widehat{X} \times_{\text{Aut } \mathcal{O}_N} \widehat{\Omega}_N$$

Thus, the fiber of  $\tilde{\Omega}_X^{\text{ch}}$  at  $x$  is the quotient of the space of pairs  $(t_{i,x}, v)$ , where  $t_{i,x}$  are formal coordinates on  $\text{Spec } \widehat{\mathcal{O}}_x$ ,  $v \in \widehat{\Omega}_N$ , by the equivalence relation

$$(t_{i,x}, v) \sim (\rho(t_{i,x}), \rho(v)), \quad \rho \in \text{Aut } \mathcal{O}_N.$$

As explained in [FB], the sheaf of sections of  $\tilde{\Omega}_X^{\text{ch}}$  is an  $\mathcal{O}_X$ -module equipped with a flat connection, i.e., it is a  $\mathcal{D}_X$ -module. The connection is constructed from the action of the translation vector fields  $\partial_{t_i}$  on  $\widehat{\Omega}_N$ . The chiral de Rham complex  $\Omega_X^{\text{ch}}$  is by definition the sheaf of horizontal sections of  $\tilde{\Omega}_X^{\text{ch}}$ . It is a sheaf of vertex superalgebras whose cohomology also possesses a vertex superalgebra structure, as explained in [MSV], [Bo].

We proceed to give an explicit description of the  $\mathcal{D}_X$ -module structure of  $\tilde{\Omega}_X^{\text{ch}}$  in local coordinates. Let  $x \in X$ , and choose formal coordinates  $t_{i,x}$  at  $x$ . Denote  $\text{Spec } \widehat{\mathcal{O}}_x$  by  $D_x$ . Then we obtain a trivialization  $\tau$  of the restriction of the bundle  $\tilde{\Omega}_X^{\text{ch}}$  to  $D_x$ . It follows from the formula (3.1) that the connection operator  $\nabla_{t_{i,x}}$  is given by

$$\partial_{t_{i,x}} - a_0^i$$

Using the trivialization  $\tau$  we can therefore identify the space of horizontal sections of  $\tilde{\Omega}_X^{\text{ch}}$  on  $D_x$  with those elements  $f(t_{i,x}, a_n^i, b_n^i, \psi_n^i, \phi_n^i)$  of

$$\mathbb{C}[[t_{1,x}, \dots, t_{N,x}]] \widehat{\otimes} \widehat{\Omega}_N$$

which are “functions of  $t_{i,x} + b_0^i$ ”, i.e. those  $f$  having the property that they are equal to 0 after the formal substitution  $t_{i,x} = -b_0^i$ ,  $i = 1, \dots, N$ . Evaluation at  $x$ , i.e., the map

$$\begin{aligned} \text{ev}_x : \Gamma_{\nabla}(D_x, \tilde{\Omega}_X^{\text{ch}}) &\mapsto (\tilde{\Omega}_X^{\text{ch}})_x, \\ f &\mapsto f|_{t_{i,x}=0} \end{aligned}$$

induces an isomorphism between the space of horizontal sections of  $\tilde{\Omega}_X^{\text{ch}}$  on  $D_x$  and the fiber of  $\tilde{\Omega}_X^{\text{ch}}$  at  $x$ .

The sheaf  $\Omega_X^{\text{ch}}$  is not an  $\mathcal{O}_X$ -module, but it carries a non-associative action of  $\mathcal{O}_X$

$$(3.6) \quad \mu : \mathcal{O}_X \times \Omega_X^{\text{ch}} \rightarrow \Omega_X^{\text{ch}}$$

defined as follows. For  $f(t_{i,x}) \in \mathbb{C}[[t_{1,x}, \dots, t_{N,x}]]$ , denote by  $f_n$  the operator

$$\text{Res}_z z^{-n-1} Y(f(b_0^i), z) \in \text{End } \widehat{\Omega}_N.$$

Let  $\tilde{f}_n \in \text{End } \widehat{\Omega}_N[[t_{i,x}]]$  be the operator obtained from  $f_n$  by formally replacing  $b_0^i$  by  $b_0^i + t_{i,x}$ . The action of  $f$  on  $\Omega_X^{\text{ch}}(D_x) = \Gamma_{\nabla}(D_x, \tilde{\Omega}_X^{\text{ch}})$  is given by multiplication by  $\tilde{f}_0$ . Alternatively, using the isomorphism  $\text{ev}_x$ , it is given by acting on  $(\tilde{\Omega}_X^{\text{ch}})_x$  by  $f_0$ . This action (3.6) is non-associative since  $f_0 g_0 \neq (fg)_0$ .



**3.5. Filtration on  $\Omega_X^{\text{ch}}$ .** In [MSV], the authors introduce a filtration on  $\Omega_X^{\text{ch}}$ , which we now recall. We begin by defining a filtration on  $\widehat{\Omega}_N$ . Note that  $\widehat{\Omega}_N$  is spanned by monomials in the letters  $a_n^i, \psi_n^i$   $n < 0$  and  $b_m^i, \phi_m^i$ ,  $m \leq 0$ . We will call these letters our *alphabet*. We define a partial order on monomials as follows.

- Define a partial order on the alphabet by the rules:  $a > b, a > \psi, a > b, \psi > \phi, \psi > b, \phi > b$  and  $x_n^i > x_m^j$ ,  $x$  being  $a, b, \phi$  or  $\psi$
- Extend the partial order  $>$  to all monomials lexicographically such that  $b_0^i$  is equivalent to the “empty” letter, i.e. all monomials that differ by products of  $b_0^i$  are considered equivalent.

This partial order uniquely determines an exhaustive increasing filtration

$$F^0 \widehat{\Omega}_N \subset F^1 \widehat{\Omega}_N \subset F^2 \widehat{\Omega}_N \cdots$$

such that  $F^i \neq F^{i+1}$  by using the following two rules (here, denote by  $\text{gr}_F \Omega_X^{\text{ch}}$  the associated graded object and  $\pi : \widehat{\Omega}_N \mapsto \text{gr}_F \Omega_X^{\text{ch}}$  the projection map):

- If  $M_1$  and  $M_2$  are two monomials, and  $M_1 < M_2$ , then  $\pi(M_1)$  lies in a strictly lower component in  $\text{gr}_F \Omega_X^{\text{ch}}$  than  $\pi(M_2)$ .
- If neither  $M_1 < M_2$  nor  $M_1 > M_2$  then  $\pi(M_1)$  and  $\pi(M_2)$  lie in the same graded component.

It follows from the formulas (3.2)–(3.5) that this filtration is  $\text{Aut } \mathcal{O}_N$ –stable, and therefore induces a filtration on  $\Omega_X^{\text{ch}}$ .

The action  $\mu$  (see formula (3.6)) induces an honest  $\mathcal{O}_X$ –module structure on  $\text{gr}_F \Omega_X^{\text{ch}}$ . Furthermore, if we fix a conformal weight  $k \geq 0$ , then the sheaves

$$(3.7) \quad F^l \Omega_X^{\text{ch}} \cap \Omega_{X,k}^{\text{ch}} / F^{l-1} \Omega_X^{\text{ch}} \cap \Omega_{X,k}^{\text{ch}}$$

are coherent  $\mathcal{O}_X$ –modules isomorphic to tensor products of exterior and symmetric powers of the tangent and cotangent bundles.

We can give a precise description of  $\text{gr}_F \Omega_X^{\text{ch}}$  by using a generating function, following the notation in [BL]. Note that  $L_0, J_0$  are  $\text{Aut } \mathcal{O}_N$ –invariant operators on  $\widehat{\Omega}_N$  and therefore give rise to global sections of  $\mathcal{E}nd \Omega_X^{\text{ch}}$  which act semi-simply, and which we also denote  $L_0$  and  $J_0$ . Furthermore,  $L_0$  and  $J_0$  preserve the filtration  $F^l \Omega_X^{\text{ch}}$ , and therefore act on  $\text{gr}_F \Omega_X^{\text{ch}}$ , which splits into eigenbundles of these operators. We will write

$$F^l \Omega_X^{\text{ch}} \cap \Omega_{X,k}^{\text{ch}} / F^{l-1} \Omega_X^{\text{ch}} \cap \Omega_{X,k}^{\text{ch}} = \sum_i q^{r_i} y^{s_i} W_{r_i, s_i}$$

if (3.7) is isomorphic to  $\oplus_i W_{r_i, s_i}$ , and if  $L_0$  (resp.  $J_0$ ) acts on  $W_{r_i, s_i}$  by  $r_i$  (resp.  $s_i$ ). If  $t$  is a formal variable and  $W$  is a vector bundle, introduce the notation

$$\Lambda_t^\bullet W = \sum_{k \geq 0} t^k \Lambda^k W$$

and similarly

$$\text{Sym}_t^\bullet W = \sum_{k \geq 0} t^k \text{Sym}^k W.$$

We then have the following equality (see [BL2])

**Proposition 3.1.**

$$\begin{aligned} \mathrm{gr}_F \Omega_X^{\mathrm{ch}} &= \bigoplus_{k,l} F^l \Omega_X^{\mathrm{ch}} \cap \Omega_{X,k}^{\mathrm{ch}} / F^{l-1} \Omega_X^{\mathrm{ch}} \cap \Omega_{X,k}^{\mathrm{ch}} \\ &= \bigotimes_{k \geq 1} \left( \Lambda_{yq^{k-1}}^\bullet \Omega_X \otimes \Lambda_{y^{-1}q^k}^\bullet T_X \otimes \mathrm{Sym}_{q^k}^\bullet \Omega_X \otimes \mathrm{Sym}_{q^k}^\bullet T_X \right), \end{aligned}$$

where  $T_X$  and  $\Omega_X$  are the holomorphic tangent and cotangent bundles of  $X$  respectively.

4. TWISTED SECTORS FOR  $\Omega_X^{\mathrm{ch}}$ 

Suppose that  $X$  carries an action of a finite group  $G$ . Then this action extends naturally to an action on  $\widehat{X}$  commuting with the actions of  $\mathrm{Aut} \mathcal{O}_N$  and  $W_N$ , and therefore gives rise to a  $G$ -equivariant structure on  $\widetilde{\Omega}_X^{\mathrm{ch}}$ . For every  $g \in G$  there exists a canonical isomorphism  $S_g : \widetilde{\Omega}_X^{\mathrm{ch}} \mapsto g^* \widetilde{\Omega}_X^{\mathrm{ch}}$ . Fibrewise, it is given by the formula

$$\begin{aligned} S_g : (\widetilde{\Omega}_X^{\mathrm{ch}})_p &\mapsto (\widetilde{\Omega}_X^{\mathrm{ch}})_{g(p)} \\ (x_i, v) &\mapsto (x_i \circ g^{-1}, v). \end{aligned}$$

These isomorphisms satisfy  $S_h \circ S_g = S_{hg}$

This action of  $G$  on  $\widetilde{\Omega}_X^{\mathrm{ch}}$  takes horizontal sections to horizontal sections, inducing an action of  $G$  on the sheaf  $\Omega_X^{\mathrm{ch}}$ . This action respects the vertex algebra structure, and thus we obtain that  $G$  acts on the sheaf  $\Omega_X^{\mathrm{ch}}$  by vertex algebra automorphisms. This action induces an action of  $G$  on  $H^*(X, \Omega_X^{\mathrm{ch}})$  by vertex algebra automorphisms. We also note that this action is compatible with all gradings as well as the filtration  $F^l \Omega_X^{\mathrm{ch}}$ .

In this section we construct on each component  $X_\alpha^g$  of the fixed point set  $X^g$ ,  $g \in G$ , a sheaf of  $g$ -twisted modules over  $\Omega_X^{\mathrm{ch}}$ .

**4.1. Special coordinates.** Let

$$(4.1) \quad X^g = \bigcup_{\alpha} X_\alpha^g$$

denote the fixed-point set of  $g$  in  $X$ , where  $X_\alpha^g$  are the connected components. Note that  $X^g$  is smooth since we are in characteristic 0, and the union (4.1) is disjoint. In this section we construct sheaves of twisted vertex algebra modules along  $X^g$ , which can be viewed as twisted chiral analogues of torsion coherent sheaves. For  $g \in G$ , denote by  $C(g)$  the centralizer of  $g$  in  $G$ . At a point  $p \in X_\alpha^g$ ,  $g$  induces an action  $\rho_p(g)$  on  $T_p X$ . Denote the order of  $g$  by  $m_g$ . The eigenvalues of  $\rho_p(g)$  are

$$\exp(2\pi i m_{1,g}^\alpha / m_g), \dots, \exp(2\pi i m_{N,g}^\alpha / m_g)$$

for some  $m_{j,g}^\alpha \in \{0, \dots, m_g - 1\}$ , and are constant on each component  $X_\alpha^g$ . Henceforth,  $g$  and a connected component  $X_\alpha^g$  will be fixed, and we write  $m_j$  for  $m_{j,g}^\alpha$ .

All of the constructions of this section have to be performed for each connected components  $X_\alpha^g$  separately.

**Definition 4.1.** We say that  $x_1, \dots, x_N$  are formal coordinates at  $p$  if they form an independent generating set for the maximal ideal of  $\widehat{\mathcal{O}}_p$ . We say they are *special* with respect to  $g$  if  $x_j \circ g^{-1} = \exp(2\pi i m_j / m_g) x_j$ .

**4.2. The group  $\text{Aut } \mathcal{O}^g$ .** Let again  $D_N = \text{Spec } \mathbb{C}[[t_1, \dots, t_N]]$  and let  $h$  be a finite-order automorphism of  $D_N$ . Such automorphisms are the same as finite order linear automorphisms of the tangent space to  $D_N$  at the origin (i.e., the vector space  $\text{span } \{\partial_{t_i}\}$ ). Let  $\text{Aut } \mathcal{O}_N^h$  denote the subgroup of  $\text{Aut } \mathcal{O}_N$  consisting of those automorphisms which commute with the action of  $h$ . Let us choose special coordinates on  $D_N$  with respect to  $h$ , which we also call  $t_1, \dots, t_N$ . Thus  $t_j \circ h^{-1} = \exp(2\pi i m_j / m_h) t_j$ . We can now give a concrete description of  $\text{Aut } \mathcal{O}_N^h$  in terms of power series. Let  $\rho \in \text{Aut } \mathcal{O}_N^h$ . We can write

$$\rho(t_1, \dots, t_N) = (\rho^1(t_1, \dots, t_N), \dots, \rho^N(t_1, \dots, t_N))$$

where the  $\rho^j$  are formal power series in the  $t_i$  without a constant term and with a non-degenerate linear part. The requirement that  $\rho \in \text{Aut } \mathcal{O}_N^h$  now becomes

$$(4.2) \quad \rho^j(\exp(2\pi i m_1 / m_h) t_1, \dots, \exp(2\pi i m_N / m_h) t_N) = \exp(2\pi i m_j / m_h) \rho^j(t_1, \dots, t_N)$$

which implies that  $\rho^j$  is a sum of monomials  $c^{i_1, \dots, i_N} t_1^{i_1} \dots t_N^{i_N}$  whose exponents satisfy

$$(4.3) \quad \sum_{l=1}^n m_l i_l \equiv m_j \pmod{m_h}$$

Let  $W_N^h$  denote the Lie algebra of all  $h$ -invariant vector fields on  $D_N$ , and  $W_N^{h,+}$  the subalgebra of those which vanish at the origin.  $W_N^h$  has as topological generators

$$t_1^{i_1} \dots t_N^{i_N} \partial_{t_j}$$

where condition (4.3) holds. The Lie algebra of  $\text{Aut } \mathcal{O}_N^h$  is isomorphic to  $W_N^{h,+}$ .

The set of special coordinates on  $D_N$  is a torsor for  $\text{Aut } \mathcal{O}_N^h$ . To see this observe that if  $\{t'_i\}_{i=1 \dots N}$  is another set of special coordinates, then it is related to the  $t_i$  by an element  $\rho \in \text{Aut } \mathcal{O}_N$ , and the condition that they are special implies that  $\rho$  satisfies the conditions of Definition 4.2.

**4.3. Twisted modules for  $\widehat{\Omega}_N$ .** Let  $\mathcal{H}_N^g$  denote the infinite-dimensional Heisenberg algebra with generators  $a_{r_i}^i, b_{s_j}^j, \quad i, j = 1, \dots, N, \quad r_i \in \frac{m_i}{m_g} + \mathbb{Z}, \quad s_j \in -\frac{m_j}{m_g} + \mathbb{Z}$ , and relations

$$[a_{r_i}^i, b_{s_j}^j] = \delta_{i,j} \delta_{r_i, -s_j}.$$

Let  $\mathcal{H}_N^{g,+}$  denote the abelian Lie subalgebra spanned by  $a_{r_i}^i, \quad r \geq 0, \quad b_{s_j}^j, \quad s > 0$ . Let

$$V_N^g = \text{Ind}_{\mathcal{H}_N^{g,+}}^{\mathcal{H}_N^g} \mathbb{C}$$

where  $\mathbb{C}$  denotes the trivial representation of  $\mathcal{H}_N^+$ . We will use the same notation  $g$  for the automorphism of  $\Omega_N$  determined by

$$a^j(z) \mapsto \exp(2\pi i m_j / m_g) a^j(z)$$

$$b^j(z) \mapsto \exp(-2\pi i m_j / m_g) b^j(z)$$

It follows from H. Li's twisted version of the reconstruction theorem (see [Li]) that  $V_N^g$  has the structure of a  $g$ -twisted  $\Omega_N$ -module. Denote the twisted vertex operator by

$$Y_{V_N}^g : V_N \mapsto \text{End } V_N^g[[z^{\frac{1}{m_g}}, z^{\frac{-1}{m_g}}]]$$

The twisted module structure on  $V_N^g$  is generated (in the sense of [Li]) by the twisted fields

$$\begin{aligned} a^{i,g}(z) &= Y_{V_N}^g(a_{-1}^i, z) = \sum_{n \in -m_i/m_g + \mathbb{Z}} a_n z^{-n-1} \\ b^{i,g}(z) &= Y_{V_N}^g(b_0^i, z) = \sum_{n \in m_i/m_g + \mathbb{Z}} b_n z^{-n} \end{aligned}$$

The remaining field assignments can be computed using the twisted iterate formula in [Li].

Clearly,  $g$  is also an automorphism of the completion  $\widehat{\Omega}_N$ . One shows as in [MSV] that the  $g$ -twisted module structure on  $V_N^g$  extends naturally to

$$\widehat{V}_N^g = \mathbb{C}[[b_0^{i_s}]] \otimes_{\mathbb{C}[b_0^{i_s}]} V_N,$$

where  $\{i_s\}$  runs through the subset of the  $i$ 's such that  $m_{i_s} = 0$ .

Let  $Cl_N^g$  be the infinite-dimensional Clifford algebra with the generators  $\psi_{r_i}^i, \phi_{s_j}^j, i, j = 1, \dots, N, r_i \in \frac{m_i}{m_g} + \mathbb{Z}, s_j \in -\frac{m_j}{m_g} + \mathbb{Z}$ , and relations

$$[\psi_{r_i}^i, \phi_{s_j}^j]_+ = \delta_{i,j} \delta_{r_i, -s_j}.$$

Denote by  $Cl_N^{g,+}$  the subalgebra generated by  $\psi_{r_i}^i, r_i \geq 0, \phi_{s_j}^j, s_j > 0$ . Let

$$\Lambda_N^g = \text{Ind}_{Cl_N^{g,+}}^{Cl_N^g} \mathbb{C}$$

where  $\mathbb{C}$  denotes the trivial representation of  $Cl_N^{g,+}$ . We will use the same notation  $g$  for the vertex superalgebra automorphism of  $\Lambda_N$  defined by

$$\psi^j(z) \mapsto \exp(2\pi i m_j / m_g) \psi^j(z)$$

$$\phi^j(z) \mapsto \exp(-2\pi i m_j / m_g) \phi^j(z)$$

Applying the results in [Li], it follows that  $\Lambda_{g,N}$  has the structure of a  $g$ -twisted  $\Lambda_N$ -module, generated by the twisted fields

$$\begin{aligned} \psi^{i,g}(z) &= Y_{\Lambda_N}^g(\psi_{-1}^i, z) = \sum_{n \in -m_i/m_g + \mathbb{Z}} \psi_n z^{-n-1} \\ \phi^{i,g}(z) &= Y_{\Lambda_N}^g(\phi_0^i, z) = \sum_{n \in m_i/m_g + \mathbb{Z}} \phi_n z^{-n} \end{aligned}$$

where we denote the twisted vertex operation by

$$Y_{\Lambda_N}^g : \Lambda_N \mapsto \text{End } \Lambda_N^g[[z^{\frac{1}{m_g}}, z^{\frac{-1}{m_g}}]]$$

The diagonal action of  $g$  induces a vertex superalgebra automorphism of  $\widehat{\Omega}_N = \widehat{V}_N \otimes \Lambda_N$ . This automorphism preserves the superconformal structure, i.e. fixes the vectors  $L, J, Q, G$ . Let

$$\widehat{\Omega}_N^g = \widehat{V}_N^g \otimes \Lambda_N^g.$$

It follows from the above that  $\widehat{\Omega}_N^g$  has the structure of a  $g$ -twisted  $\widehat{\Omega}_N$ -module.

**4.4. Twisted fields.** Let  $Y^g : \widehat{\Omega}_N \mapsto \text{End}(\widehat{\Omega}_N^g)[[z^{\pm \frac{1}{m_g}}]]$  denote the twisted vertex operator. For  $A \in \widehat{\Omega}_N$ , let  $A^g(z) = Y^g(A, z)$ . Using H. Li's twisted iterate formula (see [Li]), we obtain the following field assignments:

$$(4.4) \quad \begin{aligned} L^g(z) &= \sum_{i=1}^N (: a^{i,g}(z) \partial_z b^{i,g}(z) : + : \psi^{i,g}(z) \partial_z \phi^{i,g}(z) :) \\ J^g(z) &= \sum_{i=1}^N \left( : \phi^{i,g}(z) \psi^{i,g}(z) : + \frac{m_i}{m_g} z^{-1} \right) \end{aligned}$$

$$Q^g(z) = \sum_{i=1}^N : a^{i,g}(z) \phi^{i,g}(z) :$$

$$G^g(z) = \sum_{i=1}^N : \psi^{i,g}(z) \partial_z b^{i,g}(z) :$$

The fields  $L^g, J^g, Q^g, G^g$  satisfy the OPE's of section 3.2.

**4.5. Charges and gradings on  $\widehat{\Omega}_N^g$ .** The operators  $L_0^g$  and  $J_0^g$  commute and act semi-simply on  $\widehat{\Omega}_N^g$ . Analogously to the untwisted case, we call the  $L_0^g$  grading conformal weight, and the  $J_0^g$  grading fermionic charge. The eigenvalues of  $L_0^g$  lie in  $\frac{1}{m_g} \mathbb{Z}_{\geq 0}$ , and those of  $J_0^g$  in  $\iota(g) + \mathbb{Z}$ , where  $\iota(g)$ , the ‘‘fermionic shift’’, is defined by

$$\iota(g) = \sum_{i=1}^N m_i / m_g$$

For later reference we observe that  $\widehat{\Omega}_{N,0}^g$ , the conformal weight zero component of  $\widehat{\Omega}_N^g$ , is spanned by the monomials of the form

$$b_0^{i_1} \cdots b_0^{i_p} \phi_0^{j_1} \cdots \phi_0^{j_q},$$

where  $m_{i_k} = m_{j_l} = 0$ ,  $1 \leq k \leq p, 1 \leq l \leq q$ .

4.6.  $\text{Aut } \mathcal{O}^g$  **acts on**  $\widehat{\Omega}_N^g$ . Let  $p \in X^g$ , and choose a set of special coordinates  $\{x_i\}$  on  $D_p = \text{Spec}(\widehat{\mathcal{O}}_{X,p})$ . This identifies  $D_p$  with  $\text{Spec } \mathbb{C}[[x_1, \dots, x_N]]$ , and the action of  $g$  with the automorphism

$$x_i \mapsto \exp(2\pi i m_i / m_g) x_i$$

Let  $v$  be a  $g$ -invariant vector field on  $D_p$ . In the coordinates  $\{x_i\}$  it can be written as a linear combination of vector fields of the form:

$$f(x_1, \dots, x_N) \partial_{x_j}$$

where  $f \in \mathbb{C}[[x_1, \dots, x_N]]$  is a sum of monomials satisfying condition (4.3).

**Proposition 4.1.** *The map  $\nu^g : W_N^g \mapsto \text{End}(\widehat{\Omega}_N^g)$  defined by*

$$f(x_1, \dots, x_N) \partial_{x_j} \mapsto \text{Res } Y^g \left( f(b_0^i) a_{-1}^j + \sum_{k=1}^N (\partial_{x_k} f)(b_0^i) \phi_0^k \psi_{-1}^j, z \right)$$

*defines a representation of  $W_N^g$  on  $\widehat{\Omega}_N^g$ .*

*Proof.* For  $v = f(x_1, \dots, x_N) \partial_{x_j}$ , let

$$A_v = f(b_0^i) a_{-1}^j + \sum_{k=1}^N (\partial_{x_k} f)(b_0^i) \phi_0^k \psi_{-1}^j$$

As shown in [MSV], the map

$$\begin{aligned} W_N &\mapsto \text{End}(\widehat{\Omega}_N) \\ v &\mapsto (A_v)_{(0)} \end{aligned}$$

is a Lie algebra homomorphism, i.e.,  $(A_{[v,w]})_{(0)} = [(A_v)_{(0)}, (A_w)_{(0)}]$ . Using the vertex algebra associativity property, this implies that  $(A_{[v,w]}) = (A_v)_{(0)} \cdot A_w$ . If  $v \in W_N^g$ , then  $A_v \in (\widehat{\Omega}_N)^g$ , the  $g$ -invariant part of  $\widehat{\Omega}_N$ . The restriction of the twisted module  $\widehat{\Omega}_N^g$  to the invariant subalgebra  $(\widehat{\Omega}_N)^g$  is an untwisted module, and therefore the vertex operators  $Y^g(A_v, z)$ ,  $v \in W_N^g$  satisfy ordinary (untwisted) associativity. We have

$$\begin{aligned} Y^g(A_v, z) Y^g(A_w, u) &\sim \frac{Y^g((A_v)_{(0)} A_w, u)}{z - u} \\ &\sim \frac{Y^g(A_{[v,w]}, u)}{z - u} \end{aligned}$$

Taking residues we see that  $[(A_v^g)_{(0)}, (A_w^g)_{(0)}] = (A_{[v,w]}^g)_{(0)}$ , proving the theorem.  $\square$

Finally, the action of  $W_N^{g,+}$  on  $\widehat{\Omega}_N^g$  can be exponentiated to an action of  $\text{Aut } \mathcal{O}^g$ , so that we obtain

**Proposition 4.2.** *Restricted to  $W_N^{g,+}$ , the Lie algebra representation  $\nu^g$  may be exponentiated to yield a representation of  $\text{Aut } \mathcal{O}^g$  on  $\widehat{\Omega}_N^g$ , which we also denote by  $\nu^g$ .*

**4.7. Transformation properties of  $Y^g$  under  $\text{Aut } \mathcal{O}^g$ .** The group  $\text{Aut } \mathcal{O}^g$  is naturally a subgroup of  $\text{Aut } \mathcal{O}$ , and therefore it acts on  $\widehat{\Omega}_N$  via the representation

$$\nu : \text{Aut } \mathcal{O}^g \hookrightarrow \text{Aut } \mathcal{O}_N \rightarrow \text{End } \widehat{\Omega}_N$$

$\text{Aut } \mathcal{O}^g$  also acts on  $\widehat{\Omega}_N^g$  via the representation

$$\nu^g : \text{Aut } \mathcal{O}^g \rightarrow \text{End } \widehat{\Omega}_N^g$$

The twisted vertex operation  $Y^g$  enjoys the following invariance property with respect to the two  $\text{Aut } \mathcal{O}^g$ -actions:

**Theorem 4.1.** *For  $\rho \in \text{Aut } \mathcal{O}^g$ ,  $A \in \widehat{\Omega}_N$ ,  $B \in \widehat{\Omega}_N^g$  we have*

$$(4.5) \quad Y^g(\nu(\rho) \cdot A, z)B = \nu^g(\rho)Y^g(A, z)\nu^g(\rho^{-1})$$

*Proof.* We use the notation from the previous section. It suffices to prove the infinitesimal version of equation (4.5), which is

$$Y^g((A_v)_{(0)} \cdot A, z) = [(A_v^g)_{(0)}, Y^g(A, z)]$$

This follows from the associativity of twisted vertex operators and the fact that  $A_v \in \widehat{\Omega}_N^g$  for  $v \in W_N^g$ . □

**4.8. The sheaf  $\Omega_X^{\text{ch},g}$ .** Let  $\widehat{X}^g = (p, x_i)$  where  $p \in X^g$  and  $x_i$  form special formal coordinates at  $p$ . Thus  $\widehat{X}^g$  is an  $\text{Aut } \mathcal{O}^g$ -principal bundle over  $X^g$  whose fiber over a point  $p \in X$  consists of all formal special coordinates at  $p$ .

We can now define

$$\widetilde{\Omega}_X^g = \widehat{X}^g \times_{\text{Aut } \mathcal{O}^g} \widehat{\Omega}_N^g$$

This is an  $\mathcal{O}$ -module on  $X^g$  possessing a flat connection. Let  $\overline{\Omega}_X^g$  be the sheaf of horizontal sections of  $\widetilde{\Omega}_X^g$  on  $X^g$ , and define

$$\Omega_X^{\text{ch},g} = i_* \overline{\Omega}_X^g$$

where  $i : X^g \hookrightarrow X$  is the inclusion map.

**4.9. Filtration on  $\Omega_X^{\text{ch},g}$ .** We can define an increasing exhaustive filtration

$$F^0 \widehat{\Omega}_N^g \subset F^1 \widehat{\Omega}_N^g \subset F^2 \widehat{\Omega}_N^g \subset \dots$$

of  $\widehat{\Omega}_N^g$  in a manner identical to the one in section 3.5. As before, this filtration is  $\text{Aut } \mathcal{O}_N^g$ -stable, and therefore induces filtrations on  $\widetilde{\Omega}_X^g$ , and by pushing forward, on  $\Omega_X^{\text{ch},g}$ . As in the untwisted case,  $\overline{\Omega}_X^g$  (resp.  $\Omega_X^{\text{ch},g}$ ) carries a non-associative action  $\mu^g$  of  $\mathcal{O}_{X^g}$  (resp.  $i_* \mathcal{O}_{X^g}$ ), which induce an honest  $\mathcal{O}_{X^g}$  (resp.  $i_* \mathcal{O}_{X^g}$ )-module structures on  $\text{gr}_F \overline{\Omega}_X^g$  (resp.  $\text{gr}_F \Omega_X^{\text{ch},g}$ ). For a fixed conformal weight  $k \in \frac{1}{m_g} \mathbb{Z}_{\geq 0}$ , the sheaves

$$(4.6) \quad F^l \overline{\Omega}_X^{\text{ch},g} \cap \overline{\Omega}_{X,k}^{\text{ch},g} / F^{l-1} \overline{\Omega}_X^{\text{ch},g} \cap \overline{\Omega}_{X,k}^{\text{ch},g}$$

are coherent  $\mathcal{O}_{X^g}$ -modules.

We proceed to describe  $\text{gr}_F \overline{\Omega}_X^g$  using generating functions as in section 3.5. Note that  $L_0^g, J_0^g$  are  $\text{Aut } \mathcal{O}^g$ -invariant operators on  $\widehat{\Omega}_N^g$ , and therefore give rise to global

sections of  $\text{End} \overline{\Omega}_X^g$ , which we also denote by  $L_0^g, J_0^g$ . These operators preserve the filtration and (4.6) is a direct sum of their eigenbundles.

The restriction of the tangent bundle of  $X$  to a connected component  $X_\alpha^g$  of  $X^g$ , denoted by  $TX|_{X_\alpha^g}$ , splits under the  $g$ -action:

$$TX|_{X_\alpha^g} = \bigoplus_{\lambda \in [0,1] \cap \mathbb{Q}} V_{\lambda,\alpha},$$

where  $V_{\lambda,\alpha}$  is the sub-bundle on which  $g$  acts by multiplication by  $\exp(2\pi i\lambda)$ . Let

$$(4.7) \quad V_{g,\alpha} = \bigotimes_{k \geq 1} \left( \Lambda_{yq^{k-1}}^\bullet V_{0,\alpha}^* \otimes \Lambda_{y^{-1}q^k}^\bullet V_{0,\alpha} \otimes \text{Sym}_{q^k}^\bullet V_{0,\alpha}^* \otimes \text{Sym}_{q^k}^\bullet V_{0,\alpha} \otimes \right. \\ \left. \bigotimes_{\lambda \neq 0} \left( \Lambda_{yq^{k-1+\lambda(g)}}^\bullet V_{\lambda,\alpha}^* \otimes \Lambda_{y^{-1}q^{k-\lambda(g)}}^\bullet V_{\lambda,\alpha} \otimes \text{Sym}_{q^{k-1+\lambda(g)}}^\bullet V_{\lambda,\alpha}^* \otimes \text{Sym}_{q^{k-\lambda(g)}}^\bullet V_{\lambda,\alpha} \right) \right).$$

A straightforward calculation then shows the following

**Proposition 4.3.**

$$\text{gr}_F \overline{\Omega}_X^{\text{ch},g}|_{X_\alpha^g} = \bigoplus_{k,l} F^l \overline{\Omega}_X^{\text{ch},g} \cap \overline{\Omega}_{X,k}^{\text{ch},g} / F^{l-1} \overline{\Omega}_X^{\text{ch},g} \cap \overline{\Omega}_{X,k}^{\text{ch},g} = V_{g,\alpha}.$$

**4.10. Sheaves of twisted  $\Omega_X^{\text{ch}}$ -modules.** We wish to make  $\Omega_X^{\text{ch},g}$  into a “sheaf of twisted modules” over  $\Omega_X^{\text{ch}}$ . What this means will be explained below. We begin by constructing a map  $\tilde{\mathcal{Y}}^g$  of sheaves

$$i^* \tilde{\Omega}_X^{\text{ch}} \rightarrow \mathcal{E}nd \tilde{\Omega}_X^{\text{ch},g} \left[ \left[ z^{\frac{1}{m_g}}, z^{\frac{-1}{m_g}} \right] \right]$$

on  $X^g$  (here  $i : X^g \hookrightarrow X$ ). It suffices to do this fibrewise. Let  $p \in X^g$ , and let  $\{x_i\}$  be special coordinates at  $p$ . Define

$$\tilde{\mathcal{Y}}_p^g : (i^* \tilde{\Omega}_X^{\text{ch}})_p \mapsto \mathcal{E}nd \left( \tilde{\Omega}_X^{\text{ch},g} \right)_p \left[ \left[ z^{\frac{1}{m_g}}, z^{\frac{-1}{m_g}} \right] \right]$$

by the formula

$$\tilde{\mathcal{Y}}_p^g \left( (x_i, A), z \right) (x_i, B) = (x_i, Y^g(A, z)B)$$

That  $\tilde{\mathcal{Y}}_p^g$  is well-defined follows from the transformation property (4.5) of  $Y^g$  under  $\text{Aut } \mathcal{O}^g$ . The sheaf  $i^* \tilde{\Omega}_X^{\text{ch}}$  is equipped with a flat connection pulled back from  $\tilde{\Omega}_X^{\text{ch}}$ , and it follows from the explicit description of the flat sections of  $\tilde{\Omega}_X^{\text{ch}}$  given in section 3.4 (and similarly for  $\tilde{\Omega}_X^{\text{ch},g}$ ) that if  $A(x_i)$  and  $B(x_i)$  are horizontal sections of  $i^* \tilde{\Omega}_X^{\text{ch}}$  and  $\tilde{\Omega}_X^{\text{ch},g}$  respectively, then

$$\tilde{\mathcal{Y}}^g(A(x_i), z)B(x_i)$$

is a horizontal section of  $\tilde{\Omega}_X^{\text{ch},g} \left[ \left[ z^{\frac{1}{m_g}}, z^{\frac{-1}{m_g}} \right] \right]$  (here the connection acts trivially on the second factor). The map  $\tilde{\mathcal{Y}}^g$  therefore gives rise to a well-defined map

$$i^{-1} \Omega_X^{\text{ch}} \rightarrow \mathcal{E}nd \overline{\Omega}_X^{\text{ch},g} \left[ \left[ z^{\frac{1}{m_g}}, z^{\frac{-1}{m_g}} \right] \right]$$



where  $i^{-1}$  denotes the pull-back for sheaves of abelian groups, rather than  $\mathcal{O}_X$ -modules. Using the adjointness of  $i^{-1}$  with  $i_*$ , we obtain a map

$$\Omega_X^{\text{ch}} \rightarrow \mathcal{E}nd \Omega_X^{\text{ch},g} \left[ \left[ z^{\frac{1}{m_g}}, z^{\frac{-1}{m_g}} \right] \right],$$

which we denote by  $\mathcal{Y}^g$ .

Thus, for each open subset  $U \subset X$  we have the corresponding map

$$\mathcal{Y}_U^g : \Gamma(U, \Omega_X^{\text{ch}}) \rightarrow \text{End} \Gamma(U, \Omega_X^{\text{ch},g} \left[ \left[ z^{\frac{1}{m_g}}, z^{\frac{-1}{m_g}} \right] \right]).$$

We claim that these maps define on  $\Gamma(U, \Omega_X^{\text{ch},g})$  the structure of  $g$ -twisted modules over the vertex superalgebra  $\Gamma(U, \Omega_X^{\text{ch}})$ . These structures are compatible with respect to the inclusions of open sets  $U' \hookrightarrow U$ . It is in this sense that we speak of the structure of a sheaf of  $g$ -twisted modules on the sheaf  $\Omega_X^{\text{ch},g}$  over the sheaf of superalgebras  $\Omega_X^{\text{ch}}$ .

Indeed, it follows from the construction that the maps  $\mathcal{Y}_U^g$  satisfy the twisted associativity axiom and that  $\mathcal{Y}_U^g(|0\rangle, z) = \text{Id}$ , where  $|0\rangle$  denotes the ‘‘vacuum section’’ of  $\Omega_X^{\text{ch}}$ . Next, suppose  $h \in G$ , and let  $g' = hgh^{-1}$ . Then  $h$  maps  $X^g$  to  $X^{g'}$ . There exists a canonical isomorphism of sheaves

$$R_{g,g'}^h : \Omega_X^{\text{ch},g} \mapsto h^* \Omega_X^{\text{ch},g'}$$

defined as follows. If  $p \in X^g$  and  $\{x_i\}$  are special coordinates at  $p$ , then  $h(p) \in X^{g'}$  and  $\{x_i \circ h^{-1}\}$  are special coordinates at  $h(p)$ . We therefore obtain an isomorphism

$$\tilde{R}_{g,g'}^h : \tilde{\Omega}_X^{\text{ch},g} \mapsto h^* \tilde{\Omega}_X^{\text{ch},g'}$$

given fibrewise by

$$\tilde{R}_{g,g'}^h(x_i, A)_p = (x_i \circ h^{-1}, A)_{h(p)}$$

Taking horizontal sections and pushing forward, we obtain  $R_{g,g'}^h$ . If  $k \in G$  and  $g'' = khgh^{-1}k^{-1}$ , we have

$$R_{g',g''}^k \circ R_{g,g'}^h = R_{g,g''}^{kh}$$

$R_{g,g'}^h$  induces an isomorphism  $\mathcal{E}nd \Omega_X^{\text{ch}} \mapsto \mathcal{E}nd(h^* \Omega_X^{\text{ch},g'})$ , which we also denote  $R_{g,g'}^h$ . We have two maps

$$\Omega_X^{\text{ch}} \mapsto \mathcal{E}nd(h^* \Omega_X^{\text{ch},g'}) \left[ \left[ z^{\frac{1}{m_g}}, z^{\frac{-1}{m_g}} \right] \right]$$

The first is obtained as a composition

$$\Omega_X^{\text{ch}} \xrightarrow{S_h} h^* \Omega_X^{\text{ch}} \xrightarrow{h^* \mathcal{Y}^{g'}} \mathcal{E}nd(h^* \Omega_X^{\text{ch},g'}) \left[ \left[ z^{\frac{1}{m_g}}, z^{\frac{-1}{m_g}} \right] \right]$$

and the other by

$$\Omega_X^{\text{ch}} \xrightarrow{\mathcal{Y}^g} \mathcal{E}nd \Omega_X^{\text{ch},g} \left[ \left[ z^{\frac{1}{m_g}}, z^{\frac{-1}{m_g}} \right] \right] \xrightarrow{R_{g,g'}^h} \mathcal{E}nd(h^* \Omega_X^{\text{ch},g'}) \left[ \left[ z^{\frac{1}{m_g}}, z^{\frac{-1}{m_g}} \right] \right]$$

The two compositions are equal, i.e.

$$(4.8) \quad R_{g,g'}^h \circ \mathcal{Y}^g = h^* \mathcal{Y}^{g'} \circ S_h$$

Setting  $h = g$  in (4.8) shows that the operation (4.9) satisfies the  $g$ -equivariance axiom, i.e., if  $A \in \Gamma(U, \Omega_X^{\text{ch}})$  satisfies  $S_g(A) = \exp(2\pi i k/m_g)A$ , and

$$\mathbb{Y}^g(A, z^{\frac{1}{m_g}}) = \sum_{n \in \frac{1}{m_g}\mathbb{Z}} A_{(n)} z^{-n-1},$$

then  $A_{(n)} = 0$  unless  $n \in k/m_g + \mathbb{Z}$ . Therefore we obtain that  $\Omega_X^{\text{ch},g}$  is a sheaf of  $g$ -twisted modules over  $\Omega_X^{\text{ch}}$ .

Passing to the cohomology,  $\mathcal{Y}^g$  induces a map

$$(4.9) \quad \mathbb{Y}^g : H^*(X, \Omega_X^{\text{ch}}) \mapsto \text{End}(H^*(X, \Omega_X^{\text{ch},g}))[[z^{\frac{1}{m_g}}, z^{\frac{-1}{m_g}}]]$$

having the same properties. We have thus shown the following:

**Theorem 4.2.** *The map  $\mathcal{Y}^g : \Omega_X^{\text{ch}} \rightarrow \mathcal{E}nd \Omega_X^{\text{ch},g}[[z^{\frac{1}{m_g}}, z^{\frac{-1}{m_g}}]]$  gives  $\Omega_X^{\text{ch},g}$  the structure of a sheaf of  $g$ -twisted modules over  $\Omega_X^{\text{ch}}$ .*

*The corresponding map*

$$\mathbb{Y}^g : H^*(X, \Omega_X^{\text{ch}}) \mapsto \text{End}(H^*(X, \Omega_X^{\text{ch},g}))[[z^{\frac{1}{m_g}}, z^{\frac{-1}{m_g}}]]$$

*gives  $H^*(X, \Omega_X^{\text{ch},g})$  the structure of a  $g$ -twisted module over  $H^*(X, \Omega_X^{\text{ch}})$ .*

We can derive more information from (4.8). First, it shows that  $H^*(X, \Omega_X^{\text{ch},g})$  and  $H^*(X, \Omega_X^{\text{ch},g'})$  are isomorphic as (ordinary) modules over  $H^*(X, \Omega_X^{\text{ch}})^G$  (the  $G$ -invariant part of  $H^*(X, \Omega_X^{\text{ch}})$ ). Second, if  $h \in C(g)$ , then  $g = g'$ , and  $R_{g,g}^h$  is an automorphism of  $\Omega_X^{\text{ch},g}$ . Thus  $\Omega_X^{\text{ch},g}$  carries a  $C(g)$ -equivariant structure. Now (4.8) shows that  $H^*(X, \Omega_X^{\text{ch},g})^{C(g)}$  is a module over  $H^*(X, \Omega_X^{\text{ch}})^G$ , which by the first observation is isomorphic to  $H^*(X, \Omega_X^{\text{ch},g'})^{C(g')}$  if  $[g] = [g']$ .

**Theorem 4.3.** *If  $[g] = [g']$  then*

$$H^*(X, \Omega_X^{\text{ch},g'}) \cong H^*(X, \Omega_X^{\text{ch},g})$$

*as (untwisted)  $H^*(X, \Omega_X^{\text{ch}})^G$ -modules. Furthermore,  $H^*(X, \Omega_X^{\text{ch},g})^{C(g)}$  is a well-defined  $H^*(X, \Omega_X^{\text{ch}})^G$ -module, and*

$$H^*(X, \Omega_X^{\text{ch},g'})^{C(g')} \cong H^*(X, \Omega_X^{\text{ch},g})^{C(g)}$$

*as  $H^*(X, \Omega_X^{\text{ch}})^G$ -modules.*

## 5. ORBIFOLD DE RHAM COMPLEXES AND ORBIFOLD COHOMOLOGY

**5.1. Reminder on Orbifold Cohomology.** Let  $Y$  be a complex orbifold of complex dimension  $N$ . In [CR], the authors introduce the orbifold cohomology ring of  $Y$ , denoted  $H_{\text{orb}}^*(Y, \mathbb{Q})$ . The construction works for more general orbifolds, but we will henceforth restrict to the case of a global quotient  $Y = [X/G]$ , and work with coefficients in  $\mathbb{C}$ . As a  $\mathbb{C}$ -vector space,

$$H_{\text{orb}}^*(Y, \mathbb{C}) = \bigoplus_{[g] \in T} H^*(X^g/C(g), \mathbb{C})$$

where the summation runs over a set of representatives of the conjugacy classes of  $G$ . If  $p \in X_\alpha^g$ , then  $g$  acts on  $T_p X$ . Denote the order of  $g$  by  $m_g$ , and denote the eigenvalues of  $g$  by  $\exp(2\pi i m_1/m_g) \cdots \exp(2\pi i m_N/m_g)$ . Define the *fermionic shift of  $g$  at  $p$*  by

$$\iota(g, p) = \sum_i \frac{m_i}{m_g}$$

$\iota(g, p)$  is constant on each connected component of  $X^g$ , so we can denote it by  $\iota(g, \alpha)$ . The grading on  $H_{\text{orb}}^*(Y, \mathbb{C})$  is defined as follows: if  $a \in H^k(X_\alpha^g, \mathbb{C})$ ,  $i_\alpha : X_\alpha^g \hookrightarrow X^g$  the inclusion, and  $\tau : X^g \mapsto X^g/C(g)$  the quotient map, then  $\tau_* i_{\alpha,*}(a) \in H^*(X^g/C(g), \mathbb{C})$  is assigned degree  $k + 2\iota(g, \alpha)$ . In [CR], a product structure is defined on  $H_{\text{orb}}^*(Y, \mathbb{C})$ , making it into a graded ring.

**5.2. Realization of orbifold cohomology.** The BRST vector  $Q \in \widehat{\Omega}_N$  induces a BRST differential  $d^{\text{ch},g}$  on  $\widehat{\Omega}_N^g$ , given by

$$d^{\text{ch},g} = -\text{Res}_z Y^g(Q, z) = -Q_0^g = -\sum_{i=1}^N \sum_{n \in -\frac{m_i}{m_g} + \mathbb{Z}} : a_n^i \phi_{-n}^i : .$$

We have  $(d^{\text{ch},g})^2 = 0$ , and  $d^g$  increases the fermionic charge by 1. Furthermore,  $d^{\text{ch},g}$  is invariant under coordinate-changes, and gives rise to a global section of  $\mathcal{E}nd(\Omega_X^{\text{ch},g})$ . From the description of  $\Omega_{X,0}^{\text{ch},g}$  given at the end of section 4.5, we see that the pushforward of the ordinary de Rham complex of  $X^g$  sits inside of  $\Omega_X^{\text{ch},g}$  as the conformal weight 0 component.

**Theorem 5.1.** *The natural embedding of complexes*

$$(i_* \Omega_{dR}(X^g), d_{dR}) \mapsto (\Omega_X^{\text{ch},g}, d^{\text{ch},g})$$

*is compatible with the differentials and induces a quasiisomorphism.*

*Proof.* The proof is the same as that given in [MSV]. The operators  $G^g$ ,  $d^{\text{ch},g}$  and  $L_0^g$  are  $\text{Aut } \mathcal{O}^g$ -invariant, and so give rise to global sections of  $\mathcal{E}nd \Omega_X^{\text{ch},g}$ . We have  $[G_0^g, d^g] = L_0^g$ , and so  $L_0^g$  induces a contracting homotopy on all subcomplexes of non-zero conformal weight, since it acts semi-simply by a non-zero constant. It follows from the observation at the end of section 4.5 that  $\Omega_{X,0}^{\text{ch},g}$  is isomorphic to  $i_* \Omega_{dR}(X^g)$ . Furthermore, on  $\Omega_{X,0}^{\text{ch},g}$ ,  $d^{\text{ch},g}$  agrees with the ordinary de Rham differential, and so the theorem follows.  $\square$

Note that since  $i : X^g \mapsto X$  is a closed embedding,  $H^*(X, i_* \mathcal{F}) \cong H^*(X^g, \mathcal{F})$  for any sheaf  $\mathcal{F}$  of abelian groups on  $X^g$ .

The  $C(g)$ -action on  $\Omega_X^{\text{ch},g}$  induces a  $C(g)$ -action on the conformal weight zero component which agrees with the natural action on  $i_* \Omega(X^g)$ . It follows from Theorem 5.1 that the hypercohomology of the complex of sheaves  $(\Omega_X^{\text{ch},g}, d^{\text{ch},g})$  on  $X$ , which we denote by  $\mathbb{H}^*(X, \Omega_X^{\text{ch},g})$ , is equal to

$$\mathbb{H}^*(X, \Omega_X^{\text{ch},g}) = H^*(X, i_* \Omega_{dR}(X^g)) = H^*(X^g, \mathbb{C})$$

Taking  $C(g)$ -invariants, we obtain

$$\mathbb{H}^*(X, \Omega_X^{\text{ch},g})^{C(g)} = H^*(X^g, \mathbb{C})^{C(g)} = H^*(X^g/C(g)).$$

Moreover, under this isomorphism the natural vertex algebra module gradation on the left hand side of this isomorphism gives rise to precisely the Chen-Ruan gradation, due to formula (4.4). Thus, we obtain the following:

**Theorem 5.2.** *The direct sum of the spaces of  $C(g)$ -invariants in the cohomologies of the orbifold de Rham complexes is isomorphic as a graded vector space to the orbifold cohomology of  $[X/G]$ , i.e.,*

$$\bigoplus_{[g] \in T} \mathbb{H}^*(X, \Omega_X^{\text{ch},g})^{C(g)} = \bigoplus_{[g] \in T} H^*(X^g/C(g)).$$

Note that taking a resolution of  $(\Omega_X^{\text{ch},g}, d^{\text{ch},g})$ , we obtain a spectral sequence converging to  $\mathbb{H}^*(X, \Omega_X^{\text{ch},g})$  with the second term consisting of the cohomologies of the complexes  $\mathcal{C}^{i,\bullet} = \bigoplus_j \mathcal{C}^{i,j}$ ,  $\mathcal{C}^{i,j} = H^i(X, \Omega_X^{\text{ch},g,j})$  on  $X$ , with respect to the differential induced by  $d^{\text{ch},g}$ . Following [MS2], we call this the chiral ‘‘Hodge-to-de Rham’’ spectral sequence. We will denote the complex  $\mathcal{C}^{i,\bullet}$  simply by  $H^*(X, \Omega_X^{\text{ch},g})$  and set

$$(5.1) \quad \mathcal{H} = \bigoplus_{[g] \in T} H^*(X, \Omega_X^{\text{ch},g})^{C(g)}.$$

As shown in [MS2], the above spectral sequence coincides with the ordinary Hodge-to-de Rham spectral sequence from the  $E^2$  term on. Thus, if  $X$  is compact and Kahler, it collapses at  $E^2$ , and we have an isomorphism of graded vector spaces

$$H^*(\mathcal{H}, d^{\text{ch},g}) \cong H_{\text{orb}}^*([X/G], \mathbb{C}).$$

## 6. ORBIFOLD ELLIPTIC GENUS

The orbifold elliptic genus for global quotient orbifolds was introduced in [BL1]. Recall that we have defined in (4.7) for each component  $X_\alpha^g$  of  $X^g$  a graded vector bundle  $V_{g,\alpha}$ . By definition,

$$(6.1) \quad Ell_{\text{orb}}(X, G; y, q) = y^{-\dim(X)/2} \sum_{[g] \in T, X_\alpha^g} y^{\iota(g,\alpha)} \frac{1}{|C(g)|} \sum_{h \in C(g)} L(h, V_{g,\alpha}),$$

where the sum is over all conjugacy classes of  $G$  and connected components  $X_\alpha^g$  of  $X^g$ , and  $L(h, V_{g,\alpha})$  is the holomorphic Lefschetz number

$$L(h, V_{g,\alpha}) = \sum_i (-1)^i \text{Tr}(h, H^i(X^g, V_{g,\alpha})).$$

Using the notation (5.1), we have

**Theorem 6.1.** *We have*

$$Ell_{\text{orb}}(X, G; y, q) = y^{-\dim(X)/2} \text{Tr}_{\mathcal{H}} y^{\tilde{J}_0} q^{\tilde{L}_0},$$

where  $\tilde{J}_0$  (resp.  $\tilde{L}_0$ ) is the operator which is equal to  $J_0^g$  (resp.  $L_0^g$ ) on  $H^*(X, \Omega_X^{\text{ch},g})^{C(g)}$ .

*Proof.* The result follows from the additivity of traces and Euler characteristics and the exactness of the functor of  $C(g)$ -invariants. It follows from the Schur relations that

$$\frac{1}{|C(g)|} \sum_{h \in C(g)} \text{Tr}(h, H^i(X^g, V_{g,\alpha})) = \dim H^i(X^g, V_{g,\alpha})^{C(g)}.$$

Therefore

$$(6.2) \quad \text{Ell}_{\text{orb}}(X, G; y, q) = y^{-\dim(X)/2} \sum_{[g] \in T, X_\alpha^g} \sum_i (-1)^i \dim H^i(X_\alpha^g, V_{g,\alpha})^{C(g)}.$$

Let

$$F^{l,k} \Omega_X^{\text{ch},g} := F^l \Omega_X^{\text{ch},g} \cap \Omega_{X,k}^{\text{ch},g}$$

and for a fixed conformal weight  $k$ , consider the long exact sequence corresponding to the short exact sequence

$$0 \mapsto F^{l,k} \Omega_X^{\text{ch},g} \mapsto F^{l+1,k} \Omega_X^{\text{ch},g} \mapsto F^{l+1,k} \Omega_X^{\text{ch},g} / F^{l,k} \Omega_X^{\text{ch},g} \mapsto 0.$$

Taking  $C(g)$ -invariants keeps it exact, and applying alternating traces yields

$$\begin{aligned} & \sum_i (-1)^i \text{Tr}(y^{J_0^g} q^{L_0^g}, H^i(X, F^{l+1,k} \Omega_X^{\text{ch},g})^{C(g)}) - \sum_i (-1)^i \text{Tr}(y^{J_0^g} q^{L_0^g}, H^i(X, F^{l,k} \Omega_X^{\text{ch},g})^{C(g)}) \\ &= \sum_i (-1)^i \text{Tr}(y^{J_0^g} q^{L_0^g}, H^i(X, F^{l+1,k} \Omega_X^{\text{ch},g} / F^{l,k} \Omega_X^{\text{ch},g})^{C(g)}) \end{aligned}$$

Using the last equation and (4.3), we obtain

$$\sum_i (-1)^i \dim H^i(X^g, V_{g,\alpha})^{C(g)} = \sum_i (-1)^i \dim H^i(X^g, \Omega_\alpha^{\text{ch},g}),$$

and the statement of the theorem follows.  $\square$

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