ON THE HALL ALGEBRA OF COHERENT SHEAVES ON \( \mathbb{P}^1 \) OVER \( \mathbb{F}_1 \)

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Abstract. Using a model of schemes over \( \mathbb{F}_1 \) in the spirit of Deitmar and Haran, we study the category \( \text{Coh}(\mathbb{P}^1) \) of coherent sheaves on \( \mathbb{P}^1 \) over \( \mathbb{F}_1 \). This category resembles in most ways a finitary abelian category, but is not additive. As an application, we define and study the Hall algebra of \( \text{Coh}(\mathbb{P}^1) \). We show that it is isomorphic as a Hopf algebra to the enveloping algebra of a non-standard Borel in the loop algebra \( L_{\mathfrak{gl}2} \). This should be viewed as a \((q = 1)\) version of Kapranov’s result relating (a certain subalgebra of) the Ringel-Hall algebra of \( \mathbb{P}^1 \) over \( \mathbb{F}_q \) to a non-standard quantum Borel inside the quantum loop algebra \( \mathcal{U}_\nu(\widehat{\mathfrak{sl}2}) \), where \( \nu^2 = q \).

1. Introduction

If \( \mathcal{A} \) is an abelian category defined over a finite field \( \mathbb{F}_q \), and finitary in the sense that \( \text{Hom}(M,N) \) and \( \text{Ext}^1(M,N) \) are finite-dimensional \( \forall M, N \in \mathcal{A} \), one can attach to it an associative algebra \( \mathcal{H}(\mathcal{A}) \) defined over the field \( \mathbb{Q}(\nu) \), \( \nu = \sqrt{q} \), called the Ringel-Hall algebra of \( \mathcal{A} \). As a \( \mathbb{Q}(\nu) \)-vector space, \( \mathcal{H}(\mathcal{A}) \) is spanned by the isomorphism classes of objects in \( \mathcal{A} \), and its structure constants are expressed in terms of the number of extensions between objects. Under additional assumptions on \( \mathcal{A} \), it can be given the structure of a Hopf algebra (see [14]).

Let \( X \) be a smooth projective curve over \( \mathbb{F}_q \). It is known that the abelian category \( \text{Coh}(X) \) of coherent sheaves on \( X \) is finitary, and one can therefore consider its Ringel-Hall algebra \( \mathcal{H}(X) \). This algebra was studied by Kapranov in the important paper [10] (see also [1]), in the context of automorphic forms over the function field \( \mathbb{F}_q(X) \). Let \( L_{\mathfrak{sl}2} := \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \) be the loop algebra of \( \mathfrak{sl}_2 \), and \( \mathcal{U}_\nu(L_{\mathfrak{sl}2}) \) the corresponding quantum loop algebra (see [14]). Denote by \( L_{\mathfrak{sl}_2^+} \) the ”positive” subalgebra spanned by \( e \otimes t^k \) and \( h \otimes t^l \), \( k \in \mathbb{Z}, l \in \mathbb{N} \), and let \( \mathcal{U}_\nu(L_{\mathfrak{sl}_2^+}) \) be the corresponding deformation of the enveloping algebra \( \mathcal{U}(L_{\mathfrak{sl}_2^+}) \) inside \( \mathcal{U}_\nu(L_{\mathfrak{sl}_2}) \). In the case \( X = \mathbb{P}^1 \), Kapranov shows in [10] that there exists an embedding of bialgebras

\[
\Psi : \mathcal{U}_\nu(L_{\mathfrak{sl}_2^+}) \rightarrow \mathcal{H}(\mathbb{P}^1).
\]

In this paper, we study the category of coherent sheaves on \( \mathbb{P}^1 \) over \( \mathbb{F}_1 \), the field of one element, using a notion of schemes over \( \mathbb{F}_1 \) based on work of Deitmar ([5]) and Haran ([8]). A scheme \( X \) over \( \mathbb{F}_1 \) is defined as a topological space locally modeled on the spectrum of a commutative unital monoid with 0, carrying a structure sheaf of commutative monoids \( \mathcal{O}_X \). One defines coherent sheaves on \( X \) as sheaves of pointed sets carrying an action of \( \mathcal{O}_X \), which are locally finitely generated in a suitable sense.

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We will denote the corresponding category by $\text{Coh}(X_{\mathbb{F}_1})$. In this framework, one can make sense of most of the usual notions of algebraic geometry, such as locally free sheaves, line bundles, and torsion sheaves. The category $\text{Coh}(X_{\mathbb{F}_1})$ behaves very much like a finitary abelian category except that $\text{Hom}(\mathcal{F}, \mathcal{G})$ only has the structure of a pointed set (which corresponds to the notion of $\mathbb{F}_1$–vector space). We show that as in the case of $\mathbb{P}^1$ over a field, every coherent sheaf is a direct sum of a locally free and a torsion sheaf, and that moreover, locally free sheaves are direct sums of line bundles.

As an application, we define the Hall algebra $\mathcal{H}(\mathbb{P}^1_{\mathbb{F}_1})$ of $\text{Coh}(\mathbb{P}^1_{\mathbb{F}_1})$, and describe its structure. Letting $L\mathfrak{gl}^+_2 = (\mathfrak{d} \otimes t\mathbb{C}[t]) \oplus (e \otimes \mathbb{C}[t, t^{-1}])$, where $\mathfrak{d} := \text{span}\left\{ h_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, h_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ and $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, the main result is the following:

**Theorem 1.**

$$\mathcal{H}(\mathbb{P}^1_{\mathbb{F}_1}) \cong \mathcal{U}(L\mathfrak{gl}^+_2).$$

This result should be naturally viewed as the $q = 1$ version of Kapranov’s theorem. As seen in this paper, the category $\text{Coh}(X_{\mathbb{F}_1})$ is much simpler than $\text{Coh}(X_{\mathbb{F}_1} \otimes \mathbb{F}_q)$, where $X_{\mathbb{F}_1} \otimes \mathbb{F}_q$ denotes the base-change of the scheme $X_{\mathbb{F}_1}$ to $\mathbb{F}_q$. For instance, $\mathbb{P}^1$ over $\mathbb{F}_1$ possesses three points - two closed points $\{0, \infty\}$, and a generic point $\eta$. The category $\text{Coh}(X_{\mathbb{F}_1})$ is thus essentially combinatorial in nature. This suggests a possible application of the ideas of algebraic geometry over $\mathbb{F}_1$ to the study of Hall algebras of higher-dimensional varieties. The problem of studying the Hall algebra of $\text{Coh}(\mathbb{P}^k_{\mathbb{F}_1})$ for $k > 1$ seems somewhat difficult, while that of $\text{Coh}(\mathbb{P}^1_{\mathbb{F}_1})$ may be much more manageable. The latter should be viewed as a degenerate (i.e. $q = 1$) version of the original object, and would give some hints as to its structure.

The paper is structured as follows. In section 2 we recall basic notions related to schemes over $\mathbb{F}_1$ and coherent sheaves on them. In section 3, we study the category $\text{Coh}(\mathbb{P}^1_{\mathbb{F}_1})$ and establish basic structural results. In section 4 we introduce the Hall algebra $\mathcal{H}(\mathbb{P}^1_{\mathbb{F}_1})$ of $\text{Coh}(\mathbb{P}^1_{\mathbb{F}_1})$. Finally, in section 5 we describe the structure of $\mathcal{H}(\mathbb{P}^1_{\mathbb{F}_1})$ and prove Theorem 1.

**Remark 1.** In the remainder of this paper, unless otherwise indicated, all schemes will be over $\mathbb{F}_1$, and we will omit the subscript $\mathbb{F}_1$, which was used in the introduction for emphasis and contrast. Thus instead of writing $X_{\mathbb{F}_1}, \mathbb{P}^1_{\mathbb{F}_1}, \mathbb{A}^1_{\mathbb{F}_1}, \text{Coh}(X_{\mathbb{F}_1})$, etc. we will write simply $X, \mathbb{P}^1, \mathbb{A}^1, \text{Coh}(X)$ etc.

2. Schemes over $\mathbb{F}_1$

In this section, we briefly recall the notion of a scheme over $\mathbb{F}_1$. We essentially follow the approach of [5], with some minor differences (for instance, we work with monoids with 0, and use a somewhat different definition of the category of coherent sheaves). Our approach is also equivalent to a special case of the notion of scheme over $\mathbb{F}_1$ developed in [8]. Please consult the references for details. For other approaches to schemes over $\mathbb{F}_1$, see [2, 3, 4, 7, 9, 15, 17].
We require structure sheaf with a where which are spectra of commutative rings. A scheme over \( \mathbb{F}_1 \) is locally modeled on an affine \( \mathbb{F}_1 \)-scheme, which is the spectrum of a commutative unital monoid with 0. In the following, we will denote monoid multiplication by juxtaposition or \( \cdot \). In greater detail:

A monoid \( A \) will be a commutative associative monoid with identity \( 1_A \) and zero \( 0_A \). We require

\[
1_A \cdot a = a \cdot 1_A = a \quad \quad 0_A \cdot a = a \cdot 0_A = 0_A \quad \forall a \in A
\]

Maps of monoids are required to respect the multiplication as well as the special elements \( 1_A, 0_A \). An ideal of \( A \) is a subset \( a \subset A \) such that \( a \cdot A \subset a \). An ideal \( p \subset A \) is prime if \( xy \in p \) implies either \( x \in p \) or \( y \in p \).

Given a monoid \( A \), the topological space \( \text{Spec} A \) is defined to be the set

\[
\text{Spec} A := \{ p \mid p \subset A \text{ is a prime ideal } \},
\]

with the closed sets of the form

\[
V(a) := \{ p \mid a \subset p, p \text{ prime } \},
\]

together with the empty set. Given a multiplicatively closed subset \( S \subset A \), the localization of \( A \) by \( S \), denoted \( S^{-1}A \), is defined to be the monoid consisting of symbols \( \frac{a}{s} | a \in A, s \in S \}, with the equivalence relation

\[
\frac{a}{s} \equiv \frac{a'}{s'} \iff \exists s'' \in S \text{ such that } as'' = a's''
\]

and multiplication is given by \( \frac{a}{s} \times \frac{a'}{s'} = \frac{aa'}{ss'} \).

For \( f \in A \), let \( S_f \) denote the multiplicatively closed subset \{1, f, f^2, f^3, \cdots \}. We denote by \( A_f \) the localization \( S_f^{-1}A \), and by \( D(f) \) the open set \( \text{Spec} A \backslash V(f) \cong \text{Spec} A_f \), where \( V(f) := \{ p \in \text{Spec} A | f \in p \} \). The open sets \( D(f) \) cover \( \text{Spec} A \). Spec \( A \) is equipped with a structure sheaf of monoids \( O_A \), satisfying the property \( \Gamma(D(f), O_A) = A_f \). Its stalk at \( p \in \text{Spec} A \) is \( A_p := S_p^{-1}A \), where \( S_p = A \backslash p \).

A unital homomorphism of monoids \( \phi : A \to B \) is local if \( \phi^{-1}(B^\times) \subset A^\times \), where \( A^\times \) (resp. \( B^\times \) denotes the invertible elements in \( A \) (resp. \( B \)). A monoidal space is a pair \( (X, O_X) \) where \( X \) is a topological space and \( O_X \) is a sheaf of monoids. A morphism of monoidal spaces is a pair \( (f, f^\#) \) where \( f : X \to Y \) is a continuous map, and \( f^\# : O_Y \to f_*O_X \) is a morphism of sheaves of monoids, such that the induced morphism on stalks \( f^\#: O_{Y, f(y)} \to f_*O_{X, x} \) is local. An affine \( \mathbb{F}_1 \)-scheme is a monoidal space isomorphic to \( (\text{Spec} A, O_A) \). Thus, the category of affine \( \mathbb{F}_1 \)-schemes is opposite to the category of monoids. A monoidal space \( (X, O_X) \) is called a scheme over \( \mathbb{F}_1 \), if for every point \( x \in X \) there is an open neighborhood \( U_x \subset X \) containing \( x \) such that \( (U_x, O_{X|U_x}) \) is an affine scheme over \( \mathbb{F}_1 \).

**Example 1.** Denote by \( \langle t \rangle \) the free commutative unital monoid with zero generated by \( t \), i.e.

\[
\langle t \rangle := \{0, 1, t, t^2, t^3, \cdots, t^n, \cdots\},
\]

and let \( \mathbb{A}^1 := \text{Spec} \langle t \rangle \) - the affine line over \( \mathbb{F}_1 \). Let \( \langle t, t^{-1} \rangle \) denote the monoid

\[
\langle t, t^{-1} \rangle := \{\cdots, t^{-2}, t^{-1}, 0, t, t^2, t^3, \cdots\}.
\]
We define \( P \) and obtain the following diagram of inclusions

\[
\langle t \rangle \hookrightarrow \langle t, t^{-1} \rangle \hookrightarrow \langle t^{-1} \rangle.
\]

Taking spectra, and denoting by \( U_0 = \text{Spec} \langle t \rangle, U_\infty = \text{Spec} \langle t^{-1} \rangle \), we obtain

\[
A^1 = U_0 \hookrightarrow U_0 \cap U_\infty \hookrightarrow U_\infty = A^1
\]

We define \( \mathbb{P}^1 \), the projective line over \( \mathbb{F}_1 \), to be the scheme obtained by gluing two copies of \( A^1 \) according to the diagram 2. It has three points - two closed points \( 0 \in U_0, \infty \in U_\infty \), and the generic point \( \eta \). Denote by \( t_0 : U_0 \hookrightarrow \mathbb{P}^1, t_\infty : U_\infty \hookrightarrow \mathbb{P}^1 \) the corresponding inclusions.

Given a commutative ring \( R \), there exists a base-change functor

\[
\otimes R : \text{Sch} / \mathbb{F}_1 \rightarrow \text{Sch} / \text{Spec} R
\]

On affine schemes, \( \otimes R \) is defined by setting

\[
\otimes R(\text{Spec} A) = \text{Spec} R[A]
\]

where \( R[A] \) is the monoid algebra:

\[
R[A] := \left\{ \sum r_i a_i | a_i \in A, a_i \neq 0, r_i \in R \right\}
\]

with multiplication induced from the monoid multiplication. For a general scheme \( X \) over \( \mathbb{F}_1 \), \( \otimes R \) is defined by gluing the open affine subfunctors of \( X \). We denote \( \otimes R(X) \) by \( X \otimes R \).

2.1. **Coherent sheaves.** Let \( A \) be a monoid. An \( A \)-module is a pointed set \((M, \ast_M)\) together with an action

\[
\mu : A \times M \rightarrow M
\]

\[(a, m) \mapsto a \cdot m
\]

which is compatible with the monoid multiplication (i.e. \( 1_A \cdot m = m, a \cdot (b \cdot m) = (a \cdot b) \cdot m \), and \( 0_A \cdot m = \ast_M \forall m \in M \)). We will refer to elements of \( M \setminus \ast_M \) as nonzero elements.

A morphism of \( A \)-modules \( f : (M, \ast_M) \rightarrow (N, \ast_N) \) is a map of pointed sets (i.e. we require \( f(\ast_M) = \ast_N \)) satisfying the following two properties:

- \( f(a \cdot m) = a \cdot f(m) \) i.e. \( f \) is compatible with the action of \( A \).
- \( f|_{M \setminus \ast_N} \) is injective.

**Remark 2.** Our definition of \( A \)-module differs somewhat from that in [5].

A pointed subset \((M', \ast_M) \subset (M, \ast_M)\) is called an \( A \)-sub-module if \( \mu(A, M') \subset M' \). In this case we may form the quotient module \( M/M' \), where \( M/M' := M \setminus (M' \setminus \ast_M) \), \( \ast_{M/M'} = \ast_M \), and the action of \( A \) is defined by setting

\[
a \cdot \overline{m} = \left\{
\begin{array}{ll}
\overline{a \cdot m} & \text{if } a \cdot m \notin M' \\
\ast_{M/M'} & \text{if } a \cdot m \in M'
\end{array}
\right.
\]

where \( \overline{m} \) denotes \( m \) viewed as an element of \( M/M' \). If \( M \) is finite, we define \( |M| = \#M - 1 \), i.e. the number of non-zero elements.
Denote by $A \text{–mod}$ the category of $A$–modules. It has the following properties:

1. $A \text{–mod}$ has a zero object $\emptyset = \{\star\}$ - the one-element pointed set.
2. A morphism $f : (M, *_M) \to (N, *_N)$ has a kernel $(f^{-1}(*_N), *_N)$ and a cokernel $\mathbb{N}/ \operatorname{Im}(f)$.
3. $A \text{–mod}$ has a symmetric monoidal structure $M \oplus N := M \vee N := M \bigsqcup N/ *_M \sim *_N$ which we will call "direct sum".
4. If $R \subset M \oplus N$ is an $A$–submodule, then $R = (R \cap M) \oplus (R \cap N)$.
5. $A \text{–mod}$ has a symmetric monoidal structure $M \otimes N := M \wedge N := M \times N/ \sim$, where $\sim$ is the equivalence relation generated by $(a \cdot m, n) \sim (m, a \cdot n)$.
6. $\oplus, \otimes$ satisfy the usual associativity and distributivity properties.

$M \in A \text{–mod}$ is finitely generated if there exists a surjection $\oplus_{i=1}^n A \twoheadrightarrow M$ of $A$–modules for some $n$. Explicitly, this means that there are $m_1, \cdots, m_n \in M$ such that for every $m \in M$, $m = a \cdot m_i$, for some $1 \leq i \leq n$, and we refer to the $m_i$ as generators. $M$ is said to be free of rank $n$ if $M \cong \oplus_{i=1}^n A$. For an element $m \in M$, define $\operatorname{Ann}_A(m) := \{a \in A | a \cdot m = *_M\}$.

Obviously $0_A \subset \operatorname{Ann}_A(m) \forall m \in M$. An element $s \in S$ is torsion if $\operatorname{Ann}_A(s) \neq 0_A$. The subset of all torsion elements in $M$ forms an $A$–submodule, called the torsion submodule of $M$, and denoted $M_{\text{tor}}$. An $A$–module is torsion-free if $M_{\text{tor}} = \{*_M\}$ and torsion if $M_{\text{tor}} = M$. Every $M$ can be uniquely written $M = M_{\text{tor}} \oplus M_{\text{ff}}$, where $M_{\text{ff}}$ is torsion-free. We define the length of a torsion module $M$ to be $|M|$.

Given a multiplicatively closed subset $S \subset A$ and an $A$–module $M$, we may form the $S^{-1}A$–module $S^{-1}M$, where

$$S^{-1}M := \{ \frac{m}{s} | m \in M, s \in S \}$$

with the equivalence relation

$$\frac{m}{s} = \frac{m'}{s'} \iff \exists s'' \in S \text{ such that } s's''m = ss'm',$$

where the $S^{-1}A$–module structure is given by $\frac{a}{s} \cdot \frac{m}{s'} := \frac{am}{ss'}$. For $f \in A$, we define $M_f$ to be $S_f^{-1}M$.

Let $X$ be a topological space, and $\mathcal{A}$ a sheaf of monoids on $X$. We say that a sheaf of pointed sets $\mathcal{M}$ is an $\mathcal{A}$–module if for every open set $U \subset X$, $\mathcal{M}(U)$ has the structure of an $\mathcal{A}(U)$–module with the usual compatibilities. In particular, given a monoid $A$ and an $A$–module $M$, there is a sheaf of $\mathcal{O}_{\text{Spec} A}$–modules $\tilde{M}$ on $\text{Spec} A$, defined on basic affine sets $D(f)$ by $\tilde{M}(D(f)) := M_f$. For a scheme $X$ over $\mathbb{F}_1$, a sheaf of $\mathcal{O}_X$–modules $\mathcal{F}$ is said to be quasicoherent if for every $x \in X$ there exists an open affine $U_x \subset X$ containing $x$ and an $\mathcal{O}_X(U_x)$–module $M$ such that $\mathcal{F}|_{U_x} \cong \tilde{M}$. $\mathcal{F}$ is said to be coherent if $M$ can always be taken to be finitely generated, and locally free if $M$ can be taken to be free. Please note that here too our conventions are different from [5]. For a monoid $A$, there is an equivalence of categories between the category of quasicoherent sheaves on $\text{Spec} A$ and the category of $A$–modules, given by $\Gamma(\text{Spec} A, \cdot)$. A coherent sheaf $\mathcal{F}$
on $X$ is torsion (resp. torsion-free) if $F(U)$ is a torsion $O_X(U)$–module (resp. torsion-free $O_X(U)$–module ) for every open affine $U \subset X$. If $X$ is connected, we can define the rank of a locally free sheaf $F$ to be the rank of the stalk $F_x$ as an $O_X,x$–module for any $x \in X$. A locally free sheaf of rank one will be called a line bundle.

**Remark 3.** It follows from property (4) of the category $A - mod$ that if $F, F'$ are quasicoherent $O_X$–modules, and $G \subset F \oplus F'$ is an $O_X$–submodule, then $G = (G \cap F) \oplus (G \cap F')$, where for an open subset $U \subset X$,

$$\text{(3)} \quad (G \cap F)(U) := G(U) \cap F(U).$$

If $X$ is a scheme over $\mathbb{F}_1$, we will denote by $\text{Coh}(X)$ the category of coherent $O_X$–modules on $X$. It follows from the properties of the category $A - mod$ listed in section 2.1 that $\text{Coh}(X)$ possesses a zero object $\emptyset$ (defined as the zero module $\emptyset$ on each open affine $\text{Spec} A \subset X$), kernels and co-kernels, as well as monoidal structures $\oplus$ and $\otimes$. We may therefore talk about exact sequence in $\text{Coh}(X)$. A short exact sequence isomorphic to one of the form

$$\emptyset \to F \to F \oplus G \to G \to \emptyset$$

is called split. A coherent sheaf $F$ which cannot be written as $F = F' \oplus F''$ for non-zero coherent sheaves is called indecomposable.

**Remark 4.** Because of the property of $\text{Coh}(X)$ discussed in Remark 3, it follows that if $F$ is an indecomposable coherent sheaf, and $F \subset F' \oplus F''$, then $F \subset F'$ or $F \subset F''$.

We will make use of the gluing construction for coherent sheaves. Namely, suppose that $U = \{U_i\}_{i \in I}$ is an affine open cover of $X$, and suppose we are given for each $i \in I$ a sheaf $F_i$ on $U_i$, and for each $i, j \in I$ an isomorphism $\phi_{ij} : F_i|_{U_i \cap U_j} \to F_j|_{U_i \cap U_j}$ such that

1. $\phi_{ii} = id$
2. For each $i, j, k \in I$, $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_i \cap U_j \cap U_k$

Then there exists a unique sheaf $F$ on $X$ together with isomorphisms $\psi_i : F_i \to F$, such that for each $i, j \psi_j = \phi_{ij} \circ \psi_i$ on $U_i \cap U_j$. If moreover, $F_i$ are coherent and $\phi_{ij}$ isomorphisms of coherent $O_X$–modules, then $F$ is itself coherent.

### 3. Coherent sheaves on $\mathbb{P}^1$

In this section, we study coherent sheaves on $\mathbb{P}^1$. Recall from example 1 that $\mathbb{P}^1$ is obtained by gluing two copies of $\mathbb{A}^1$ labeled $U_0, U_\infty$, and has three points - $0, \infty$, and $\eta$, where the first two are closed, and $\eta$ is the generic point. Since the only open set of $\mathbb{A}^1$ containing the closed point is all of $\mathbb{A}^1$, any locally free sheaf on $\mathbb{A}^1$ is trivial. Observe furthermore that $\text{Aut}_{\mathbb{A}^1}(\mathcal{O}_{\mathbb{A}^1}) = S_n$ (i.e. any automorphism of a free module is determined by a permutation of the generators).

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3.1. Building blocks of \( \text{Coh}(\mathbb{P}^1) \). We proceed to introduce the building blocks of \( \text{Coh}(\mathbb{P}^1) \) - the line bundles \( \mathcal{O}(n) \) and the torsion sheaves \( \mathcal{T}_{x,n}, \ x = 0, \infty \).

- Since \( \text{Aut}(\mathcal{O}_A) = \{1\} \), the data of a line bundle on \( \mathbb{P}^1_{\mathbb{F}_1} \) corresponds to an \( \mathcal{O}_{\mathbb{P}^1} \)-linear clutching isomorphism
  \[
  \tau : \mathcal{O}_{U_0}|_{U_0 \cap U_\infty} \to \mathcal{O}_{U_\infty}|_{U_0 \cap U_\infty},
  \]
  i.e. an automorphism \( \tau \) of \( \{\cdots t^{-2}, t^{-1}, 0, 1, t, t^2, \cdots\} \) equivariant with respect to the action of \( \langle t, t^{-1} \rangle \) - these are of the form \( \tau_m(t^k) = t^{k - m} \), and are therefore determined by an integer \( m \in \mathbb{Z} \). Denote by \( \mathcal{O}(m) \) the line bundle obtained from the clutching map \( \tau_m \) (we have chosen the convention so that \( \mathcal{O}(\mathbb{P}^1, \mathcal{O}(m)) \neq \emptyset \) for \( m \geq 0 \).

- For \( n \in \mathbb{N} \), let \( C_n \) denote the \( \langle t \rangle \)-submodule \( \{t^n, t^{n+1}, \cdots\} \) of \( \langle t \rangle \). Let \( T_n \) denote the quotient \( \langle t \rangle \)-module \( \langle t \rangle/C_n \), and \( \mathcal{T}_n \) the corresponding coherent sheaf on \( \mathbb{A}^1 \). This is a torsion sheaf of length \( n \). Denote by \( \mathcal{T}_{0,n} \) (resp. \( \mathcal{T}_{\infty,n} \)) the torsion sheaves \( t_{0,\infty}(\mathcal{T}_n) \) (resp. \( t_{\infty,\infty}(\mathcal{T}_n) \)) supported at 0 (resp. \( \infty \)).

**Remark 5.** If \( m \leq n \), then the \( \langle t \rangle \)-module \( T_n \) contains a unique submodule isomorphic to \( T_m \) (given by \( \{t^m, t^{m+1}, \cdots, t^{n-1}, 0\} \)) and we have a short exact sequence
\[
\emptyset \to T_m \to T_n \to T_{n-m} \to \emptyset \tag{4}
\]
which yields corresponding short exact sequences of torsion sheaves:
\[
\emptyset \to \mathcal{T}_{x,m} \to \mathcal{T}_{x,n} \to \mathcal{T}_{x,n-m} \to \emptyset \tag{5}
\]
for \( n \geq m \), and \( x = 0, \infty \).

We proceed to prove a few short lemmas establishing the structure of the category \( \text{Coh}(\mathbb{P}^1) \).

**Lemma 1.** Let \( X \) be a scheme over \( \mathbb{F}_1 \), and \( \mathcal{F} \) a coherent sheaf on \( X \). Then
\[
\mathcal{F} = \mathcal{F}_{\text{tor}} \oplus \mathcal{F}_{\text{tf}}, \tag{6}
\]
where \( \mathcal{F}_{\text{tor}} \) is a torsion sheaf and \( \mathcal{F}_{\text{tf}} \) is torsion-free.

**Proof.** If \( X \) is affine, the statement is trivial. It suffices to check that the decomposition \( 6 \) persists under the gluing of coherent sheaves, but this is clear since it is compatible with localization - i.e. if \( M \) is an \( A \)-module, \( M = M_{\text{tor}} \oplus M_{\text{tf}} \) its decomposition into a direct sum of torsion and torsion-free components, and \( S \) a multiplicatively closed subset of \( A \), then \( (S^{-1}M)_{\text{tor}} = S^{-1}M_{\text{tor}} \) and \( (S^{-1}M)_{\text{tf}} = S^{-1}M_{\text{tf}} \). \( \square \)

**Lemma 2.** On \( \mathbb{P}^1 \) every torsion-free coherent sheaf is locally free.

**Proof.** The statement will follow if we can show that every torsion-free coherent sheaf on \( \mathbb{A}^1_{\mathbb{F}_1} \) is free. Coherent sheaves on \( \mathbb{A}^1 \) are of the form \( \mathcal{M} \) for some finitely generated \( \langle t \rangle \)-module \( M \). Let
\[
\emptyset \to K \to \oplus_{i=1}^n \langle t_i \rangle \to M \to \emptyset
\]
be a resolution of $M$. By property (4) of section 2.1, we have

$$K = \phi_{i=1}^n (K \cap \langle t_i \rangle),$$

and so

$$M = \phi_{i=1}^n (\langle t_i \rangle) / (K \cap \langle t_i \rangle).$$

The only $\langle t \rangle$-submodules of $\langle t \rangle$ are of the form $C_n$, and $\langle t \rangle / C_n \cong T_n$. Since $M$ is torsion-free, we must have

$$K \cap \langle t_i \rangle = 0 \text{ or } K \cap \langle t_i \rangle = \langle t_i \rangle,$$

which proves that $M$ is free. \hfill \Box

**Lemma 3.** Every torsion sheaf on $\mathbb{P}^1$ is a direct sum of sheaves $\mathcal{T}_{0,n}$ and $\mathcal{T}_{\infty,n}$ for $m, n \geq 0$.

**Proof.** The proof of the preceding lemma shows that every coherent torsion sheaf on $\mathbb{A}^1 = \text{Spec}(\langle t \rangle)$ is of the form $\phi_{i=1}^k \mathcal{T}_{0,n_i}$. Since $\mathcal{T}_{0,n} |_{U_0 \cap U_\infty} = 0$, the gluing data for torsion sheaves on $U_0$ and $U_\infty$ is trivial, and so the statement follows. \hfill \Box

**Lemma 4.** Every locally free sheaf on $\mathbb{P}^1$ is of the form $\mathcal{O}(n_1) \oplus \mathcal{O}(n_2) \oplus \cdots \oplus \mathcal{O}(n_k)$ for integers $n_1, \ldots, n_k \in \mathbb{Z}$.

**Proof.** A locally free sheaf $\mathcal{F}$ on $\mathbb{P}^1$ of rank $k$ is determined by its gluing data on $U_0 \cap U_\infty$, which is a $\langle t, t^{-1} \rangle$-linear automorphism $\phi_{0,\infty}$ of $\phi_{i=1}^k \mathcal{T}_{0,n_i}$. Every such is of the form $\phi_{0,\infty}(t_i^m) = t_i^{m_i-n_i}$, where $\sigma \in S_k$ is a permutation, and $n_i \in \mathbb{Z}$. We thus have $\mathcal{F} \cong \phi_{i=1}^k \mathcal{O}(n_i)$. \hfill \Box

Combining the results of Lemmas 4 and 3, we obtain:

**Corollary 1.** The only indecomposable sheaves on $\mathbb{P}^1$ are $\mathcal{O}(k)$, $k \in \mathbb{Z}$, and $\mathcal{T}_{0,n}, \mathcal{T}_{\infty,m}$, $m, n \in \mathbb{N}$.

**Lemma 5.**

(7) \hspace{1cm} Hom($\mathcal{T}_{x,n}, \mathcal{O}(m)$) = $\emptyset$

(8) \hspace{1cm} Hom($\mathcal{T}_{x,n}, \mathcal{T}_{x',m}$) = $\emptyset$ if $x \neq x'$

(9) \hspace{1cm} $|\text{Hom}(\mathcal{O}(n), \mathcal{O}(m))| = \begin{cases} 0 \text{ if } n > m \\ m - n + 1 \text{ if } n \leq m \end{cases}$

**Proof.** (7) is trivial. (8) follows from the observation that $\mathcal{T}_{x,n}$ and $\mathcal{T}_{x',m}$ have disjoint supports.

For (9), let $\rho \in \text{Hom}(\mathcal{O}(n), \mathcal{O}(m))$, and let $\rho_0$ (resp. $\rho_\infty$) denote the restriction of $\rho$ to $U_0$ (resp. $U_\infty$). Trivializing $\mathcal{O}(n), \mathcal{O}(m)$ on $U_0 \cap U_\infty$, we may write $\rho_0(t^k) = t^{k+a_0}$, $\rho(t^l) = t^{l+a_\infty}$, where $a_0 \geq 0, a_\infty \leq 0$. The condition $\rho_0 |_{U_0 \cap U_\infty} = \rho_\infty |_{U_0 \cap U_\infty}$ becomes

$$\sigma_m \circ \rho_0 = \rho_\infty \circ \sigma_n,$$
which reduces to $a_0 - a_\infty = m - n$. It is clear that the number of solutions to this equation with $a_0 \geq 0, a_\infty \leq 0$ is $m - n + 1$ if $m \geq n$ and 0 otherwise. □

**Remark 6.** It follows from the proof of the last lemma that if

$$0 \to O(n) \to O(m) \to \mathcal{T} \to 0$$

is a short exact sequence, then $\mathcal{T} \cong \mathcal{T}_{0,k_0} \oplus \mathcal{T}_{\infty,k_\infty}$ with $k_0 + k_\infty = m - n$ (i.e. $|\mathcal{T}| = m - n$). There is precisely one such short exact sequence with fixed cokernel.

If

$$0 \to \mathcal{F} \xrightarrow{j_1} \mathcal{F}' \xrightarrow{j_2} \mathcal{F}'' \to 0$$

and

$$0 \to \mathcal{G} \xrightarrow{j_1} \mathcal{G}' \xrightarrow{j_2} \mathcal{G}'' \to 0$$

are two short exact sequences in $\text{Coh}(X)$, we say that the short exact sequence

$$0 \to \mathcal{F} \oplus \mathcal{G} \xrightarrow{j_1 \oplus j_2} \mathcal{F}' \oplus \mathcal{G}' \xrightarrow{j_1 \oplus j_2} \mathcal{F}'' \oplus \mathcal{G}'' \to 0$$

is their direct sum.

**Theorem 2.** Every short exact sequence in $\text{Coh}(\mathbb{P}^1)$ is a direct sum of short exact sequences of the form

1. $0 \to \mathcal{T}_{x,m} \to \mathcal{T}_{x,n} \to \mathcal{T}_{x,n-m} \to 0$ as in Remark 5
2. $0 \to O(n) \to O(m) \to \mathcal{T} \to 0$ as in Remark 6

**Proof.** Let

$$0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$$

be a short exact sequence in $\text{Coh}(\mathbb{P}^1)$. $\mathcal{F}$ can be uniquely written as a sum of indecomposable factors $O(k), \mathcal{T}_{0,m}, \mathcal{T}_{\infty,m}$ and we proceed by induction on the number $r$ of these. If $r = 1$, then $\mathcal{F} = O(k)$ or $\mathcal{F} = \mathcal{T}_{x,m}$ and the statement is clearly true. Suppose now the theorem holds for $r \leq p, p \geq 1$, and $\mathcal{F}$ has $p + 1$ indecomposable factors. Write $\mathcal{F} = I \oplus \mathcal{H}$ with $I$ indecomposable. By Remark 4, $I$ maps into an indecomposable factor $\mathcal{J}$ of $\mathcal{F}'$, and is moreover the only factor of $\mathcal{F}$ to map into $\mathcal{J}$. Writing $\mathcal{F}' = \mathcal{J} \oplus \mathcal{K}$, it follows that 10 is a direct sum of

$$0 \to I \to \mathcal{J} \to \mathcal{J}/I \to 0$$

and

$$0 \to \mathcal{H} \to \mathcal{K} \to \mathcal{K}/\mathcal{H} \to 0$$

and the result follows from the inductive hypothesis. □

**Corollary 2.** Let $\mathcal{F}, \mathcal{G}$ be coherent sheaves on $\mathbb{P}^1$. Then
(1) $|\text{Hom}(\mathcal{F}, \mathcal{G})| < \infty$
(2) The number of extensions of $\mathcal{F}$ by $\mathcal{G}$ is finite.

Denote by $\text{Iso}(\text{Coh}(\mathbb{P}^1))$ the set of isomorphism classes of coherent sheaves on $\mathbb{P}^1$. Define the Grothendieck group $K_0(\mathbb{P}^1)$ of $\text{Coh}(\mathbb{P}^1)$ by

$$K_0(\mathbb{P}^1) := \mathbb{Z}[\text{Iso}(\text{Coh}(\mathbb{P}^1))]/\sim$$

where $\sim$ is the subgroup generated by $[\mathcal{F}] + [\mathcal{F}'] - [\mathcal{F}'']$ for short exact sequences. Define a homomorphism of abelian groups

$$\Psi : \mathbb{Z}[\text{Iso}(\text{Coh}(\mathbb{P}^1))] \to \mathbb{Z} \oplus \mathbb{Z}$$
on generators by

$$(11) \quad \Psi([O(k)]) = (1, k)
(12) \quad \Psi([\mathcal{F}_{x,n}]) = (0, n)$$

Since $\Psi$ is additive on every short exact sequence in Theorem 2, it follows that $\Psi$ descends to $K_0(\mathbb{P}^1)$, and it’s easy to see that it is an isomorphism. We have thus shown:

**Theorem 3.** $K_0(\mathbb{P}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$

We call the first factor rank and the second degree in analogy with the case of $\mathbb{P}^1$ over a field.

4. **Hall algebras**

In this section, we introduce the Hall algebra $\mathbb{H}(\mathbb{P}^1)$ of the category $\text{Coh}(\mathbb{P}^1)$. For more on Hall algebras see [14]. As a vector space:

$$(13) \quad \mathbb{H}(\mathbb{P}^1) := \{ f : \text{Iso}(\text{Coh}(\mathbb{P}^1)) \to \mathbb{C} \mid \#\text{supp}(f) < \infty \}.$$ We equip $\mathbb{H}(\mathbb{P}^1)$ with the convolution product

$$(14) \quad f * g(\mathcal{F}) = \sum_{\mathcal{F}' \subset \mathcal{F}} f(\mathcal{F}/\mathcal{F}') g(\mathcal{F}'),$$

where the sum is over all coherent sub-sheaves $\mathcal{F}'$ of the isomorphism class $\mathcal{F}$ (in what follows, it is conceptually helpful to fix a representative of each isomorphism class). Note that Corollary 2 and the finiteness of the support of $f, g$ ensures that the sum in 14 is finite.

**Lemma 6.** The product $*$ is associative.

**Proof.** Suppose $f, g, h \in \mathbb{H}(\mathbb{P}^1)$. Then

$$\begin{align*}
(f * (g * h))(\mathcal{F}) &= \sum_{\mathcal{F}' \subset \mathcal{F}} f(\mathcal{F}/\mathcal{F}')(g * h)(\mathcal{F}') \\
&= \sum_{\mathcal{F}' \subset \mathcal{F}} f(\mathcal{F}/\mathcal{F}')(\sum_{\mathcal{F}'' \subset \mathcal{F}'} g(\mathcal{F}' / \mathcal{F}'') h(\mathcal{F}'')) \\
&= \sum_{\mathcal{F}'' \subset \mathcal{F} \cap \mathcal{F}'} f(\mathcal{F} / \mathcal{F}') g(\mathcal{F}' / \mathcal{F}'') h(\mathcal{F}'')
\end{align*}$$
whereas
\[
((f \star g) \star h)(\mathcal{F}) = \sum_{\mathcal{F}' \subset \mathcal{F}} (f \star g)(\mathcal{F} / \mathcal{F}'') h(\mathcal{F}'')
\]
\[
= \sum_{\mathcal{F}' \subset \mathcal{F}} \left( \sum_{\mathcal{K} \subset \mathcal{F} / \mathcal{F}''} f((\mathcal{F} / \mathcal{F}'') / \mathcal{K}) \cdot g(\mathcal{K}) \right) h(\mathcal{F}'')
\]
\[
= \sum_{\mathcal{F}' \subset \mathcal{F}} f(\mathcal{F} / \mathcal{F}') \cdot g(\mathcal{F}' / \mathcal{F}'') h(\mathcal{F}''),
\]
where in the last step we have used the fact that there is an inclusion-preserving bijection between sub-sheaves \( \mathcal{K} \subset \mathcal{F} / \mathcal{F}'' \) and sub-sheaves \( \mathcal{F}' \subset \mathcal{F} \) containing \( \mathcal{F}'' \), under which \( \mathcal{F}' / \mathcal{F}'' \cong \mathcal{K} \). This bijection is compatible with taking quotients in the sense that \( (\mathcal{F} / \mathcal{F}'') / \mathcal{K} \cong \mathcal{F} / \mathcal{F}' \).

We may also equip \( \mathcal{H}(\mathbb{P}^1) \) with a coproduct
\[
\Delta : \mathcal{H}(\mathbb{P}^1) \to \mathcal{H}(\mathbb{P}^1) \otimes \mathcal{H}(\mathbb{P}^1)
\]
given by
\[
\Delta(f)(\mathcal{F}, \mathcal{F}') := f(\mathcal{F} \oplus \mathcal{F}')
\]
\( \Delta \) is clearly co-commutative.

**Lemma 7.** The following holds in \( \mathcal{H}(\mathbb{P}^1) \):

1. \( \Delta \) is co-associative: \( (\Delta \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \Delta) \circ \Delta \)
2. \( \Delta \) is compatible with \( \star \): \( \Delta(f \star g) = \Delta(f) \star \Delta(g) \).

**Proof.** The proof of both parts is the same as the proof of the corresponding statements for the Hall algebra of the category of quiver representations over \( \mathbb{F}_1 \), given in [16]. □

We may equip \( \mathcal{H}(\mathbb{P}^1) \) with a grading by \( K^+_0(\mathbb{P}^1) \) - the effective cone inside \( K_0(\mathbb{P}^1) \), (which by Theorem 3 is isomorphic to \( \mathbb{N} \times \mathbb{N} \)) as follows. \( \mathcal{H}(\mathbb{P}^1) \) is spanned by \( \delta \)-functions \( \delta_\mathcal{F} \) supported on individual isomorphism classes, and we define
\[
\deg(\delta_\mathcal{F}) = \overline{\mathcal{F}} \in K_0(\mathbb{P}^1)
\]
where \( \overline{\mathcal{F}} \) denotes the class of \( \mathcal{F} \) in the Grothendieck group. With this grading, \( \mathcal{H}(\mathbb{P}^1) \) becomes a graded, connected, co-commutative Hopf algebra. By the Milnor-Moore Theorem, \( \mathcal{H}(\mathbb{P}^1) \) is isomorphic to \( \mathfrak{U}(\mathfrak{q}) \) - the universal enveloping algebra of \( \mathfrak{q} \), where the latter is the Lie algebra of its primitive elements. The definition of the co-product 16 implies that \( \mathfrak{q} \) is spanned by \( \delta_\mathcal{F} \) for isomorphism classes \( \mathcal{F} \) of indecomposable coherent sheaves, which by corollary 1 are \( \mathcal{O}(k), \mathcal{T}_{0,n}, \mathcal{T}_{\infty,m}, \) for \( k \in \mathbb{Z}, n, m \in \mathbb{N} \).

5. The structure of \( \mathcal{H}(\mathbb{P}^1) \)

In this section we compute the structure of the Hopf algebra \( \mathcal{H}(\mathbb{P}^1) \). We will use the shorthand notation \([\mathcal{F}]\) for \( \delta_\mathcal{F} \) - the delta-function supported on the isomorphism class of \( \mathcal{F} \). The following theorem follows from Lemma 5 and Theorem 2:
Theorem 4. The following identities hold in $\mathcal{H}(\mathbb{P}^1)$:

\begin{align*}
[O(n)] \cdot [O(m)] &= [O(n) \oplus O(m)] \\
[T_{x,n}] \cdot [O(m)] &= [O(n + m)] + [T_{x,n} \oplus O(m)] \\
[O(m)] \cdot [T_{x,n}] &= [T_{x,n} \oplus O(m)] \\
[T_{x,n}] \cdot [T_{x,m}] &= [T_{x,n} \oplus T_{x,m}] \text{ for } x = 0, \infty
\end{align*}

From the previous theorem we deduce the following commutation relations:

\begin{align*}
[[O(n)], [O(m)]] &= 0 \\
[[T_{x,n}], [O(m)]] &= [O(m + n)] \\
[[T_{x,n}], [T_{x',m}]] &= 0
\end{align*}

Let $L_{\mathfrak{gl}_2} := \mathfrak{gl}_2 \otimes \mathbb{C}[t, t^{-1}]$, and $L_{\mathfrak{gl}_2}^+ = (\mathfrak{h} \otimes t\mathbb{C}[t]) \oplus (e \otimes \mathbb{C}[t, t^{-1}])$ where

$\mathfrak{h} := \text{span} \left\{ h_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, h_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ and $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Define

$\rho : L_{\mathfrak{gl}_2}^+ \to \mathcal{H}(\mathbb{P}^1)$

by setting $\rho(e \otimes t^k) = [O(k)], \rho(h_1 \otimes t^n) = [T_{0,n}], \text{ and } \rho(h_2 \otimes t^m) = -[T_{0,m}]$. The following proposition now follows from the commutation relations 22, 23, 24:

Theorem 5. $\rho$ is an isomorphism of Lie algebras. Consequently, $\mathcal{H}(\mathbb{P}^1) \simeq U(L_{\mathfrak{gl}_2}^+)$. 

The sub-algebra $\widehat{\mathcal{H}}(\mathbb{P}^1)$ of $\mathcal{H}(\mathbb{P}^1)$ generated by $[O(k)], [T_{0,n}] + [T_{\infty,n}]$ is analogous to the sub-algebra of the Ringel-Hall algebra studied by Kapranov in [10]. It is easily seen to be isomorphic to $U(L_{\mathfrak{sl}_2}^+)$, with $L_{\mathfrak{sl}_2}^+ = (\mathfrak{h} \otimes t\mathbb{C}[t]) \oplus (e \otimes \mathbb{C}[t, t^{-1}])$, where

$\mathfrak{h} = \mathbb{C} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. 

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ON THE HALL ALGEBRA OF COHERENT SHEAVES ON $\mathbb{P}^1$ OVER $\mathbb{F}_1$

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REFERENCES


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