

PROTO-EXACT CATEGORIES OF MODULES OVER SEMIRINGS AND HYPERRINGS

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ABSTRACT. *Proto-exact categories*, introduced by Dyckerhoff and Kapranov, are a generalization of Quillen exact categories which provide a framework for defining algebraic K-theory and Hall algebras in a *non-additive* setting. This formalism is well-suited to the study of categories whose objects have strong combinatorial flavor.

In this paper, we show that the categories of modules over semirings and hyperrings - algebraic structures which have gained prominence in tropical geometry - carry proto-exact structures.

In the first part, we prove that the category of modules over a semiring is equipped with a proto-exact structure; modules over an idempotent semiring have a strong connection to matroids. We also prove that the category of algebraic lattices \mathcal{L} has a proto-exact structure, and furthermore that the subcategory of \mathcal{L} consisting of finite lattices is equivalent to the category of finite \mathbb{B} -modules as proto-exact categories, where \mathbb{B} is the *Boolean semifield*. We also discuss some relations between \mathcal{L} and geometric lattices (simple matroids) from this perspective.

In the second part, we prove that the category of modules over a hyperring has a proto-exact structure. In the case of finite modules over the *Krasner hyperfield* \mathbb{K} , a well-known relation between finite \mathbb{K} -modules and finite incidence geometries yields a combinatorial interpretation of exact sequences.

1. Introduction

Recent years have seen several attempts to formulate notions of algebraic geometry in characteristic one based on monoids, semirings, hyperrings, and blueprints [Dei05, CC10b, CC11b, Lor12, Sou04]. One reason for this effort is the desire to develop scheme-theoretic foundations for tropical geometry [GG16, Lor15]. In these “exotic” theories, just as in “ordinary” algebraic geometry, an affine scheme corresponds one of the aforementioned algebraic structures, which we generically denote A for the moment. Proceeding by analogy, one expects a quasi-coherent sheaf on $\mathrm{Spec}(A)$ to correspond to some appropriate notion of A -module, and is therefore led to study the category $A\text{-mod}$ of A -modules and its homological properties in general. Here, the situation is complicated by the fact that unlike the case of commutative rings, $A\text{-mod}$ is typically not abelian or even additive, and so what should be considered an “exact sequence” does not have an immediate answer.

Proto-exact categories, introduced by T. Dyckerhoff and M. Kapranov [DK12] as a generalization of Quillen exact categories, provide a flexible framework for exact sequences in (potentially) non-additive categories. Roughly speaking, a proto-exact category is a pointed category with two distinguished classes of morphisms (*admissible monomorphisms* and *admissible epimorphisms*) satisfying certain conditions on pullback and pushout diagrams from which one can obtain a notion of *admissible exact sequences*.¹ Several interesting “combinatorial” categories are equipped with a proto-exact structure, for instance, the category of matroids [EJS20], the category of representations over a quiver (and more generally any monoid) over “the field with one element” [Szc12], [JS20a], [JS21]. Categories with more algebro-geometric flavors, which are not additive, have been explored in [Szc18], [JS20b], [ELY20].

There are at least two “features” associated with a proto-exact category \mathcal{A} :

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¹See Definition 2.1 for the precise definition.

- (1) If \mathcal{A} is *finitary*, in the sense that the number $|\text{Ext}^1(X, Y)|$ of inequivalent short exact sequences is finite for each pair of objects $X, Y \in \mathcal{A}$, then one may define the *Hall algebra* $H_{\mathcal{A}}$. This is an associative (and in “good” cases Hopf) algebra spanned by the isomorphism classes of objects of \mathcal{A} , whose structure coefficients count the number of extensions between objects. Classically - for instance when \mathcal{A} is the category of quiver representations or coherent sheaves on a curve over a finite field [Rin90, Kap97], $H_{\mathcal{A}}$ yields quantum groups and related objects and is an important tool in their representations theory. When \mathcal{A} is non-additive, $H_{\mathcal{A}}$ typically has a combinatorial flavor. In this case, the Hall algebra often becomes a Hopf algebra, where the product of two objects is obtained by “assembling” two objects into a new object and coproduct encodes all possible ways to “disassemble” the given object into two objects. As an application, one may study various operations and identities for combinatorial objects from the Hall algebra perspective.²
- (2) One can define a well-behaved version of algebraic K-theory for \mathcal{A} (either through Quillen’s Q-construction or Waldhausen’s S-construction - see [DK12, ELY20, Hek17]). Even for relatively simple combinatorial categories \mathcal{A} this is a rich and interesting invariant [CLS12, EJS20].

The main goal of the current paper is to enlarge the catalogue of non-additive proto-exact categories by showing these include the categories of modules over *semirings* as well as *hyperrings*. Modules over an idempotent semiring are closely related to matroid theory [GG18] and modules over a hyperring have an interesting connection to finite incidence geometries [CC10a], [Jun18b] and matroids [BB19]. We also examine the category of algebraic lattices in relation to finite modules over \mathbb{B} , and discuss how the proto-exact structure of algebraic lattices is related to the proto-exact structure of the category of matroids in [EJS20] via geometric lattices. Along the way, we also investigate whether the proto-exact categories constructed are finitary, and thus whether the Hall algebra $H_{\mathcal{A}}$ is defined.

Let R be a semiring and M be an R -module. We define admissible monomorphisms (resp. admissible epimorphisms) to be equalizers (resp. coequalizers) of some morphisms and the zero map. We first prove the following.

Theorem A (Theorem 3.15). *Let R be a semiring. With the admissible monomorphisms and admissible epimorphisms as above, the category Mod_R of modules over R is a proto-exact category.*

Next, we turn our attention to the category of *algebraic lattices* (not necessarily finite). Recall that an algebraic lattice is a complete lattice such that any element is a join of compact elements. In particular, any finite lattice is an algebraic lattice. For algebraic lattices, roughly we define admissible monomorphisms (resp. admissible epimorphisms) to be downward closed subsets (resp. upward closed subsets). See Definition 4.5 for the precise definition. We prove the following.

Theorem B (Theorem 4.15). *With the admissible monomorphisms and admissible epimorphisms as above, the category \mathcal{L} of algebraic lattices is a proto-exact category.*

Let \mathcal{L}^c be the subcategory of \mathcal{L} consisting of finite lattices. \mathcal{L}^c is a proto-exact category with the induced proto-exact structure (from \mathcal{L}). Also, from Theorem A, one can prove that the category $\text{Mod}_{\mathbb{B}}^{\text{fin}}$ of finite \mathbb{B} -modules is proto-exact with the induced proto-exact structure (from $\text{Mod}_{\mathbb{B}}$). We prove the following.

Theorem C (Corollary 4.24). *$\text{Mod}_{\mathbb{B}}^{\text{fin}}$ is equivalent to \mathcal{L}^c as proto-exact categories.*

²See [IJ19] for this line of ideas.

A *geometric lattice* is a finite semimodular lattice in which every element is a join of atoms. Geometric lattices provide another cryptomorphic definition for (simple) matroids. In fact, the subcategory \mathcal{G} of \mathcal{L} consisting of geometric lattices is a proto-exact subcategory of \mathcal{L} (Proposition 4.29). Note that the category \mathcal{G} contains “more morphisms” than the category of matroids with strong maps as in [EJS20]. In particular, the proto-exact structure on \mathcal{G} is different from the proto-exact structure for the category of matroids studied in [EJS20]. For more details, see Section 4.3.

Next, we move to *hyperrings*. A morphism of hyperrings preserves multi-valued addition in a “weak sense” (see Definition 2.10). We define admissible monomorphisms (resp. admissible epimorphisms) to be injective (resp. surjective) morphisms which preserve multi-valued addition in a “strong sense”. Then, we prove the following.

Theorem D (Theorem 5.11). *Let H be a hyperring. With the admissible monomorphisms and admissible epimorphisms as above, the category Mod_H of modules over H is a proto-exact category.*

Finally, we recall a well-known equivalence between finite modules over the Krasner hyperfield \mathbb{K} (Example 2.13) and finite incidence geometries which could potentially allow us to describe the structure constants for the Hall algebra associated $\text{Mod}_{\mathbb{K}}$ by using finite incidence geometries. This is the topic of a future paper.

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2. Preliminaries

2.1. Proto-exact categories. In [Rin90], Ringel defined the Hall algebra of the category $\text{Rep}(Q, \mathbb{F}_q)$ of representations of a quiver Q over a finite field \mathbb{F}_q , and proved that in the case of a simply laced Dynkin quiver Q , the associated Hall algebra is the upper triangular part of the quantum group which is classified by the same Dynkin diagram. Later in [Kap97], Kapranov investigated $\text{Coh}(X)$, the category of coherent sheaves on a smooth projective curve X over a finite field \mathbb{F}_q . and proved when X is a projective line, some subalgebra of $H_{\text{Coh}(X)}$ is isomorphic to a positive part of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$.

Notice that both $\text{Rep}(Q, \mathbb{F}_q)$ and $\text{Coh}(X)$ are examples of a finitary abelian category, and one can indeed associate a Hall algebra $H_{\mathcal{A}}$ to *any* finitary abelian category as follows: the underlying vector space of $H_{\mathcal{A}}$ is spanned by the set $\text{Iso}(\mathcal{A})$ of isomorphism classes of \mathcal{A} . For each $A, B \in \text{Iso}(\mathcal{A})$, multiplication is defined as follows:

$$A \cdot B := \sum_{C \in \text{Iso}(\mathcal{A})} \mathbf{a}_{A,B}^C C, \quad (1)$$

where

$$\mathbf{a}_{A,B}^C = \#\{D \subseteq C \mid D \simeq B \text{ and } C/D \simeq A\}. \quad (2)$$

Then one linearly extends (1) to $H_{\mathcal{A}}$. If \mathcal{A} satisfies some extra conditions, it becomes a Hopf algebra.

As a first step into a non-additive setting, one may notice from (1) and (2) that to define the Hall algebra $H_{\mathcal{A}}$ of an abelian category, one does not need the full strength of the abelian categories; one only needs a notion of “exact sequences” (or proper definitions of “subobjects” and “quotient objects”). For instance, Quillen exact categories would suffice to define the algebraic structures as in (1) and (2).

In [DK12], Dyckerhoff and Kapranov introduced the notions of proto-exact and proto-abelian categories which allow one to construct the Hall algebra in a more general setting beyond abelian categories. In particular, this allows one to define and study Hall algebras for a non-additive category in a systemic way. In the following, we recall basic definitions for proto-exact categories which will be used in sequel.

Definition 2.1. A proto-exact category is a pointed category \mathcal{C} equipped with two distinguished classes of morphisms \mathfrak{M} (admissible monomorphisms) and \mathfrak{E} (admissible epimorphisms) which satisfy the following conditions.

- (1) For each object $A \in \mathcal{C}$, a morphism $0 \rightarrow A$ is in \mathfrak{M} , and a morphism $A \rightarrow 0$ is in \mathfrak{E} .
- (2) \mathfrak{M} and \mathfrak{E} contain all isomorphisms and are closed under composition.
- (3) A commutative square as in (3), with $i, i' \in \mathfrak{M}$ and $j, j' \in \mathfrak{E}$, is Cartesian (pullback) if and only if it is co-Cartesian (pushout).

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ j \downarrow & & \downarrow j' \\ A' & \xrightarrow{i'} & B' \end{array} \quad (3)$$

- (4) A diagram as in (4), with $i' \in \mathfrak{M}$ and $j' \in \mathfrak{E}$, can be completed to a bi-Cartesian square (3) with $i \in \mathfrak{M}$ and $j \in \mathfrak{E}$.

$$\begin{array}{ccc} & & B \\ & & \downarrow j' \\ A' & \xrightarrow{i'} & B' \end{array} \quad (4)$$

- (5) A diagram as in (5), with $i \in \mathfrak{M}$ and $j \in \mathfrak{E}$, can be completed to a bi-Cartesian square (3) with $i' \in \mathfrak{M}$ and $j' \in \mathfrak{E}$.

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ j \downarrow & & \\ A' & & \end{array} \quad (5)$$

Let \mathcal{C} be a proto-exact category. By an *admissible short exact sequence* in \mathcal{C} , we mean a bi-Cartesian square as in (6) with $i \in \mathfrak{M}$ and $j \in \mathfrak{E}$.

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow & & \downarrow j \\ 0 & \longrightarrow & C \end{array} \quad (6)$$

We sometimes call the diagram (6) an *admissible extension of C by A* , and it will also be denoted by

$$A \hookrightarrow B \twoheadrightarrow C. \quad (7)$$

When there is no potential confusion, we will simply call (7) a short exact sequence in \mathcal{C} . In (7) we denote the object C , which is unique up to a unique isomorphism, by B/A .

Definition 2.2. Let \mathcal{C} and \mathcal{D} be proto-exact categories.

- (1) A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is said to be *exact* if F preserves admissible short exact sequences.
- (2) In \mathcal{C} , two extensions $A \hookrightarrow B \twoheadrightarrow C$ and $A \hookrightarrow B' \twoheadrightarrow C'$ of C by A are *equivalent* if there is a commutative diagram

$$\begin{array}{ccccc} A & \hookrightarrow & B & \twoheadrightarrow & C \\ \text{id} \downarrow & & \downarrow \cong & & \downarrow \text{id} \\ A & \hookrightarrow & B' & \twoheadrightarrow & C \end{array} \quad (8)$$

The set of equivalence classes of such sequences is denoted $\text{Ext}_{\mathcal{C}}(C, A)$.

The following is the finiteness condition that we need to ensure the existence of the Hall algebra of a proto-exact category.

Definition 2.3. A proto-exact category \mathcal{C} is *finitary* if, for every pair of objects A and B , the sets $\text{Hom}_{\mathcal{C}}(A, B)$ and $\text{Ext}_{\mathcal{C}}(A, B)$ are finite sets.

Example 2.4. The following are typical examples of proto-exact categories.

- (1) Any Quillen exact category is proto-exact with the same exact structure.
- (2) A simple example of proto-exact categories (which is not an additive category) is the category \mathbf{Set}_{\bullet} of pointed sets; \mathfrak{M} consists of all pointed injections, and \mathfrak{E} consists of all pointed surjections

$$f : (A, *_A) \rightarrow (B, *_B)$$

such that $f|_{A \setminus f^{-1}(*_B)}$ is injective. The full subcategory $\mathbf{Set}_{\bullet}^{\text{fin}}$ of finite pointed sets is finitary. See [Dyc18, Section 1.3] for the Hall algebra of $\mathbf{Set}_{\bullet}^{\text{fin}}$ in perspective of algebraic geometry over “the field with one element” along with other interesting examples. Also, the first and the second authors further explored certain categories of coherent sheaves arising from algebraic geometry over monoids in [JS20b].

- (3) In [EJS20], together with C. Eppolito, the first and the second authors proved that the category \mathbf{Mat}_{\bullet} of pointed matroids is equipped with a finitary proto-exact structure under which the Hall algebra is isomorphic to the dual of a matroid-minor Hopf algebra.

Now, one can define the Hall algebra $H_{\mathcal{C}}$ of a finitary proto-exact category \mathcal{C} as in the case of abelian categories; it has the underlying set as a vector space spanned by the set $\text{Iso}(\mathcal{C})$ of isomorphism classes of \mathcal{C} . For each $M, N \in \text{Iso}(\mathcal{C})$, multiplication is as in (1), where $D \subseteq C$ is a subobject of C and C/D is the object (unique up to a unique isomorphism) such that the following is a short exact sequence in \mathcal{C} :

$$D \hookrightarrow C \twoheadrightarrow C/D. \quad (9)$$

As in the classical case, when \mathcal{C} satisfies further conditions, $H_{\mathcal{C}}$ becomes a Hopf algebra. For instance, this is the case of matroids, finite pointed sets, and representations of a quiver over “the field with one element”. See [EJS20], [Szc12], [JS20a], [JS21], [Szc18].

2.2. Semirings and hyperrings.

2.2.1. Semirings. By a *semiring*, we mean a nonempty set R with two binary operations $+, \cdot$ such that $(R, +, 1_R)$ and $(R, \cdot, 0_R)$ are commutative monoids and two binary operations are compatible in the sense that $(a+b) \cdot c = a \cdot c + a \cdot b$ for any $a, b, c \in R$. When $(R \setminus \{0_R\}, \cdot)$ is a group, R is said to be a *semifield*. By an R -module for a semiring R , we mean a monoid M with the usual axioms.

Definition 2.5. Let R be a semiring.

- (1) Let M be an R -module. A submodule N is said to be *saturated* if $x, x+y \in N$ implies $y \in N$ for all $x, y \in M$.
- (2) A semiring R is *additively idempotent* if $a+a = a$ for all $a \in R$.

Example 2.6. Let $\mathbb{B} = \{0, 1\}$. With the following addition and multiplication, \mathbb{B} is a semifield.

$$0 \cdot 0 = 0, \quad 1 \cdot 1 = 1, \quad 0 \cdot 1 = 0, \quad 0 + 0 = 0, \quad 1 + 0 = 1, \quad 1 + 1 = 1.$$

\mathbb{B} is said to be the *Boolean semifield*.

Example 2.7. Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ be the set of real numbers together with the symbol $-\infty$. With the following addition \oplus and multiplication \odot , \mathbb{R}_{\max} is a semifield.

$$a \oplus b = \max\{a, b\}, \quad a \odot b = a + b,$$

where $+$ is the usual addition of real numbers. \mathbb{R}_{\max} is said to be the *tropical semifield* and plays an important role in tropical geometry; it plays the role that is classically played by the complex numbers.

The following example illustrates that idempotent semirings are neither restrictive nor exotic.

Example 2.8. Let K be a valued field whose value group is \mathbb{R} , \mathcal{O}_K the ring of integers, and A a K -algebra. Let $\mathbb{S}_{\text{fg}}(A)$ be the set of \mathcal{O}_K -submodules of A . Then $\mathbb{S}_{\text{fg}}(A)$ is equipped with a natural idempotent semiring structure: for $N_1 = \langle S_1 \rangle, N_2 = \langle S_2 \rangle \in \mathbb{S}_{\text{fg}}(A)$, we define addition and multiplication as follows:

$$N_1 + N_2 := \langle S_1 \cup S_2 \rangle, \quad N_1 N_2 := \langle S_1 S_2 \rangle.$$

For more details in relation to tropical geometry, we refer the readers to [JMT20]. This correspondence was also employed in [JRT20] to enrich Hochster’s theorem on *spectral spaces* in [Hoc69] by showing that a topological space X is spectral if and only if X is the “saturated prime spectrum” of an idempotent semiring. In fact, in [JRT20], with S. Ray, the first and the third authors proved that the category of spectral spaces is equivalent to a certain subcategory of the category of idempotent semirings.

Remark 2.9. In this paper, by an idempotent semiring we always mean an additively idempotent semiring. Equivalently, an idempotent semiring is an \mathbb{B} -algebra, where \mathbb{B} is the Boolean semifield.

Let M be an R -module and N be a submodule of M . N defines a congruence relation on M as follows: for $x, y \in M$

$$x \sim y \iff x + n = y + n' \text{ for some } n, n' \in N. \quad (10)$$

The quotient M/N is defined to be the set of equivalence classes under (10) with the induced operations. One can easily check that M/N is an R -module.

2.2.2. Hyperrings. Next, we recall the definition of hyperrings. Let M be a nonempty set and $\mathcal{P}^*(M)$ be the set of all nonempty subsets of M . By a *hyperaddition*, we mean a binary function $\phi : M \times M \rightarrow \mathcal{P}^*(M)$ such that $\phi(a, b) = \phi(b, a)$. We simply denote $\phi(a, b)$ by $a + b$ or $a +_M b$. If $A, B \subseteq M$, we use the following notation:

$$A + B := \bigcup_{a \in A, b \in B} (a + b). \quad (11)$$

In what follows, we will identify an element a and the singleton $\{a\}$.

Definition 2.10. A *hypergroup* is a nonempty set M with a hyperaddition satisfying the following:

- (1) For any $a, b, c \in M$, we have $(a + b) + c = a + (b + c)$.
- (2) There exists a unique element $0 \in M$ such that $a + 0 = a$ for all $a \in M$.
- (3) For each $a \in M$, there exists a unique element b , denoted by $-a$, such that $0 \in a + b$.
- (4) For any $a, b, c \in M$, if $a \in b + c$, then $c \in a + (-b)$.

A morphism $f : M_1 \rightarrow M_2$ between hypergroups is a function such that $f(0) = 0$ and for all $a, b \in M_1$,

$$f(a + b) \subseteq f(a) + f(b). \quad (12)$$

A morphism f is said to be *strict* if (12) is an equality.

Definition 2.11. By a *hyperring*, we mean a hypergroup H equipped with (usual) multiplication \cdot satisfying the following two conditions:

- (1) $(H, \cdot, 1_H)$ is a commutative monoid,
- (2) Hyperaddition and multiplication are compatible in the sense that $a \cdot (b + c) = a \cdot b + a \cdot c$.

If $(H \setminus \{0_H\}, \cdot)$ is a group, then H is said to be a hyperfield. A morphism $f : H_1 \rightarrow H_2$ between hyperrings is a morphism of hypergroups which also preserves multiplication. A morphism is said to be *strict* if it is strict as a morphism of hypergroups.

Definition 2.12. Let H be a hyperring. By an H -module, we mean a hypergroup M together with a map $\phi : H \times M \rightarrow M$ such that (we denote $\phi(r, m)$ by rm .)

- (1) $1m = m$ for all $m \in M$, $0m = 0$ for all $m \in M$,
- (2) $(xy)m = x(ym)$ for all $x, y \in H$ and $m \in M$ and
- (3) $x(m_1 + m_2) = xm_1 + xm_2$ for all $x \in H$ and $m_i \in M$,

(4) $(x+y)m = xm + ym$ for all $x, y \in H$ and $m \in M$.

By a morphism of H -modules, we mean a morphism $f : M_1 \rightarrow M_2$ of hypergroups such that $f(rm) = rf(m)$ for all $r \in H$ and $m \in M_1$.

Example 2.13. Let $\mathbb{K} = \{0, 1\}$. With the following hyperaddition and multiplication \mathbb{K} becomes a hyperfield.

$$1 \cdot 1 = 1, \quad 1 \cdot 0 = 0, \quad 0 \cdot 0 = 0, \quad 0 + 0 = 0 \quad 1 + 0 = 1, \quad 1 + 1 = \mathbb{K}.$$

\mathbb{K} is said to be the *Krasner hyperfield*.

Example 2.14. Let $\mathbb{S} = \{-1, 0, 1\}$. With the following hyperaddition and multiplication \mathbb{S} becomes a hyperfield.

$$\begin{aligned} 1 \cdot 1 &= (-1) \cdot (-1) = 1, & 1 \cdot (-1) &= -1, & 0 \cdot (-1) &= 0 \cdot 0 = 0 \cdot 1 = 0, \\ 0 + 0 &= 0, & 1 + 1 &= 1 + 0 = 1, & (-1) + (-1) &= (-1) + 0 = -1, & 1 + (-1) &= \mathbb{S}. \end{aligned}$$

\mathbb{S} is said to be the *sign hyperfield*.

Example 2.15. Let $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$, the set of real numbers together with the symbol $-\infty$. With the following hyperaddition \oplus and multiplication \odot , \mathbb{T} becomes a hyperfield.

$$a \odot b = a + b, \quad a \oplus b = \begin{cases} \max\{a, b\} & \text{if } a \neq b, \\ [-\infty, a] & \text{if } a = b, \end{cases}$$

where $+$ is the usual addition of real numbers and $[-\infty, a] = \{x \in \mathbb{T} \mid x \leq a\}$. \mathbb{T} is said to be the *tropical hyperfield*.

The following example shows that although hyperrings look exotic one can easily construct them from a commutative ring A and a multiplicative subgroup G of A .

Example 2.16. Let A be a commutative ring and G be a subgroup of the group of multiplicative units of A . Then G naturally acts on A by multiplication. Let A/G be the set of equivalence classes. We let $[x]$ be the equivalence class of $x \in A$. Then, A/G becomes a hyperring with the following hyperaddition and multiplication: for any $[a], [b] \in A/G$,

$$[a] \cdot [b] = [ab], \quad [a] + [b] = \{[c] \mid c = g_1a + g_2b \text{ for some } g_1, g_2 \in G\}.$$

A hyperring of the form A/G is said to be a *quotient hyperring*.

Definition 2.17. Let H be a hyperring and M be an H -module. By a submodule N of M , we mean a subset of M which is itself an H -module with the induced hyperaddition and scalar multiplication.

Remark 2.18. We remark that in Definition 2.17, for any $a, b \in N$ the hyperaddition $a +_N b$ taken in N is a subset of the hyperaddition $a +_M b$ taken in M , and they do not have to be same. For example, the tropical hyperfield \mathbb{T} is an \mathbb{K} -module, where \mathbb{K} is the Krasner hyperfield. In this case \mathbb{K} is a submodule of \mathbb{T} , but, $1 +_{\mathbb{K}} 1 = \{0, 1\}$ whereas $1 +_{\mathbb{T}} 1 = [-\infty, 1]$.

3. The category of modules over a semiring as a proto-exact category

In this section, we prove that the category of modules over a semiring is a proto-exact category. We define admissible monomorphisms (resp. admissible epimorphisms) to be equalizers (resp. coequalizers) in certain forms. Then, we give an equivalent description of admissible monomorphisms and admissible epimorphisms in terms of saturated submodules.

Let M be a module over a semiring R . Recall that a submodule N of M is said to be saturated if $x + y \in N$ and $y \in N$ implies that $x \in N$ for any $x, y \in M$. It is clear that the intersection of a family of saturated submodules of M is again saturated. In particular, for each subset S of M , there exists a smallest saturated submodule of M containing S . We call this the saturation closure (or simply the saturation) of S , denoted by \bar{S} or $\langle S \rangle$. The following is proved in [JRT20] for ideals, but the same proof works for submodules. We include the proof for completeness.

Proposition 3.1. *Let R be a semiring. Let M be an R -module and N be a submodule of M . Then the smallest saturated submodule of M containing N is $\{x \in M \mid x + a = b \text{ for some } a, b \in N\}$.*

Proof. Let $L = \{x \in M \mid x + a = b \text{ for some } a, b \in N\}$. It is clear that L is a submodule of M containing N . If $x + y, y \in L$, then we have $x + y + a = b$ and $y + a' = b'$ for some $a, a', b, b' \in N$. Then, we have $x + a + (y + a') = x + a + b'$ and $(x + y + a) + a' = b + a'$. It follows that $x + a + b' = b + a'$, showing that $x \in L$. In particular, L is saturated. Finally it is clear that any saturated submodule of M containing N should contain L . \square

When R is an idempotent semiring, one has the following equivalent description of saturated submodules. Recall that any module M over an idempotent semiring R is equipped with a canonical partial order as follows: for any $x, y \in M$

$$x \leq y \iff x + y = y.$$

Proposition 3.2. *(c.f. [JRT20]) Let R be an idempotent semiring. Let M be an R -module and N be a submodule of M . Then N is saturated if and only if for all $x \in N$ and $y \leq x$ it follows that $y \in N$.*

Proof. Suppose that N is saturated. If $x \in N$ and $y \leq x$, then we have $x + y = x \in N$. Now $y \in N$ since N is saturated. Conversely, suppose N satisfies the given condition. Let $x \in N$ and $x + y \in N$. Then, we have $(x + y) + y = x + y \in N$ and hence $y \leq (x + y)$, implying that $y \in N$ from the given condition. \square

Lemma 3.3. *Let R be a semiring and $f : M \rightarrow N$ be a morphism of R -modules. If L is a saturated submodule of N , then $f^{-1}(L)$ is a saturated submodule of M*

Proof. Let $K = f^{-1}(L)$. It is clear that K is a submodule of M . If $x + y, y \in K$, then we have $f(x + y) = f(x) + f(y), f(y) \in L$. Since L is saturated, we have $f(x) \in L$, in particular $x \in K$, showing that K is saturated. \square

Remark 3.4. Let M be an R -module and N be a submodule of M . Then M/N is isomorphic to $M/\langle N \rangle$, where $\langle N \rangle$ is the saturation closure of N . To see this, we only have to show that N and $\langle N \rangle$ define the same equivalence relation. Let \sim (resp. \equiv) be the equivalence relation defined by $\langle N \rangle$ (resp. N). Since $N \subseteq \langle N \rangle$, we only have to prove that if $x \sim y$ for $\langle N \rangle$, then $x \equiv y$ for N . Now, if $x \sim y$ for $\langle N \rangle$, then $x + a = y + b$ for some $a, b \in \langle N \rangle$. It follows from Proposition 3.1 that $a + n_1 = n_2$ and $b + n'_1 = n'_2$ for some $n_1, n'_1, n_2, n'_2 \in N$. By combining these, we have

$$x + a + n_1 + n'_1 = x + n_2 + n'_1 = y + b + n_1 + n'_1 = y + n_1 + n'_2,$$

showing that $x \equiv y$ for N .

Definition 3.5. Let R be a semiring and $f : M \rightarrow N$ be a morphism of R -modules.

- (1) f is said to be a *normal monomorphism* if f is the equalizer of some morphism and the zero map.
- (2) f is said to be a *normal epimorphism* if f is the coequalizer of some morphism and the zero map.

Remark 3.6. Let R be a semiring and $f : M \rightarrow N$ be a morphism of R -modules.

- (1) Since an equalizer in the category of modules over a semiring is just a set-theoretic equalizer, if f is a normal monomorphism, then f is injective. In particular, f is a monomorphism.
- (2) One can easily observe that the following is a coequalizer diagram in the category of modules over R :

$$L \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{0} \end{array} M \xrightarrow{\pi} M/g(L), \quad (13)$$

where π is a canonical projection. In fact, if $h : M \rightarrow K$ is an R -module morphism such that $hg = 0$. Then, we can define a map $u : M/g(L) \rightarrow K$ sending $[m]$ to $h(m)$, where $[m]$ is the equivalence class of $m \in M$ in $M/g(L)$. Note that u is well-defined; if $[m] = [n]$, then $m + x = n + y$ for some $x, y \in g(L)$. Hence, $h(m + x) = h(m) + h(x) = h(m)$ and similarly $h(n + y) = h(n)$. It is also clear that u is unique once it exists. This shows that any normal epimorphism is surjective, in particular, it is an epimorphism.

The following provides an equivalent description for normal monomorphisms and normal epimorphisms.

Lemma 3.7. *Let R be a semiring and $f : M \rightarrow N$ be a morphism of R -modules.*

- (1) *f is a normal monomorphism if and only if f is injective and $f(M)$ is isomorphic to a saturated submodule of N .*
- (2) *f is a normal epimorphism if and only if N is isomorphic to the quotient of M by a saturated submodule and f is a natural projection.*

Proof. (1) : Suppose that f is a normal monomorphism. From Remark 3.6, we may assume that

$$M = \{n \in N \mid g(n) = 0\}$$

for some $g : N \rightarrow L$ and $f : M \rightarrow N$ is an inclusion. In particular, $M = g^{-1}(0)$ which is clearly saturated from Lemma 3.3 since $\{0\}$ is a saturated submodule. It follows that $f(M)$ is a saturated submodule of N . Conversely, we may assume that M is a saturated submodule of N . Consider the natural projection $\pi : N \rightarrow N/M$. One can easily see that $f : M \rightarrow N$ is the equalizer of π and the zero map since the equalizer should be the smallest saturated submodule of N containing M in this case.

(2) : Suppose that f is a normal epimorphism. It follows from Remarks 3.4 and 3.6 that N is isomorphic to the quotient of M by a saturated submodule and f is a natural projection. The converse is clear. \square

Definition 3.8. Let Mod_R be the category of modules over a semiring R . Let \mathfrak{M} be the class of normal monomorphisms in Mod_R and \mathfrak{E} the class of normal epimorphisms in Mod_R .

In the following, we prove that $(\text{Mod}_R, \mathfrak{M}, \mathfrak{E})$ is a proto-exact category. We will use the terms normal monomorphisms (resp. normal epimorphisms) and admissible monomorphisms (resp. admissible epimorphisms) interchangeably. We first need the following version of the third isomorphism theorem for modules over a semiring.³

Lemma 3.9. *Let R be a semiring, M an R -module, and N a saturated submodule of M . Let $\pi : M \rightarrow M/N$ be a natural projection and K be a saturated submodule of M/N . Then we have*

$$M/\pi^{-1}(K) \simeq (M/N)/K.$$

Proof. It is enough to show that the following natural map is well-defined and injective:

$$\varphi : M/\pi^{-1}(K) \rightarrow (M/N)/K, \quad \bar{m} \mapsto [\tilde{m}],$$

where \bar{m} (resp. \tilde{m}) is the equivalence class of $m \in M$ in $M/\pi^{-1}(K)$ (resp. in M/N) and $[\tilde{m}]$ is the equivalence class of \tilde{m} in $(M/N)/K$.

We first show that φ is well-defined. For $m_1, m_2 \in M$, suppose that $\bar{m}_1 = \bar{m}_2$. Then we have

$$m_1 + c_1 = m_2 + c_2 \text{ for some } c_1, c_2 \in \pi^{-1}(K). \quad (14)$$

It follows that

$$\pi(m_1) + \pi(c_1) = \pi(m_2) + \pi(c_2). \quad (15)$$

Since $\pi(c_1), \pi(c_2) \in K$, (15) implies that $\pi(m_1) = \tilde{m}_1$ and $\pi(m_2) = \tilde{m}_2$ are equivalent under K . In particular, $[\tilde{m}_1] = [\tilde{m}_2]$, showing that φ is well-defined.

³Since we cannot locate the proof of the statement, we include our proof here.

Next, we show that φ is injective. Suppose that $\varphi(\overline{m_1}) = \varphi(\overline{m_2})$. So, m_1 and m_2 induce the same element of $(M/N)/K$, or there exist $k_1, k_2 \in K$ such that

$$\pi(m_1) + k_1 = \pi(m_2) + k_2. \quad (16)$$

By surjectivity of π , we may choose d_1 and d_2 which map to k_1 and k_2 under π . This implies that

$$\pi(m_1) + \pi(d_1) = \pi(m_2) + \pi(d_2), \quad (17)$$

in particular, $\overline{m_1 + d_1} = \overline{m_2 + d_2}$. Since $d_1, d_2 \in \pi^{-1}(K)$, we have that

$$\overline{m_1} = \overline{m_1 + d_1} = \overline{m_2 + d_2} = \overline{m_2},$$

showing that φ is injective. \square

Lemma 3.10. *\mathfrak{M} and \mathfrak{E} in Definition 3.8 are closed under composition and contain all isomorphisms.*

Proof. It is clear that \mathfrak{M} and \mathfrak{E} contain all isomorphisms. From the descriptions in Lemma 3.7, it is clear that \mathfrak{M} is closed under composition. So, we prove that \mathfrak{E} is closed under composition. Let $g : L \rightarrow M$ and $f : M \rightarrow N$ be normal epimorphisms and $h := fg$. From Lemma 3.7, we may assume that $M = L/T$ for some saturated submodule T of L and $g : L \rightarrow L/T$ is a projection. Now, since $f : L/T \rightarrow N$ is a normal epimorphism, again from Lemma 3.7, we may assume that $N = (L/T)/H$ for some saturated submodule H of (L/T) and f is a projection. Let $\pi : L \rightarrow (L/T)/H$ be a projection. It follows from Lemmas 3.3 and 3.9 that $\pi^{-1}(H)$ is a saturated submodule of L and $L/\pi^{-1}(H) = (L/T)/H$. Hence, we conclude that

$$h = fg : L \rightarrow N = L/\pi^{-1}(H)$$

is a projection. This shows that \mathfrak{E} is closed under composition. \square

Remark 3.11. One may be also tempted to use regular monomorphisms and regular epimorphisms to impose a proto-exact structure on Mod_R . However, these classes do not behave well for our purpose. More explicitly, regular mono/epi-morphisms are not closed under composition in general. In particular, Lemma 3.10 does not hold for regular morphisms. Note that even normal mono/epi-morphisms are not closed under composition in arbitrary pointed categories although they are closed under composition for Mod_R .

Lemma 3.12. *Let R be a semiring and M an R -module. Let $i \in \mathfrak{M}$, $\pi, \pi' \in \mathfrak{E}$, and L a saturated submodule of M . Then, the following hold.*

- (1) *Projections $\pi : M \rightarrow M/L$ and $\pi' : N \rightarrow N/L$ are normal epimorphisms.*
- (2) *The induced map $i' : M/L \rightarrow N/L$ is a normal monomorphism.*
- (3) *The following commutative diagram is a bi-Cartesian diagram in Mod_R .*

$$\begin{array}{ccc} M & \xrightarrow{i} & N \\ \pi \downarrow & & \downarrow \pi' \\ M/L & \xrightarrow{i'} & N/L \end{array} \quad (18)$$

Proof. We first note that since L is a saturated submodule of M and M is a saturated submodule of L (as $i \in \mathfrak{M}$), L is a saturated submodule of N . One can easily see from Lemma 3.7 that π and π' are normal epimorphisms.

Next, we claim that if $i : M \rightarrow N$ is a normal monomorphism, then the induced map $i' : M/L \rightarrow N/L$ is a normal monomorphism. In fact, it is clear that i' is injective. Hence, it is enough to prove that $i'(M/L)$ is saturated in N/L . We identify M/L and $i'(M/L)$. Let $[x]$ and $[y]$ be the equivalence classes of $x, y \in N$ in N/L . Suppose that $[x] + [y] = [x + y] \in M/L$ and $[x] \in M/L$, that is, $[x + y] = [m]$ and $[x] = [m']$ for some $m, m' \in M$. It follows that

$$x + y + \ell_1 = m + \ell_2, \quad x + \ell_3 = m' + \ell_4, \quad \ell_i \in L.$$

Hence we have

$$(m + \ell_2) + \ell_3 = x + \ell_3 + (y + \ell_1) = m' + \ell_4 + (y + \ell_1) = y + (m' + \ell_1 + \ell_4).$$

Therefore $y + (m' + \ell_1 + \ell_4) \in M$ and $(m' + \ell_1 + \ell_4) \in M$. Since i is a normal monomorphism, M is saturated submodule of N , so we conclude that $y \in M$, showing that $[y] \in M/L$. In particular, (18) is well-defined and $i' \in \mathfrak{M}$.

(co-Cartesian) Consider the following commutative diagram, where K is an R -module and α, β are morphisms in Mod_R :

$$\begin{array}{ccc}
 M & \xleftarrow{i} & N \\
 \pi \downarrow & & \downarrow \pi' \\
 M/L & \xleftarrow{i'} & N/L \\
 & \searrow \alpha & \downarrow \beta \\
 & & K
 \end{array}
 \quad (19)$$

(Note: A dashed arrow γ points from N/L to K , and a solid arrow α points from M/L to K .)

Since $L \subseteq M$, for any $\ell \in L$, we have

$$\pi' i(\ell) = i' \pi(\ell) = 0_{N/L}.$$

Since i' is injective (as it is a normal monomorphism), we have $\pi(\ell) = 0_{M/L}$. It follows that

$$0_K = \alpha \pi(\ell) = \beta i(\ell) = \beta(\ell). \quad (20)$$

Now, if $\pi'(x) = \pi'(y)$, then we have $x + \ell_1 = y + \ell_2$ for some $\ell_i \in L$. Then, from (20), we have

$$\beta(x) = \beta(y).$$

Therefore, the function $\gamma: N/L \rightarrow K$ sending $[x]$ to $\beta(x)$ is a well-defined morphism of R -modules. Furthermore, it is clear that with γ , (19) commutes, and if a morphism $\delta: N/L \rightarrow K$ makes (19) to commute, then $\delta = \gamma$. Hence (18) is a co-Cartesian diagram.

(Cartesian) Consider the following diagram:

$$\begin{array}{ccc}
 & & K \\
 & & \downarrow \beta \\
 & & N \\
 & \swarrow \gamma & \downarrow \pi' \\
 & & M \\
 & \swarrow \alpha & \downarrow \pi \\
 & & M/L \\
 & & \xleftarrow{i'} N/L
 \end{array}
 \quad (21)$$

(Note: A solid arrow α points from M/L to K , and a solid arrow β points from N to K . A dashed arrow γ points from N to K .)

where K is an R -module and α, β are morphisms of R -modules. We claim that $\beta(K) \subseteq i(M)$. In fact, for any $x \in K$, we have $i' \alpha(x) = \pi' \beta(x)$. We may write $\alpha(x) = \pi(m)$ for some $m \in M$ since π is surjective (as it is a normal epimorphism). Hence we have

$$\pi' i(m) = i' \pi(m) = \pi' \beta(x).$$

In particular, $\beta(x) + \ell_1 = i(m) + \ell_2$ for some $\ell_i \in L$. But, since $L \subseteq M$ and M is assumed to be saturated in N (since i is a normal monomorphism), we have that $\beta(x) \in i(M)$. Hence, the function $\gamma: K \rightarrow M$ sending x to $\beta(x)$ is a well-defined morphism of R -modules, which makes the diagram (21) to commute. It is clear that such γ is unique. This proves that (18) is a Cartesian diagram. \square

Lemma 3.13. *Every diagram in Mod_R*

$$\begin{array}{ccc}
 & & N \\
 & & \downarrow j' \\
 M' & \xleftarrow{i'} & N'
 \end{array}
 \quad (22)$$

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with $i' \in \mathfrak{M}$ and $j' \in \mathfrak{E}$ can be completed to a bi-Cartesian square (3) with $i \in \mathfrak{M}$ and $j \in \mathfrak{E}$.

Proof. Let $M = (j')^{-1}(i'(M'))$. Since $i'(M')$ is a saturated submodule of N' , it follows from Lemma 3.3, M is a saturated submodule of N . In particular the inclusion $i : M \rightarrow N$ is a normal monomorphism.

We also have a canonical surjective homomorphism $\pi : M \rightarrow M'$ sending any $m \in M$ to $((i')^{-1}j'i)(m)$. Note that π is well-defined since $j'i(M) \subseteq i'(M')$ by definition. We claim that π is a normal epimorphism. In fact, since $j' : N \rightarrow N'$ is a normal epimorphism, we may assume that $j' : N \rightarrow N/L$, where L is a saturated submodule of N . Then, by definition of M , we have $L \subseteq M$. In particular, we may assume that

$$\pi : M \rightarrow M' = M/L$$

and hence π is a normal epimorphism since L is a saturated submodule of M . Therefore, we can complete the diagram (22) as follows:

$$\begin{array}{ccc} M & \xrightarrow{i} & N \\ \pi \downarrow & & \downarrow j' \\ M/L & \xrightarrow{i'} & N/L \end{array} \quad (23)$$

Now it follows from Lemma 3.12 that (23) is a bi-Cartesian diagram. \square

Lemma 3.14. *Every diagram in Mod_R*

$$\begin{array}{ccc} M & \xrightarrow{i} & N \\ j \downarrow & & \\ M' & & \end{array} \quad (24)$$

with $i \in \mathfrak{M}$ and $j \in \mathfrak{E}$ can be completed to a bi-Cartesian square (3) with $i' \in \mathfrak{M}$ and $j' \in \mathfrak{E}$.

Proof. Since j is a normal epimorphism, we may assume that $j : M \rightarrow M' = M/L$, where L is a saturated submodule of M . We let $j' : N \rightarrow N/L$ be a natural projection map, which is clearly a normal epimorphism since L is also a saturated submodule of N .⁴ As in the proof of Lemma 3.12, i induces a natural map

$$i' : M/L \rightarrow N/L$$

which is a normal monomorphism. In particular, we can complete the diagram (24) as follows:

$$\begin{array}{ccc} M & \xrightarrow{i} & N \\ \pi \downarrow & & \downarrow j' \\ M/L & \xrightarrow{i'} & N/L \end{array} \quad (25)$$

Now, it follows from Lemma 3.12 that (25) is a bi-Cartesian diagram. \square

Theorem 3.15. *Let R be a semiring. Then the triple $(\text{Mod}_R, \mathfrak{M}, \mathfrak{E})$ as in Definition 3.8 defines a proto-exact category.*

Proof. Mod_R is clearly pointed. The axioms (2), (4), and (5) in Definition 2.1 directly follow from Lemmas 3.10, 3.13, and 3.14. Finally, the axiom (3) is a direct consequence of (4) and (5) along with the uniqueness of pullbacks and pushouts. \square

A finitely generated module N over a semiring R is said to be *Noetherian* if any submodule of N is also finitely generated. The following is obvious from Theorem 3.15.

⁴Since M is a saturated submodule of N , if $x, y \in N$ and $x + y, y \in L$, then $x + y, y \in M$. So, this follows from the fact that L is a saturated submodule of M .

Corollary 3.16. *Let R be a semiring such that any finitely generated module N over R is Noetherian.⁵ Let Mod_R^{fg} be a subcategory of Mod_R consisting of finitely generated R -modules with the same exact structure, then Mod_R^{fg} is also a proto-exact category.*

Remark 3.17. When one has a proto-exact category \mathcal{C} , then one may study the Hall algebra $H_{\mathcal{C}}$ of \mathcal{C} or develop K -theory of \mathcal{C} . When \mathcal{C} is the category of matroids, these two applications were considered in [EJS20]. In this paper, with these two applications in mind, we equip the category of modules over a semiring with a proto-exact structure. In [JMT20], some ingredients of K -theory for semirings were introduced without appealing to any proto-exact structure.

4. Algebraic lattices, \mathbb{B} -modules, and matroids

In this section, we focus on the case when $R = \mathbb{B}$, the Boolean semifield. It is well-known that any finite \mathbb{B} -module is naturally equipped with a lattice structure.⁶ See, for instance, [Les12, §2].

We prove that the category \mathcal{L} of algebraic lattices is proto-exact, and explore certain connections to \mathbb{B} -modules and the subcategory of geometric lattices (i.e. simple matroids).

4.1. Algebraic lattices. In this subsection, we prove that the category of algebraic lattices is equipped with a proto-exact structure. We first recall some necessary definitions.

Definition 4.1. Let L be a lattice.

- (1) By a *cover* of an element $x \in L$, we mean a family of elements $\{y_i\}_{i \in I}$ such that

$$x \leq \bigvee_{i \in I} y_i.$$

- (2) For a complete lattice L , an element $x \in L$ is said to be *compact* if every cover of x has a finite subcover.
(3) An *algebraic lattice* is a complete lattice such that any element is a join of compact elements.

Note that any finite lattice is algebraic, but we will use the term “finite algebraic lattices” to emphasize that we work with algebraic lattices in general rather than lattices.

Remark 4.2. Equivalently, one may define an algebraic lattice as a poset which is locally finitely presentable as a category.

Definition 4.3. Let L_1 and L_2 be algebraic lattices. A function $f : L_1 \rightarrow L_2$ is said to be a *morphism of algebraic lattices* if the following hold:

- (1) f preserve arbitrary joins: for any $\{y_i\}_{i \in I} \subseteq L_1$ we have

$$f\left(\bigvee_{i \in I} y_i\right) = \bigvee_{i \in I} f(y_i).$$

- (2) If $x \in L_1$ is compact, then $f(x)$ is compact element in L_2 .

Remark 4.4. We note that in our definition of morphisms for algebraic lattices, we do not require them to preserve meets. In particular, a morphism of algebraic lattices as in Definition 4.3 does not have to be a morphism of underlying lattices.

Definition 4.5. Let L_1 and L_2 be algebraic lattices and $f : L_1 \rightarrow L_2$ be a morphism of algebraic lattices as in Definition 4.3.

- (1) f is said to be a *normal monomorphism* if
(a) f is an injection;
(b) $f(L_1)$ is downward closed, i.e., $\forall x \in f(L_1)$ and $\forall y \in L_2$ if $y \leq x$ then $y \in f(L_1)$.
(2) f is said to be a *normal epimorphism* if there exists $x_0 \in L_1$ such that

⁵For instance, this is true when $R = \mathbb{B}$.

⁶To be precise, any \mathbb{B} -module defines a join semilattice.

- (a) $L_2 \simeq \{y \in L_1 \mid y \geq x_0\}$;
- (b) $f : L_1 \rightarrow L_2$ is given by joining with x_0 under the above isomorphism.

Definition 4.6. Let L be an algebraic lattice. A subset $K \subseteq L$ is said to be an *algebraic sub-lattice* if K itself is an algebraic lattice and the inclusion $i : K \rightarrow L$ is a normal monomorphism.

Let L be an algebraic lattice and K be an algebraic sub-lattice of L . Let 1_K be the unique maximal element of K . By the quotient lattice L/K , we mean the following algebraic sub-lattice of L :

$$L/K := \{x \in L \mid x \geq 1_K\}.$$

The canonical map $f : L \rightarrow L/K$ sending x to $(x \vee 1_K)$ is a normal epimorphism.

Definition 4.7. Let \mathcal{L} be the category of algebraic lattices. Let \mathfrak{M} be the class of normal monomorphisms in \mathcal{L} and \mathfrak{E} the class of normal epimorphisms in \mathcal{L} .

Lemma 4.8. \mathfrak{M} and \mathfrak{E} in Definition 4.5 are closed under composition and contain all isomorphisms.

Proof. It is clear that \mathfrak{M} and \mathfrak{E} contain all isomorphisms.

To show that \mathfrak{M} is closed under composition, suppose that $f_1 : L_1 \rightarrow L_2$ and $f_2 : L_2 \rightarrow L_3$ are normal monomorphisms and $g := f_2 f_1$. Clearly, g is an inclusion. Suppose further that $x \in g(L_1)$, $y \in L_3$, and $y \leq x$. We write $x = f_2(f_1(z))$. Since $y \leq f_2(f_1(z))$ and f_2 is a normal monomorphism, we have that $y \in f_2(L_2)$. Let $y = f_2(t)$ for some $t \in L_2$. Then, we have $f_2(t) = y \leq x = f_2(f_1(z))$. Since f_2 is an inclusion, we have $t \leq f_1(z)$. Again, since $f_1(L_1)$ is downward closed, we have that $t \in f_1(L_1)$, and hence $y = f_2(t) \in g(L_1)$.

To show that \mathfrak{E} is closed under composition, suppose that $f_1 : L_1 \rightarrow L_2$ and $f_2 : L_2 \rightarrow L_3$ are normal epimorphisms. From the definition, we may assume that f_1 is the composition of the map $L_1 \rightarrow L'_1 = \{x \in L_1 \mid x \geq x_1\}$ given by joining with some element x_1 and an isomorphism $u : L'_1 \rightarrow L_2$. Similarly, f_2 is the composition of the map $L_2 \rightarrow L'_2$ given by joining with some element x_2 and an isomorphism $v : L'_2 \rightarrow L_3$. Then we have

$$f_2(f_1(x)) = f_2(u(x_1 \vee x)) = v(x_2 \vee u(x_1 \vee x)) = v(u(u^{-1}(x_2) \vee x_1 \vee x)).$$

Thus $f_2 f_1$ is the composition of the join map of $u^{-1}(x_2) \vee x_1$ with the map $vu : u^{-1}(L_2) \rightarrow L_3$.⁷ vu is an isomorphism because both v and the map $u : u^{-1}(L_2) \rightarrow L_2$ are isomorphisms. Now, it follows from the definition that $f_2 f_1$ is a normal epimorphism. \square

Lemma 4.9. Let M be an algebraic lattice and L an algebraic sub-lattice of M . Then, any compact element in M/L is the join of a compact element of M with 1_L .

Proof. Let x be a compact element of M/L , and by viewing x as an element of M , we can write it as the join of compact elements $\{y_\alpha\}$ of M (where α ranges over some indexing set). Then x is also the join of elements of the form $y_\alpha \vee 1_L$, and since these lie in M/L we can pick a finite subcover. This allows one to write $x = y_1 \vee \cdots \vee y_n \vee 1_L$, showing that $y_1 \vee \cdots \vee y_n$ is compact. \square

Now, we prove that $(\mathcal{L}, \mathfrak{M}, \mathfrak{E})$ is a proto-exact category. As in the case of semirings, we will interchangeably use the terms normal monomorphisms (resp. normal epimorphisms) and admissible monomorphisms (resp. admissible epimorphisms). We will make use of the following lemma whose proof is similar to Lemma 3.12.

Lemma 4.10. Let M and N be algebraic lattices and L an algebraic sub-lattice of M . Let $i \in \mathfrak{M}$, $\pi, \pi' \in \mathfrak{E}$. Then, the induced map $i' : M/L \rightarrow N/L$ is a normal monomorphism and the following

⁷Of course, we have to restrict the domain of u in order for vu to make sense.

commutative diagram is a bi-Cartesian diagram in \mathcal{L} .

$$\begin{array}{ccc}
 M & \xrightarrow{i} & N \\
 \pi \downarrow & & \downarrow \pi' \\
 M/L & \xrightarrow{i'} & N/L
 \end{array} \tag{26}$$

Proof. One can easily observe that if $i : M \rightarrow N$ is a normal monomorphism, then the induced map $i' : M/L \rightarrow N/L$ is also a normal monomorphism. In fact, the only nontrivial part is to show that i' preserves compact elements which directly follows from Lemma 4.9.

(co-Cartesian) Consider the following commutative diagram, where K is an algebraic lattice and α, β are morphisms in \mathcal{L} :

$$\begin{array}{ccc}
 M & \xrightarrow{i} & N \\
 \pi \downarrow & & \downarrow \pi' \\
 M/L & \xrightarrow{i'} & N/L
 \end{array}
 \begin{array}{l}
 \searrow \beta \\
 \downarrow \gamma \\
 \searrow \alpha
 \end{array}
 \rightarrow K
 \tag{27}$$

From the definition of N/L , we can define the function $\gamma : N/L \rightarrow K$ such that $\gamma(x) = \beta(x)$. We first claim that γ is a morphism of algebraic lattices as in Definition 4.3. If $\{y_i\} \subseteq N/L$, we may assume that $\{y_i\} = \{x_i \vee 1_L\}$ for some $x_i \in N$. In particular, for a nonempty join, we have

$$\gamma(\vee y_i) = \gamma(\vee(x_i \vee 1_L)) = \beta(\vee(x_i \vee 1_L)) = \vee(\beta(x_i \vee 1_L)) = \vee\beta(y_i) = \vee\gamma(y_i),$$

showing that γ preserves nonempty joins. For the empty join, since $\beta i = \alpha \pi$ and $\pi(1_L) = 0_{M/L}$, we have that $\beta(1_L) = 0$. It follows from the definition that $\gamma(1_L) = \beta(1_L) = 0$. In particular, since the empty join in N/L is 1_L , γ preserves the empty join. Hence, γ preserves arbitrary joins. Furthermore, since β is a morphism of algebraic lattices, β preserves compact elements. It follows from Lemma 4.9 that γ preserves compact elements.

Next, one may observe that $\gamma \pi' = \beta$ since $\beta(1_L)$ is the minimal element in N/L and γ is a morphism of algebraic lattices. One can also easily check that $\alpha = \gamma i'$. This shows the commutativity of (27) with γ .

Finally, if a morphism $\delta : N/L \rightarrow K$ makes (27) to commutes, then $\delta = \gamma$. Hence (26) is a co-Cartesian diagram.

(Cartesian) Consider the following diagram:

$$\begin{array}{ccc}
 & & K \\
 & \searrow \beta & \\
 & & \downarrow \gamma \\
 & & M \xrightarrow{i} N \\
 & \searrow \alpha & \downarrow \pi \\
 & & M/L \xrightarrow{i'} N/L \\
 & & \downarrow \pi'
 \end{array}
 \tag{28}$$

where K is an algebraic lattice and α, β are morphisms in \mathcal{L} . We claim that $\beta(K) \subseteq i(M)$. In fact, for any $x \in K$, we have $i' \alpha(x) = \pi' \beta(x)$. We may write $\alpha(x) = \pi(m)$ for some $m \in M$ since π is surjective (as it is a normal epimorphism). Hence we have

$$\pi' i(m) = i' \pi(m) = \pi' \beta(x).$$

In other words, we have

$$\beta(x) \vee 1_L = i(m) \vee 1_L.$$

Now, since L is an algebraic sub-lattice of M , we have that $i(m) \vee 1_L \in i(M)$, and hence $\beta(x) \vee 1_L \in i(M)$. However, since i is a normal monomorphism, $i(M)$ is downward closed, showing that $\beta(x) \in i(M)$. Hence, the function $\gamma : K \rightarrow M$ sending x to $\beta(x)$ is a well-defined morphism in \mathcal{L} , which makes the diagram (28) to commute. It is clear that such γ is unique. This proves that (26) is a Cartesian diagram. \square

Lemma 4.11. *Let M be an algebraic lattice and K be a subset of M which is closed under arbitrary joins and also is downward closed. Then K is an algebraic sub-lattice.*

Proof. To show that K is a complete lattice, it is well-known that one only needs to show that K contains arbitrary joins. In particular, it follows from our assumption that K is a complete lattice. Furthermore, every element of K is a join of compact elements of M ; since K is downward closed, these are elements of K as well (and are easily seen to be compact inside K). So K is an algebraic lattice.

Next, we prove that the inclusion is a normal monomorphism. If $x \in K$ is compact and the $S = \{y_i\}_{i \in I}$ is a cover of x in M , then $S' = \{y_i \wedge 1_K\}$ is a cover of x in K . By picking a finite subcover of S' , we obtain a finite subcover of S for x in M . It follows that x is compact in M , so the inclusion preserves compactness. Since it clearly preserves joins, it is a morphism of algebraic lattices. Finally, since K is downward closed, it is an algebraic sub-lattice. \square

Lemma 4.12. *Let $f : M \rightarrow N$ be a morphism in \mathcal{L} . If L is an algebraic sub-lattice of N (as in Definition 4.6), then $f^{-1}(L)$ is an algebraic sub-lattice of M .*

Proof. From Lemma 4.11, we only have to prove that the inclusion $i : f^{-1}(L) \rightarrow M$ is closed under joins and is downward closed.

All joins will take place in M, N , or L (there is no ambiguity since joins in L and N are the same). If $\{x_i\}_{i \in I} \subseteq f^{-1}(L)$, then we have

$$f(\vee x_i) = \vee f(x_i) \in L.$$

Hence $f^{-1}(L)$ is closed under joins.

To check that $f^{-1}(L)$ is downward closed, suppose that $y \in f^{-1}(L)$ and $x \leq y$. Then $f(x) \leq f(y)$, and hence $f(x) \in L$, showing that $f^{-1}(L)$ is downward closed. \square

Lemma 4.13. *Every diagram in \mathcal{L}*

$$\begin{array}{ccc} & N & \\ & \downarrow j' & \\ M' & \xrightarrow{i'} N' & \end{array} \quad (29)$$

with $i' \in \mathfrak{M}$ and $j' \in \mathfrak{E}$ can be completed to a bi-Cartesian square (3) with $i \in \mathfrak{M}$ and $j \in \mathfrak{E}$.

Proof. Let $M = (j')^{-1}(i'(M'))$. Since $i'(M')$ is an algebraic sub-lattice of N' , it follows from Lemma 4.12, M is an algebraic sub-lattice of N . In particular the inclusion $i : M \rightarrow N$ is a normal monomorphism.

Next, consider the function $\pi : M \rightarrow M'$ sending any $m \in M$ to $((i')^{-1} j'i)(m)$. Note that π is well-defined since $j'i(M) \subseteq i'(M')$ by definition. We claim that π is a normal epimorphism. In fact, since $j' : N \rightarrow N'$ is a normal epimorphism, we may assume that $j' : N \rightarrow N/L$, where L is an algebraic sub-lattice of N . Then, by definition of M , we have $L \subseteq M$ (we identify $i(M)$ and M). In particular, we may assume that

$$\pi : M \rightarrow M' = M/L$$

and hence π is a normal epimorphism since L is an algebraic sub-lattice of M . Therefore, we can complete the diagram (29) as follows:

$$\begin{array}{ccc} M & \xrightarrow{i} & N \\ \pi \downarrow & & \downarrow j' \\ M/L & \xrightarrow{i'} & N/L \end{array} \quad (30)$$

Now it follows from Lemma 4.10 that (30) is a bi-Cartesian diagram. \square

Lemma 4.14. *Every diagram in \mathcal{L}*

$$\begin{array}{ccc} M & \xrightarrow{i} & N \\ j \downarrow & & \\ M' & & \end{array} \quad (31)$$

with $i \in \mathfrak{M}$ and $j \in \mathfrak{E}$ can be completed to a bi-Cartesian square (3) with $i' \in \mathfrak{M}$ and $j' \in \mathfrak{E}$.

Proof. Since j is a normal epimorphism, we may assume that $j : M \rightarrow M' = M/L$, where L is an algebraic sub-lattice of M . We let $j' : N \rightarrow N/L$ be the map sending x to $(x \vee 1_L)$, which is a normal epimorphism by definition. One can easily see that i induces a natural map

$$i' : M/L \rightarrow N/L, \quad x \mapsto i(x),$$

which is a normal monomorphism. In particular, we can complete the diagram (31) as follows:

$$\begin{array}{ccc} M & \xrightarrow{i} & N \\ \pi \downarrow & & \downarrow j' \\ M/L & \xrightarrow{i'} & N/L \end{array} \quad (32)$$

Now, it follows from Lemma 4.10 that (32) is a bi-Cartesian diagram. \square

Theorem 4.15. *The triple $(\mathcal{L}, \mathfrak{M}, \mathfrak{E})$ as in Definition 4.7 defines a proto-exact category.*

Proof. \mathcal{L} is clearly pointed. The axioms (2), (4), and (5) in Definition 2.1 directly follow from Lemmas 4.8, 4.13, and 4.14. Finally, the axiom (3) is a direct consequence of (4) and (5) along with the uniqueness of pullbacks and pushouts. \square

4.2. Connection to \mathbb{B} -modules and geometric lattices. In this subsection, we study relations between the category of \mathbb{B} -modules and the category of algebraic lattices \mathcal{L} . We refer the reader to [JRT20, §3] more on this line of ideas.

Let M be a \mathbb{B} -module and $S(M)$ be the set of all saturated submodules of M . It is well-known that $S(M)$ is an algebraic lattice. Let $S(M)^c$ be an algebraic lattice of $S(M)$ consisting of compact elements of $S(M)$. The following is also well-known.

Theorem 4.16. *With the same notation as above, we have the following.*

- (1) $M \simeq S(M)^c$ (as \mathbb{B} -modules).
- (2) Let L be an algebraic lattice. Then L^c is equipped with a \mathbb{B} -module structure with the order induced from L . Moreover, L is isomorphic to $S(L^c)$ (as lattices), where we consider L^c as a \mathbb{B} -module.

The following lemma will be handy.

Lemma 4.17. [JRT20] *Let M be a \mathbb{B} -module and N be a submodule of M . The saturated submodule of M generated by N is given as follows:*

$$\{x \in M \mid \exists y \in N \text{ such that } x + y \in N\}.$$

For the following lemma, we let $\langle N \rangle$ be the saturated submodule generated by N .

Lemma 4.18. *Let $f : M \rightarrow N$ be a morphism of \mathbb{B} -modules. Let $\{M_i\}$ be a family of submodules of M , T be the submodule of M generated by $\cup M_i$, and $L = \langle T \rangle$. Then we have*

$$\langle f(L) \rangle = \langle \cup f(M_i) \rangle.$$

Proof. Let $K = \langle \cup f(M_i) \rangle$. It is clear that $K \subseteq \langle f(L) \rangle$. Conversely, suppose that $y \in \langle f(L) \rangle$. It follows from Lemma 4.17 that there exists $x \in f(L)$ such that

$$y + x \in f(L).$$

Take $z \in L$ such that $f(z) = y + x$. Again, from Lemma 4.17, there exists $q \in T$ such that $z + q \in T$. In particular, we can find $x_i, y_i \in \cup M_i$ such that

$$z + q = \sum x_i, \quad q = \sum y_i.$$

In particular, we have

$$y + x + f(q) = f(z) + f(q) = f(z + q) = \sum f(x_i), \quad f(q) = \sum f(y_i). \quad (33)$$

On the other hand, since $x \in f(L)$, we may write $x = f(w)$ for some $w \in L$. So, there exists $q' \in T$ such that $w + q' \in T$. In particular, we can find $a_i, b_i \in \cup M_i$ such that

$$w + q' = \sum a_i, \quad q' = \sum b_i.$$

Hence we have

$$f(w + q') = f(w) + f(q') = x + f(q') = \sum f(a_i) = x + \sum f(b_i). \quad (34)$$

By combining (33) and (34), we have

$$y + x + f(q) + f(q') = y + \sum f(a_i) + f(q) = y + \sum f(a_i) + \sum f(y_i) = \sum f(x_i) + \sum f(b_i).$$

Since the addition is idempotent, we further obtain:

$$y + (\sum f(a_i) + \sum f(b_i) + \sum f(x_i) + \sum f(y_i)) = (\sum f(a_i) + \sum f(b_i) + \sum f(x_i) + \sum f(y_i)),$$

showing that $y \in K$. \square

Lemma 4.19. *Let $f : M \rightarrow N$ be a morphism of \mathbb{B} -modules. Let $K = \{m \in M \mid m \leq b\}$ for some b . Then K is a saturated submodule of M and $\langle f(K) \rangle = \{n \in N \mid n \leq f(b)\}$.*

Proof. From Proposition 3.2, one may easily observe that K is a saturated submodule of M . Let $L = \{n \in N \mid n \leq f(b)\}$. It is clear that $f(K) \subseteq L$ and L is a saturated submodule of N (again from Proposition 3.2). So, it is enough to show that any saturated submodule T of N containing $f(K)$ contains L as well. However, as $b \in K$, we have $f(b) \in f(K) \subseteq T$. It follows that if $n \leq f(b)$ then $n \in T$ since T is saturated. Hence $L \subseteq T$. \square

Lemma 4.20. *Let M be a \mathbb{B} -module and N be a saturated submodule of M . Then N is compact (as an element of the algebraic lattice $S(M)$) if and only if $N = \{a \in M \mid a \leq b\}$ for some $b \in M$.*

Proof. First, suppose that N is compact. Let $S = \{\langle n \rangle\}_{n \in N}$ be a cover of N in $S(M)$. Since N is compact, there exists a finite subcover, say $\{\langle n_1 \rangle, \dots, \langle n_r \rangle\}$. Let $b = \sum_{i=1}^r n_i$. Then for any $a \in N$, we have $a \leq b$. Now, from Proposition 3.2, $N = \{a \in M \mid a \leq b\}$.

Conversely, suppose that $N = \{a \in M \mid a \leq b\}$ for some $b \in M$. Let $S = \{K_i\}_{i \in I}$ be a cover of N . Since S is a cover, after relabeling if needed, we may assume that there exist $k_i \in K_i$ (for $i = 1, \dots, t$) such that $b \leq \sum_{i=1}^t k_i$. It follows that $\{K_1, \dots, K_t\}$ is a finite subcover, showing that N is compact. \square

Proposition 4.21. *Theorem 4.16 is functorial and hence one obtains the following functor:*

$$S : \text{Mod}_{\mathbb{B}} \rightarrow \mathcal{L}, \quad M \mapsto S(M).$$

Proof. Let M, N be \mathbb{B} -modules. For each $f \in \text{Hom}(M, N)$, consider the function $S(f) : S(M) \rightarrow S(N)$ sending any saturated submodule L of M to the saturated submodule of N generated by $f(L)$. It follows from Lemma 4.18 that $S(f)$ preserves arbitrary joins. Next, suppose that $y \in S(M)$ is compact. We claim that $z = S(f)(y)$ is compact. In fact, from Lemma 4.20 a saturated submodule y is compact if and only if it has the form $y = \{a \mid a \leq b\} \subseteq M$ for some $b \in M$. Furthermore, from Lemma 4.19, we have that $z = \{d \mid d \leq f(b)\}$, showing that z is compact. This proves that $S(f)$ is a morphism of algebraic lattices. It remains to check that $S(fg) = S(f)S(g)$ for morphisms $f : M \rightarrow N$ and $g : L \rightarrow M$ in $\text{Mod}_{\mathbb{B}}$. But, this is clear from the definition. \square

Lemma 4.22. *Let $f : M \rightarrow N$ be a morphism of \mathbb{B} -modules.*

- (1) *If f is a normal monomorphism in $\text{Mod}_{\mathbb{B}}$, then $S(f)$ is a normal monomorphism in \mathcal{L} .*
- (2) *If f is a normal epimorphism in $\text{Mod}_{\mathbb{B}}$, then $S(f)$ is a normal epimorphism in \mathcal{L} .*

Proof. (1) : From Lemma 3.7, f is a normal monomorphism in $\text{Mod}_{\mathbb{B}}$ if and only if $f(M)$ is a saturated submodule of N . In particular, this implies that $S(f) : S(M) \rightarrow S(N)$ is an inclusion of algebraic lattices which is downward closed, showing that $S(f)$ is a normal monomorphism in \mathcal{L} .

(2) : We first claim that if L is a saturated submodule of a \mathbb{B} -module M , then we have

$$S(M/L) = S(M)/S(L).$$

In fact, the algebraic lattice $S(M/L)$ consists of saturated submodules of M containing L . This is precisely the definition of $S(M)/S(L)$ as an algebraic lattice. Now, from Lemma 3.7, if $f : M \rightarrow N$ is a normal epimorphism, then we may assume that $f : M \rightarrow M/L$ for some saturated submodule L of M , where f is a natural projection. In particular, one can easily check that $S(f) : S(M) \rightarrow S(M/L) = S(M)/S(L)$ sends any saturated submodule K to $\langle K \cup L \rangle$, i.e., $S(f)$ is given by joining with L (as an element of $S(M)$). This shows that $S(f)$ is a normal epimorphism. \square

Proposition 4.23. *The functor $S : \text{Mod}_{\mathbb{B}} \rightarrow \mathcal{L}$ is fully faithful and essentially surjective. Moreover, S is an exact functor.*

Proof. It follows from Theorem 4.16 that S is essentially surjective. Once we prove that S is also fully faithful, it follows from Lemma 4.22 that the functor S is exact. So, we only have to prove that S is fully faithful.

Suppose that $S(f) = S(g)$ for $f, g \in \text{Hom}(M, N)$ for \mathbb{B} -modules M, N . Note that for any $x \in M$ the saturated submodule of M generated by x is $\{m \in M \mid m \leq x\}$. In particular, this implies that if $S(f) = S(g)$, then $f = g$, showing that the functor S is faithful.

Next, consider $\alpha : S(M) \rightarrow S(N)$. This restricts to $\alpha^c : S(M)^c \rightarrow S(N)^c$ since α sends compact elements to compact elements. With the isomorphism in Theorem 4.16, we have $S(M)^c \simeq M$ and $S(N)^c \simeq N$. One can easily see that $\alpha^c \in \text{Hom}(M, N)$ and $S(\alpha^c) = \alpha$, showing that the functor S is full. \square

Corollary 4.24. *Let $\text{Mod}_{\mathbb{B}}^{\text{fin}}$ be the category of finite \mathbb{B} -modules and \mathcal{L}^c be the subcategory of \mathcal{L} consisting of finite algebraic lattices. Then, $\text{Mod}_{\mathbb{B}}^{\text{fin}}$ is equivalent to \mathcal{L}^c as proto-exact categories.*

Proof. The equivalence of categories in Proposition 4.23 restricts to the equivalence of the categories $\text{Mod}_{\mathbb{B}}^{\text{fin}}$ and \mathcal{L}^c . Hence, we have a quasi-inverse:

$$S^{-1} : \mathcal{L}^c \rightarrow \text{Mod}_{\mathbb{B}}^{\text{fin}}.$$

It is enough to prove that S^{-1} is exact. In fact, we only have to prove that S^{-1} sends normal monomorphisms (resp. normal epimorphisms) in \mathcal{L}^c to normal monomorphisms (resp. normal epimorphisms) in $\text{Mod}_{\mathbb{B}}^{\text{fin}}$.

Suppose that $S(f) : S(M) \rightarrow S(N)$ is a normal monomorphism of finite algebraic lattices. From Lemma 3.7, it is enough to show that f is an injection and $f(M)$ is isomorphic to a saturated submodule of N . It follows from Lemmas 4.17 and 4.18 that for any $x \in M$ we have

$$S(f)(\langle x \rangle) = \langle f(x) \rangle \text{ and } \langle x \rangle = \{y \in M \mid y \leq x\}. \quad (35)$$

In particular, f is an injection; if $\langle f(x) \rangle = \langle f(y) \rangle$, then since $S(f)$ is injective, we have $\langle x \rangle = \langle y \rangle$. This implies that $x \leq y$ and $y \leq x$ and hence $x = y$.

Furthermore, $f(M)$ is a saturated submodule of N . To see this, suppose that we have $x \in N$ and $y \in f(M)$ such that $x + y \in f(M)$. We have that $x \leq x + y$, hence $\langle x \rangle \leq \langle x + y \rangle$ in $S(N)$. But, since $S(f)$ is normal, the image $S(f)(S(M))$ is downward closed. It follows that $\langle x \rangle = S(f)(L)$ for some saturated submodule L of M , in particular, $\langle x \rangle$ is the saturation closure of $f(L)$. This implies that there is some $q \in L$ such that $x \leq f(q)$. But, on the other hand, $f(q)$ is in $f(L)$ and hence is in $\langle x \rangle$, showing that $f(q) \leq x$. This implies that $x \in f(M)$. Therefore $f(M)$ is a saturated submodule of N .

Next, suppose that $S(f) : S(M) \rightarrow S(N)$ is a normal epimorphism of finite algebraic lattices. Write $S(N) = S(M)/L$ for some algebraic sub-lattice L . Let H be the maximal element of L . We note that since $S(M)$ and $S(N)$ are finite algebraic lattices, any element is compact, and hence from Theorem 4.16 any element of $S(M)$ (resp. $S(N)$) corresponds to a unique element in M (resp. N).

We first claim that $f : M \rightarrow N$ is surjective. Let $y \in N$. Then, for the corresponding compact element $\langle y \rangle \in S(N)$, it follows from Lemma 4.9 that there is some compact element $\langle x \rangle \in S(M)$ such that $\langle y \rangle$ is the join of H and $\langle x \rangle$. In other words, we have

$$\langle y \rangle = H \vee \langle x \rangle = S(f)(\langle x \rangle)$$

In particular, $\langle y \rangle = \langle f(x) \rangle$, and hence $y = f(x)$. This shows that f is surjective.

Now, if $a \in H$, then $\langle a \rangle$ is a subset of H , hence is in L since L is downwards closed. This implies that $S(f)(\langle a \rangle) = 0$, and hence $f(a) = 0$. In particular, f factors through M/H . Suppose $x, y \in M$ satisfy $f(x) = f(y)$. Then we have $f(\langle x \rangle) = f(\langle y \rangle)$, i.e., $\langle x, H \rangle = \langle y, H \rangle$. Therefore, there exist $h_1, h_2 \in H$ such that

$$x \leq y + h_1, \quad y \leq x + h_2,$$

which implies that

$$x + (h_1 + h_2) = y + (h_1 + h_2).$$

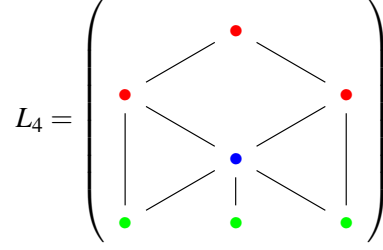
It follows that x and y determine the same equivalence class in M/H , and hence the induced map $M/H \rightarrow N$ is injective, showing that $N = M/H$. This shows that f is a normal epimorphism. \square

Example 4.25. Consider the following algebraic lattices.

$$L' = \left(\begin{array}{ccc} & \bullet & \\ & / \quad \backslash & \\ \bullet & & \bullet \\ & | & \\ & \bullet & \end{array} \right), \quad L'' = \left(\begin{array}{ccc} & \bullet & \\ & / \quad \backslash & \\ \bullet & & \bullet \\ & \backslash \quad / & \\ & \bullet & \end{array} \right)$$

The following are some possible algebraic lattices L which fit into a short exact sequence $L' \rightarrow L \rightarrow L''$, where green dots belong to L' , the red dots belong to L'' , and the blue dot belongs to both L' and L'' .

$$L_1 = \left(\begin{array}{ccc} & \bullet & \\ & / \quad \backslash & \\ \bullet & & \bullet \\ & \backslash \quad / & \\ & \bullet & \\ & | & \\ & \bullet & \\ & / \quad \backslash & \\ \bullet & & \bullet \\ & | & \\ & \bullet & \end{array} \right), \quad L_2 = \left(\begin{array}{ccc} & \bullet & \\ & / \quad \backslash & \\ \bullet & & \bullet \\ & \backslash \quad / & \\ & \bullet & \\ & | & \\ & \bullet & \\ & / \quad \backslash & \\ \bullet & & \bullet \\ & | & \\ & \bullet & \end{array} \right), \quad L_3 = \left(\begin{array}{ccc} & \bullet & \\ & / \quad \backslash & \\ \bullet & & \bullet \\ & \backslash \quad / & \\ & \bullet & \\ & | & \\ & \bullet & \\ & / \quad \backslash & \\ \bullet & & \bullet \\ & | & \\ & \bullet & \end{array} \right)$$

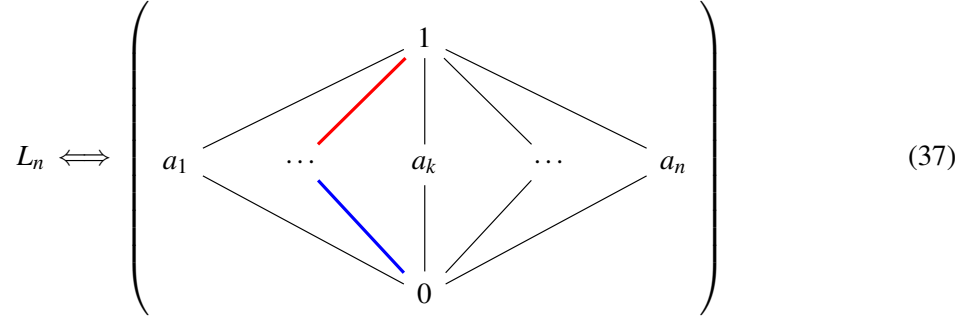


Even though the category \mathcal{L}^c (or $\text{Mod}_{\mathbb{B}}^{\text{fin}}$) concerns only finite sets, surprisingly they are not finitary as the following simple example illustrates.

Example 4.26. Let n be a positive integer, and $L_n = \{0, a_1, \dots, a_n, 1\}$ be an algebraic lattice such that 0 is the smallest element, 1 is the largest element, and a_i are incomparable. Then $K = \{0, a_1\}$ is an algebraic sub-lattice of L_n which is isomorphic to \mathbb{B} . The quotient L_n/K consists of elements greater than or equal to a_1 , that is $\{a_1, 1\}$. So, one has the following short exact sequence for any positive integer n :

$$\mathbb{B} \hookrightarrow L_n \twoheadrightarrow \mathbb{B}. \quad (36)$$

In other words, we have



and each \mathbb{B} corresponds to either a red lattice or a blue lattice in (37). Since for $n \neq m$ L_n is not isomorphic to L_m , the set $\text{Ext}_{\mathcal{L}^c}(\mathbb{B}, \mathbb{B})$ is not finite, showing that \mathcal{L}^c is not finitary.

Finally, we consider geometric lattices. Since geometric lattices are finite, they are algebraic lattices. In particular, the proto-exact structure of \mathcal{L} may provide the induced proto-exact structure for geometric lattices. We first recall the definition of geometric lattices, which provides a cryptomorphic definition for simple matroids. For details, we refer the reader to [Oxl06].

Definition 4.27. Let P be a finite poset.

- (1) A *chain* in P from x_0 to x_n is a subset $\{x_0, x_1, \dots, x_n\}$ of P such that

$$x_0 < x_1 < \dots < x_n.$$

The length of a chain $\{x_0, \dots, x_n\}$ is n . A chain is said to be maximal if there is no element z such that $x_i < z < x_{i+1}$ for all $i = 0, \dots, n-1$.

- (2) P is said to satisfy the *Jordan-Dedekind chain condition* if for any pair $\{x, y\} \subseteq P$ with $x < y$, all maximal chains from x to y have the same length.

Let P be a poset with the minimal element 0 satisfying the Jordan-Dedekind chain condition. One can define the *height function*

$$h : P \rightarrow \mathbb{Z}$$

such that $h(x)$ is the length of a maximal chain from 0 to x . An *atom* of P is an element of height 1.

Definition 4.28. Let L be a finite lattice.

- (1) L is said to be a *semimodular lattice* if L satisfies the Jordan-Dedekind chain condition, and the height function on L satisfies the following condition: for $x, y \in L$,

$$h(x) + h(y) \geq h(x \vee y) + h(x \wedge y).$$

- (2) L is said to be a *geometric lattice* if L is semimodular in which every element is a join of atoms.

Proposition 4.29. *Let \mathcal{G} be the subcategory of \mathcal{L} consisting of geometric lattices. Then, \mathcal{G} is a proto-exact subcategory of \mathcal{L} with the same proto-exact structure.*

Proof. Clearly, the zero object (the empty lattice) is in \mathcal{G} . Hence, we only have to show that \mathcal{G} is closed under taking subobjects and quotients. But, this is clear since any interval of a geometric lattice is again a geometric lattice. \square

4.3. Further directions. In [EJS20], together with C. Eppolito, the first and the second authors showed that the category of pointed matroids with strong maps is finitary and proto-exact, where admissible monomorphisms are strong maps that can be factored as

$$N \xrightarrow{\sim} M|S \hookrightarrow M$$

and admissible epimorphism are strong maps that can be factored as

$$M \twoheadrightarrow M/S \xrightarrow{\sim} N$$

In [Cra67, Section 2], H. Crapo showed that a strong map between matroids corresponds to a function $f : L_1 \rightarrow L_2$ between geometric lattices satisfying the following three conditions:

- (1) (non-singular) $f(x) = 0_{L_2}$ implies $x = 0_{L_1}$;
- (2) (join-homomorphism) $f(\sup X) = \sup f(X)$ for any $X \subseteq L_1$;
- (3) (cover-preserving) The image of each atom in L_1 is either 0 or an atom of L_2 .

One can easily that Definition 4.3 is equivalent to a function being a join-homomorphism for geometric lattices. In particular, our subcategory \mathcal{G} of geometric lattices contains more morphisms than the category of geometric lattices with strong maps, and as a result we have a pathological examples such as Example 4.26. This leads us to the following question.

Question 4.30. Is there a “better” notion of morphisms (as well as admissible morphisms) for algebraic lattices in such a way that the category of algebraic lattices is finitary and proto-exact? If so, when restricted to geometric lattices, do they define strong maps for matroids?

5. The category of modules over a hyperring as a proto-exact category

In this section, we study the category of modules over a hyperring from the perspective of proto-exact categories. Throughout this section, let H be a hyperring and Mod_H be the category of H -modules. We note that in [Mad06], certain categorical aspects of modules over a hyperring are studied, however it does not show that the category of modules over a hyperring is proto-exact.

Definition 5.1. Let \mathfrak{M} be the class of strict injective homomorphisms in Mod_H and \mathfrak{E} be the class of strict surjective homomorphisms in Mod_H .

In the following, we show that $(\text{Mod}_H, \mathfrak{M}, \mathfrak{E})$ is a proto-exact category. It is clear that \mathfrak{M} and \mathfrak{E} contain all isomorphisms and are closed under composition.

Let B be an H -module, and A a submodule of B . Then, A defines an equivalence relation \equiv on B as follows: for $b_1, b_2 \in B$,

$$b_1 \equiv b_2 \iff b_1 + A = b_2 + A, \tag{38}$$

where $b_i + A = \bigcup_{a \in A} (b_i + a)$ and $b_1 + A = b_2 + A$ is equality of sets.

Definition 5.2. Let M be an H -module and \equiv be an equivalence relation on M . For subsets $X, Y \subseteq M$, we write $X \equiv Y$ if the following two conditions holds:

- (1) For any $x \in X$, there exists $y \in Y$ such that $x \equiv y$.
- (2) For any $y \in Y$, there exists $x \in X$ such that $x \equiv y$.

Definition 5.3. By a congruence relation on M , we mean an equivalence relation \equiv on M such that for any $x_i, y_i \in M$ for $i = 1, 2$ and $r \in H$, the following holds:

$$\text{If } x_i \equiv y_i \text{ for } i = 1, 2, \text{ then, } (x_1 + x_2) \equiv (y_1 + y_2) \text{ and } (rx_i) \equiv (ry_i),$$

where the notation $(x_1 + x_2) \equiv (y_1 + y_2)$ is as in Definition 5.2.

Proposition 5.4. Let B be an H -module and A be a submodule of B . Then, B/\equiv (with \equiv as in (38)) is an H -module, with the following hyperaddition:

$$[x] + [y] := \{[z] \mid z \in x' + y' \text{ such that } [x'] = [x], [y'] = [y]\},$$

and an H -action $r[x] := [rx]$, where $r[x] = \{rx' \mid x' \in [x]\}$.

Proof. One may apply a similar argument as in [Jun18a, Proposition 3.16] to prove that B/\equiv is a hypergroup.

Next, we claim that H -action is well-defined, i.e., if $[x'] = [x]$, then $[rx'] = [rx]$. In fact, if $[x'] = [x]$, then we have

$$x' + A = x + A,$$

in particular, $x' \in x + a$ for some $a \in A$. It follows that for any $r \in H$, we have $rx' \in rx + ra$, showing that $rx' + A \subseteq rx + A$ since $ra \in A$. A similar argument shows that $rx + A \subseteq rx' + A$, and hence $[rx'] = [rx]$.

Finally, it is clear that $0[x] = [0x] = [0]$ and $1[x] = [x]$. Suppose that $[m], [n] \in B/\equiv$. Then, we have

$$[m] + [n] = [m + n]. \quad (39)$$

In fact, it is clear from the definition that $[m + n] \subseteq [m] + [n]$. Conversely, if $[x] \in [m] + [n]$, then we have $x \in m' + n'$ for some $[m'] = [m]$ and $[n'] = [n]$. But, since $[m'] = [m]$ and $[n'] = [n]$, there exist $a_1, a_2 \in A$ such that

$$m' \in m + a_1, \quad n' \in n + a_2.$$

It follows that $x \in (m + n) + A$. Hence $[x] \in [m + n]$, showing that $[m] + [n] \subseteq [m + n]$. This implies that for any $r \in H$, we have $r([m] + [n]) = r[m + n] = [rm + rn] = r[m] + r[n]$. Furthermore, for $r, s \in R$ and $[x] \in B/\equiv$ we have

$$(r + s)[m] = [(r + s)m] = [rm + sm] = [rm] + [sm] = r[m] + s[m].$$

This shows that B/\equiv is an H -module. □

In what follows, we will denote the H -module (B/\equiv) as in Proposition 5.4 by B/A .

Corollary 5.5. Let B be an H -module and A be a submodule of B . Then, the canonical map $\pi : B \rightarrow B/A$ sending x to $[x]$ is a strict surjective homomorphism of H -modules.

Proof. This is proved in Proposition 5.4 in (39). □

Lemma 5.6. Let $(\text{Mod}_H, \mathfrak{M}, \mathfrak{C})$ be as in Definition 5.1. Let $i, i' \in \mathfrak{M}$ and $j, j' \in \mathfrak{C}$, A an H -module, and C a submodule of A . Then the following commutative diagram is a bi-Cartesian diagram in Mod_H .

$$\begin{array}{ccc} A & \xleftarrow{i} & B \\ j \downarrow & & \downarrow j' \\ A/C & \xleftarrow{i'} & B/C \end{array} \quad (40)$$

Proof. We first note that since C is a submodule of A and A is a submodule of B , C is also a submodule of B . In particular the quotient B/C is well-defined.

Next, we claim that the induced map $i' : A/C \rightarrow B/C$ sending $[x]$ to $[i(x)]$ is well-defined. Indeed, if $[x] = [x']$, then we have $x + C = x' + C$, in particular, $x' \in x + c$ for some $c \in C$. It follows that $i(x') \in i(x) + i(c)$, hence $i(x') + C \subseteq i(x) + C$. By the same argument, one sees that $i(x) + C \subseteq i(x') + C$, showing that $[i(x)] = [i(x')]$. Hence i' is well-defined. Furthermore i' is an injection since for $[i(x)] = [i(y)]$, we have $i(x) \in i(y) + i(c) = i(y + c)$ for some $c \in C$ implying $[x] = [y]$. Finally, i' is strict since, by Corollary 5.5,

$$i'([x] + [y]) = i'([x + y]) = [i(x + y)] = [i(x) + i(y)] = [i(x)] + [i(y)].$$

This shows that i' is an admissible monomorphism. In particular, the diagram (40) is well-defined.

(co-Cartesian) Suppose that we have the following commutative diagram with an H -module D and H -module homomorphisms α and β :

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \pi \downarrow & & \downarrow \pi' \\ A/C & \xrightarrow{i'} & B/C \end{array} \quad \begin{array}{c} \beta \\ \searrow \\ \gamma \\ \searrow \\ \alpha \end{array} \quad \begin{array}{c} \\ \\ \\ \\ D \end{array} \quad (41)$$

We define $\gamma : B/C \rightarrow D$ by $\gamma([x]) = \beta(x)$. We claim that γ is well-defined. Indeed, if $[x] = [y]$ in B/C , then we have $x + C = y + C$, hence $x \in y + n$ for some $n \in C$. From the commutativity of the diagram (41), we have

$$\beta(n) = \beta(i(n)) = \alpha(\pi(n)) = 0.$$

In particular, we have

$$\beta(x) \in \beta(y + n) \subseteq \beta(y) + \beta(n) = \beta(y),$$

showing that $\beta(x) = \beta(y)$, and hence $\gamma([x]) = \gamma([y])$. Next, we claim that γ is a homomorphism of R -modules. For $r \in R$, we have

$$\gamma(r[x]) = \gamma([rx]) = \beta(rx) = r\beta(x) = r\gamma([x]).$$

Also, suppose that $[x], [y] \in B/C$ and $[z] \in [x] + [y]$. In other words, $z \in x' + y'$ for some $[x'] = [x]$ and $[y'] = [y]$ in B/C . It follows that

$$\gamma([z]) = \beta(z) \in \beta(x') + \beta(y') = \beta(x) + \beta(y) = \gamma([x]) + \gamma([y]),$$

showing that γ is a homomorphism of H -modules. It is clear that with γ , (41) commutes. Furthermore, one can easily check that if a morphism $\delta : B/C \rightarrow D$ makes (41) to commute, then $\delta = \gamma$. Hence (40) is a co-Cartesian diagram.

(Cartesian) Now, suppose that we have the following commutative diagram:

$$\begin{array}{ccc} & & D \\ & \beta & \nearrow \\ & & \gamma \\ & & \searrow \\ & & A \\ & i & \rightarrow B \\ \pi \downarrow & & \downarrow \pi' \\ A/C & \xrightarrow{i'} & B/C \end{array} \quad \begin{array}{c} \alpha \\ \searrow \\ \alpha \end{array} \quad \begin{array}{c} \\ \\ \\ \\ D \end{array} \quad (42)$$

We claim that $\beta(D) \subseteq i(A)$. In fact, for any $x \in D$, we have $i'\alpha(x) = \pi'\beta(x)$. Since π is surjective, we may write $\alpha(x) = \pi(a)$ for some $a \in A$. Therefore, we have

$$\pi'i(a) = i'\pi(a) = \pi'\beta(x).$$

It follows that $i(a) + C = \beta(x) + C$, in particular, $\beta(x) \in i(a) + c$ for some $c \in C$. Since C is a submodule of A , we have that $\beta(x) \in i(A)$, and hence $\beta(D) \subseteq i(A)$.

From the above claim, the function $\gamma : D \rightarrow A$ sending $x \in D$ to $\beta(x)$ is well-defined (here we identify A with $i(A)$). It is clear that γ is a homomorphism of H -modules. Furthermore, with γ , the diagram (42) commutes. Finally, if we have $\delta : D \rightarrow A$ making (42) to commutes, then we have $i(\delta(x)) = \beta(x)$, and hence $\delta(x) = \gamma(x)$ for all $x \in D$, showing that (42) is a Cartesian diagram. \square

Lemma 5.7. *Let A and B be H -modules, and $f : A \rightarrow B$ be a strict homomorphism of H -modules. If C is a submodule of B , then $f^{-1}(C)$ is a submodule of A .*

Proof. Let $D = f^{-1}(C)$. We first claim that if $x, y \in D$, then $x + y \in D$. In fact, let $x \in f^{-1}(a)$ and $y \in f^{-1}(b)$ for some $a, b \in C$. Suppose that $z \in x + y$. Then we have

$$f(z) \in f(x) + f(y) = a + b \subseteq C$$

showing that $z \in D$.

Next, for $r \in H$ and $x \in D$ we have $rx \in D$. In fact, we write $f(x) = a \in C$, then we have $f(rx) = rf(x) = ra \in C$. Hence $rx \in f^{-1}(C) = D$. Now, it is clear that D is a submodule of A . \square

Lemma 5.8. *Let $j' : B \rightarrow B'$ be a strict surjective homomorphism of H -modules. Then, B' is isomorphic to $B/\ker(j')$.*

Proof. This follows from the first isomorphism theorem of H -modules which holds when one has strict homomorphisms. \square

Lemma 5.9. *Every diagram in Mod_H of the following form*

$$\begin{array}{ccc} & B & \\ & \downarrow j' & \\ A' & \xrightarrow{i'} & B' \end{array} \quad (43)$$

with $i' \in \mathfrak{M}$ and $j' \in \mathfrak{E}$ can be completed to a bi-Cartesian square (3) with $i \in \mathfrak{M}$ and $j \in \mathfrak{E}$.

Proof. Let $A = (j')^{-1}(i'(A'))$. Since $i'(A')$ is a submodule of B' , it follows from Lemma 5.7, A is a submodule of B . We claim that the inclusion $i : A \rightarrow B$ is a strict homomorphism. For us to show this, it is enough that show that for any $a, b \in A$, we have $a + b \subseteq A$, where $a + b$ is computed in B .⁸ Let $c \in a + b$. Then, there exist $x, y \in A'$ such that $j'(a) = i'(x)$ and $j'(b) = i'(y)$. In particular, since j' and i' are strict, we have

$$j'(c) \in j'(a) + j'(b) = i'(x) + i'(y) = i'(x + y).$$

It follows that $j'(c) \in i'(A')$, and hence $c \in A$. This implies that $i : A \rightarrow B$ is a strict injective homomorphism, and hence $i \in \mathfrak{M}$.

Next, consider the function $\pi : A \rightarrow A'$ sending any $a \in A$ to $((i')^{-1}j'i)(a)$. Note that π is well-defined since $j'i(A) \subseteq i'(A')$ by definition. We claim that π is a strict surjective homomorphism (admissible epimorphism). First, one can easily see that π is a homomorphism. To show that π is strict, suppose that $a, b \in A$ and $d \in \pi(a) + \pi(b)$. Since i', i, j' are strict, this implies that

$$i'(d) \in j'i(a) + j'i(b) = j'i(a + b).$$

In particular, $d \in (i')^{-1}j'i(a + b) = \pi(a + b)$, showing that π is strict. Finally, it is clear from the definition that π is surjective, showing that π is an admissible epimorphism.

Moreover, since $j' : B \rightarrow B'$ is an admissible epimorphism, from Lemma 5.8, we may assume that $j' : B \rightarrow B/C$, where C is a submodule of B . Then, by definition of A , we have $C \subseteq A$ (we identify $i(A)$ and A). In particular, we may assume that

$$\pi : A \rightarrow A' = A/C.$$

⁸This is equivalent to $i(a + b) = i(a) + i(b)$.

Therefore, we can complete the diagram (43) as follows:

$$\begin{array}{ccc}
A & \xleftarrow{i} & B \\
\pi \downarrow & & \downarrow j' \\
A/C & \xleftarrow{i'} & B/C
\end{array} \tag{44}$$

Now it follows from Lemma 5.6 that (44) is a bi-Cartesian diagram. \square

Lemma 5.10. *Every diagram in Mod_H of the following form*

$$\begin{array}{ccc}
A & \xleftarrow{i} & B \\
j \downarrow & & \\
A' & &
\end{array} \tag{45}$$

with $i \in \mathfrak{M}$ and $j \in \mathfrak{E}$ can be completed to a bi-Cartesian square (3) with $i' \in \mathfrak{M}$ and $j' \in \mathfrak{E}$.

Proof. Since j is an admissible epimorphism, from Lemma 5.8, we may assume that $j : A \rightarrow A' = A/C$ for some submodule C of A . We let $j' : B \rightarrow B/C$ be the map sending x to $[x]$, which is an admissible epimorphism by Corollary 5.5. Moreover, i induces a natural map

$$i' : A/C \rightarrow B/C, \quad [x] \mapsto [i(x)].$$

From the proof of Lemma 5.6, we know that i' is well-defined and $i' \in \mathfrak{M}$. In particular, we can complete the diagram (45) as follows:

$$\begin{array}{ccc}
A & \xleftarrow{i} & B \\
j \downarrow & & \downarrow j' \\
A/C & \xleftarrow{i'} & B/C
\end{array} \tag{46}$$

Now, it follows from Lemma 5.6 that (46) is a bi-Cartesian diagram. \square

Theorem 5.11. *The triple $(\text{Mod}_H, \mathfrak{M}, \mathfrak{E})$ is a proto-exact category.*

Proof. Mod_H is clearly pointed. The axiom (2) in Definition 2.1 is clear. The axioms (4) and (5) in Definition 2.1 directly follow from Lemmas 5.9 and 5.10. Finally, the axiom (3) is a direct consequence of (4) and (5) along with the uniqueness of pullbacks and pushouts. \square

5.1. Finite \mathbb{K} -modules and incidence geometries. Let \mathbb{K} be the Krasner hyperfield. One can easily check that a hypergroup E is an \mathbb{K} -module if and only if $x+x = \{0, x\}$ for all $x \neq 0$. In [CC11a], Connes and Consani proved that each finite \mathbb{K} -module E uniquely determines a projective geometry \mathcal{P}_E whose set of points is $E \setminus \{0_E\}$ and the line $\ell(x, y)$ passing through two distinct points $x, y \in \mathcal{P}$ is defined by using the hyperaddition of E . To be a bit more precise, the points on the line $\ell(x, y)$ is given as follows:

$$\ell(x, y) = (x+y) \cup \{x, y\}.$$

In this case, any line in \mathcal{P}_E contains at least four points. Conversely, any projective geometry such that each line contains at least four points uniquely arises in this way.

Let $\text{Mod}_{\mathbb{K}}^{\text{fin}}$ be the category of finite \mathbb{K} -modules. Let E be a finite \mathbb{K} -module, and \mathcal{P}_E the corresponding projective geometry. Let \mathcal{L}_E be the lattice of submodules E . Then one can easily observe the following:

- (1) The atoms of \mathcal{L}_E have the form $\{0, x\}$ for $x \neq 0$, and are in one-to-one correspondence with the points of \mathcal{P}_E .

- (2) A submodule of height 2^9 must contain $\{0, x\}$ for some x and must contain y for some nonzero $y \neq x$. Thus such a submodule must contain $x + y$, so contains all points of the line passing through x and y . Since such a line (together with 0) is a submodule, a submodule of height 2 is a line in \mathcal{P}_E . Moreover, since a line cannot properly contain another line, all lines in \mathcal{P}_E have height 2.
- (3) Consider a short exact sequence¹⁰ in $\text{Mod}_{\mathbb{K}}^{\text{fin}}$ as follows:

$$E' \hookrightarrow E \twoheadrightarrow E''. \quad (47)$$

Recall that the submodules of $E/E' (\simeq E'')$ are in one-to-one correspondence with those of E which contain E' as in the classical case. In particular, a point in E'' is a submodule (or projective subspace) which is minimal among those which properly contain E' , while a line is minimal among submodules which are not points of E'' but properly contain E' , and so on. For example, when $E' = \{0, x\}$ (or a point in the corresponding projective geometry), the projective geometry $\mathcal{P}_{E''}$ is the quotient geometry as in [BAR98, Section 1.4].

- (4) With the short exact sequence 47, in the special case where E is the projective space associated to a vector space V over some field, and E' corresponds to a subspace W of V , E'' corresponds to the projectivization of V/W .

From the above, one may be tempted to use this correspondence to define the Hall algebra of $\text{Mod}_{\mathbb{K}}^{\text{fin}}$ as it appears to be finitary. However, the following example shows that unfortunately $\text{Mod}_{\mathbb{K}}^{\text{fin}}$ is not finitary.

Example 5.12. Let $n \geq 3$, and $E_n = \{0, a_1, \dots, a_n\}$. We impose hyperaddition on E as follows:

$$a_i + a_j = \begin{cases} \{0, a_i\} & \text{if } i = j, \\ \{a_k \mid k \neq j \text{ and } k \neq i\} & \text{if } i \neq j. \end{cases}$$

One can think of \mathcal{P}_{E_n} as a projective space with only a single line but many points. If we consider the quotient projective space by a_1 as in the above observation (3), the points of the quotient are lines through a_1 , of which there are only one. In particular, there is an exact sequence $\mathbb{K} \hookrightarrow E_n \twoheadrightarrow \mathbb{K}$ for each $n \geq 3$. This shows that $\text{Mod}_{\mathbb{K}}^{\text{fin}}$ is not finitary since if $n \neq m$, then E_n and E_m are not isomorphic. In fact, one may apply a similar idea to show that $\text{Mod}_{\mathbb{K}}^{\text{fin}}$ is not closed under extensions.

5.2. Further directions. Let $H_{\mathbb{K}}$ be the vector space over complex numbers spanned by the set of isomorphism class in $\text{Mod}_{\mathbb{K}}^{\text{fin}}$. Let E_1, E_2 be finite \mathbb{K} -modules. Suppose that in $H_{\mathbb{K}}$ we have

$$E_1 \cdot E_2 := \sum_{E \in \text{Iso}(\text{Mod}_{\mathbb{K}}^{\text{fin}})} \mathbf{a}_{E_1, E_2}^E E, \quad (48)$$

where

$$\mathbf{a}_{E_1, E_2}^E = \#\{L \subseteq E \mid L \simeq E_2 \text{ and } E/L \simeq E_1\}. \quad (49)$$

From the above discussion we see that the number \mathbf{a}_{E_1, E_2}^E counts the number of different embeddings of projective geometries L , which is isomorphic to \mathcal{P}_{E_2} , into \mathcal{P}_E such that the ‘‘quotient geometry’’ $\mathcal{P}_{E/L}$ is isomorphic to E_1 . Here is a more explicit example.

Example 5.13. Let $E_2 = \mathbb{K}$ (as a hypergroup) and Let $E_1 = \mathbb{F}_{5^2}/\mathbb{F}_5^\times$. Then \mathcal{P}_{E_2} is just a point and $\mathcal{P}_{E_1} = \mathbb{P}_{\mathbb{F}_5}^1$, the projective line over \mathbb{F}_5 . Hence, when $E = \mathbb{P}_{\mathbb{F}_5}^2$, the structure constant \mathbf{a}_{E_1, E_2}^E is the number of flags (a, ℓ) in $\mathbb{P}_{\mathbb{F}_5}^2$, where a is a point and ℓ is a line containing a .

⁹By this we mean minimal among those which properly contain an atom

¹⁰Recall that by a short exact sequence, we mean a commutative diagram of the form (6) with the proto-exact structure of $\text{Mod}_{\mathbb{K}}^{\text{fin}}$ induced from $\text{Mod}_{\mathbb{K}}$.

Remark 5.14. In fact, the aforementioned correspondence between finite \mathbb{K} -module and projective geometries are functorial, where the category of projective geometries is defined as in [FF94]. Moreover, when $H = \mathbb{S}$, the sign hyperfield, finite \mathbb{S} -modules correspond to *spherical geometries* and the above discussions are carried over to this case.

As Example 5.12 shows, the category $\text{Mod}_{\mathbb{K}}^{\text{fin}}$ is not finitary, in particular, the multiplication (48) does not have to be a finite sum in general. This leads us to the following questions.

- Question 5.15.** (1) Can one find a proto-exact subcategory \mathcal{C} of $\text{Mod}_{\mathbb{K}}^{\text{fin}}$ which is finitary?
 (2) If so, can one describe the subcategory of projective geometries through the categorical correspondence mentioned in Remark 5.14. In this case, do structure constants for $H_{\mathcal{C}}$ encode some interesting combinatorial identities for projective geometries?

References

- [BAR98] Albrecht Beutelspacher, Beutelspacher Albrecht, and Ute Rosenbaum. *Projective geometry: from foundations to applications*. Cambridge University Press, 1998.
- [BB19] Matthew Baker and Nathan Bowler. Matroids over partial hyperstructures. *Advances in Mathematics*, 343:821–863, 2019.
- [CC10a] Alain Connes and Caterina Consani. From monoids to hyperstructures: in search of an absolute arithmetic. *Casimir Force, Casimir Operators and the Riemann Hypothesis, de Gruyter*, pages 147–198, 2010.
- [CC10b] Alain Connes and Caterina Consani. Schemes over \mathbb{F}_1 and zeta functions. *Compos. Math*, 146(6):1383–1415, 2010.
- [CC11a] Alain Connes and Caterina Consani. The hyperring of adèle classes. *Journal of Number Theory*, 131(2):159–194, 2011.
- [CC11b] Alain Connes and Caterina Consani. On the notion of geometry over \mathbb{F}_1 . *Journal of Algebraic Geometry*, 20(3):525–557, 2011.
- [CLS12] Chenghao Chu, Oliver Lorscheid, and Rekha Santhanam. Sheaves and k-theory for fl-schemes. *Advances in Mathematics*, 229(4):2239–2286, 2012.
- [Cra67] Henry H Crapo. Structure theory for geometric lattices. *Rendiconti del Seminario Matematico della Università di Padova*, 38:14–22, 1967.
- [Dei05] Anton Deitmar. Schemes over \mathbb{F}_1 . In *Number fields and function fields-two parallel worlds*, pages 87–100. Springer, 2005.
- [DK12] Tobias Dyckerhoff and Mikhail Kapranov. Higher Segal spaces I. *arXiv preprint arXiv:1212.3563*, 2012.
- [Dyc18] Tobias Dyckerhoff. Higher categorical aspects of Hall algebras. In *Building Bridges Between Algebra and Topology*, pages 1–61. Springer, 2018.
- [EJS20] Chris Eppolito, Jaiung Jun, and Matt Szczesny. Proto-exact categories of matroids, Hall algebras, and K-theory. *Math. Z.*, 296:147–167, 2020.
- [ELY20] Jens Niklas Eberhardt, Oliver Lorscheid, and Matthew B Young. Group completion in the K-theory and Grothendieck-Witt theory of proto-exact categories. *arXiv preprint arXiv:2009.12635*, 2020.
- [FF94] Claude-Alain Faure and Alfred Frölicher. Morphisms of projective geometries and semilinear maps. *Geometriae Dedicata*, 53(3):237–262, 1994.
- [GG16] Jeffrey Giansiracusa and Noah Giansiracusa. Equations of tropical varieties. *Duke Mathematical Journal*, 165(18):3379–3433, 2016.
- [GG18] Jeffrey Giansiracusa and Noah Giansiracusa. A Grassmann algebra for matroids. *manuscripta mathematica*, 156(1):187–213, 2018.
- [Hek17] J. Hekking. Segal objects in homotopical categories & K-theory of proto-exact categories. *Master’s Thesis, Univ. Leiden*, 2017.
- [Hoc69] Melvin Hochster. Prime ideal structure in commutative rings. *Transactions of the American Mathematical Society*, 142:43–60, 1969.
- [IJ19] Miodrag Iovanov and Jaiung Jun. On the Hopf algebra of multi-complexes. *arXiv preprint arXiv:1912.10645 (to appear in Journal of Algebraic Combinatorics)*, 2019.
- [JMT20] Jaiung Jun, Kalina Mincheva, and Jeffrey Tolliver. Vector bundles on tropical schemes. *arXiv preprint arXiv:2009.03030*, 2020.
- [JRT20] Jaiung Jun, Samarpita Ray, and Jeffrey Tolliver. Lattices, spectral spaces, and closure operations on idempotent semirings. *arXiv preprint arXiv:2001.00808 (to appear in Journal of Algebra)*, 2020.
- [JS20a] Jaiung Jun and Alex Sistko. On quiver representations over \mathbb{F}_1 . *arXiv preprint arXiv:2008.11304 (to appear in Algebras and Representation Theory)*, 2020.

- [JS20b] Jaiung Jun and Matt Szczesny. Toric Hall algebras and infinite-dimensional Lie algebras. *arXiv preprint arXiv:2008.11302*, 2020.
- [JS21] Jaiung Jun and Alex Sistko. Coefficient quivers, \mathbb{F}_1 -representations, and Euler characteristics of quiver Grassmannians. *arXiv preprint arXiv:2112.06291*, 2021.
- [Jun18a] Jaiung Jun. Algebraic geometry over hyperrings. *Advances in Mathematics*, 323:142–192, 2018.
- [Jun18b] Jaiung Jun. Association schemes and hypergroups. *Communications in Algebra*, 46(3):942–960, 2018.
- [Kap97] MM Kapranov. Eisenstein series and quantum affine algebras. *Journal of Mathematical Sciences*, 84(5):1311–1360, 1997.
- [Les12] Paul Lescot. Absolute algebra III—The saturated spectrum. *Journal of Pure and Applied Algebra*, 216(5):1004–1015, 2012.
- [Lor12] Oliver Lorscheid. The geometry of blueprints: Part I: Algebraic background and scheme theory. *Advances in Mathematics*, 229(3):1804–1846, 2012.
- [Lor15] Oliver Lorscheid. Scheme theoretic tropicalization. *arXiv preprint arXiv:1508.07949*, 2015.
- [Mad06] A Madanshekaf. Exact category of hypermodules. *International journal of mathematics and mathematical sciences*, 2006.
- [Oxl06] James G Oxley. *Matroid theory*, volume 3. Oxford University Press, USA, 2006.
- [Rin90] Claus Michael Ringel. Hall algebras and quantum groups. *Inventiones mathematicae*, 101(1):583–591, 1990.
- [Sou04] Chr Soulé. Les variétés sur le corps à un élément. *Mosc. Math. J*, 4(1):217–244, 2004.
- [Szc12] Matt Szczesny. Representations of quivers over \mathbb{F}_1 and Hall algebras. *International Mathematics Research Notices*, 2012(10):2377–2404, 2012.
- [Szc18] Matt Szczesny. The Hopf algebra of skew shapes, torsion sheaves on $\mathbb{A}^n/\mathbb{F}_1$, and ideals in Hall algebras of monoid representations. *Advances in Mathematics*, 331:209–238, 2018.

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