PRE-LIE ALGEBRAS AND INCIDENCE CATEGORIES OF COLORED ROOTED TREES

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ABSTRACT. The incidence category $\mathcal{C}_{\mathcal{F}}$ of a family $\mathcal{F}$ of colored posets closed under disjoint unions and the operation of taking convex sub-posets was introduced by the author in [12], where the Ringel-Hall algebra $\mathrm{H}_{\mathcal{F}}$ of $\mathcal{C}_{\mathcal{F}}$ was also defined. We show that if the Hasse diagrams underlying $\mathcal{F}$ are rooted trees, then the subspace $\mathfrak{n}_{\mathcal{F}}$ of primitive elements of $\mathrm{H}_{\mathcal{F}}$ carries a pre-Lie structure, defined over $\mathbb{Z}$, and with positive structure constants. We give several examples of $\mathfrak{n}_{\mathcal{F}}$, including the nilpotent subalgebras of $\mathfrak{sl}_n$, $L\mathfrak{gl}_n$, and several others.

1. INTRODUCTION

A *left pre-Lie algebra* is a k–vector space A endowed with a binary bilinear operation $\triangleright$ satisfying the identity

$$(a \triangleright b) \triangleright c - a \triangleright (b \triangleright c) = (b \triangleright a) \triangleright c - b \triangleright (a \triangleright c) \tag{1.1}$$

It follows easily from 1.1 that anti-symmetrizing $\triangleright$ yields a Lie bracket

$$[a, b] = a \triangleright b - b \triangleright a$$

on A. However, not every Lie algebra arises from a pre-Lie algebra. Pre-Lie algebras first appeared in the works of E.B. Vinberg [13] and M. Gerstenhaber [3], and have since found applications in several areas. One prominent example is perturbative quantum field theory [4], where insertion of Feynman graphs into each other equips them with a pre-Lie structure which controls the combinatorics of the renormalization procedure.

In this paper, we show that pre-Lie algebras arise naturally from *incidence categories* introduced by that author in [12]. An incidence category is built from a collection $\mathcal{F}$ of colored posets, which is closed under the operations of disjoint union and convex subposet - we will denote it by $\mathcal{C}_{\mathcal{F}}$. The objects of $\mathcal{C}_{\mathcal{F}}$ are the posets in $\mathcal{F}$, and for $P_1, P_2 \in \mathcal{F}$

$$\mathrm{Hom}(P_1, P_2) := \{(I_1, I_2, f) | I_j \text{ is an order ideal in } P_j, f : P_1 \backslash I_1 \to I_2 \text{ an isomorphism } \}$$

Here, the poset I_1 should be viewed as the kernel of the morphism, and I_2 as the image. All morphisms in $\mathcal{C}_{\mathcal{F}}$ have kernels and cokernels, and so the notion of exact sequence makes sense. In [12], the Ringel-Hall algebra $\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$ of $\mathcal{C}_{\mathcal{F}}$ was defined.

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$\mathrm{H}_{\mathcal{C}_\mathcal{F}}$ is the $\mathbb{Q}$–vector space of finitely supported functions on isomorphism classes of $\mathcal{C}_\mathcal{F}$:

$$\mathrm{H}_{\mathcal{C}_\mathcal{F}} := \{f : \mathrm{Iso}(\mathcal{C}_\mathcal{F}) \rightarrow \mathbb{Q} | |supp(f)| < \infty\}$$

with product given by convolution:

$$f \star g(M) = \sum_{A \subset M} f(A) g(M/A). \tag{1.2}$$

$\mathrm{H}_{\mathcal{C}_\mathcal{F}}$ possesses a co-commutative co-product given by

$$\Delta(f)(M, N) = f(M \oplus N) \tag{1.3}$$

(where $M \oplus N$ denotes the disjoint union of M and N) as well as an antipode, making it a Hopf algebra. $\mathrm{H}_{\mathcal{C}_\mathcal{F}}$ is graded, connected, and co-commutative, and so by the Milnor-Moore theorem isomorphic to $U(\mathfrak{n}_\mathcal{F})$, where $\mathfrak{n}_\mathcal{F}$ is the Lie algebra of its primitive elements. It follows from 1.3 that

$$\mathfrak{n}_\mathcal{F} = \mathrm{span}\{\delta_P | P \in \mathcal{F}, P \text{ connected }\}$$

We show that if $\mathcal{F}$ consists of posets whose Hasse diagrams are rooted trees, then $\mathfrak{n}_\mathcal{F}$ carries a pre-Lie structure $\triangleright$, with

$$\delta_{P_1} \triangleright \delta_{P_2} := \delta_{P_1} \star \delta_{P_2} - \delta_{P_1 \oplus P_2}.$$

A more concrete description of $\triangleright$ is the following: $\delta_{P_1} \triangleright \delta_{P_2}$ is a sum of delta-functions supported on connected posets $P \in \mathcal{F}$ whose Hasse diagram is obtained by grafting the root of P_1 onto a vertex of P_2. It follows from the definition of $\triangleright$ that the structure constants are non-negative integers.

The paper is organized as follows. Section 2 recalls the definition of pre-Lie algebra and introduces the universal example, namely the pre-Lie algebra of colored rooted trees. In section 3 we recall the construction of the incidence category $\mathcal{C}_\mathcal{F}$ as well as its main properties. The Ringel-Hall algebra of $\mathcal{C}_\mathcal{F}$ is introduced in section 4. In section 5 we define the pre-Lie structure $\triangleright$ on $\mathfrak{n}_\mathcal{F}$ and verify that it satisfies the identity 1.1. Finally, section 6 is devoted to examples - among these are pre-Lie structures on nilpotent Lie subalgebras of $\mathfrak{sl}_n$ and $L\mathfrak{gl}_n$.

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2. Pre-Lie algebras

In this section, we recall the definition and some examples of (left) pre-Lie algebras. Let k be a field.

Definition 1. A *left pre-Lie algebra* is a k–vector space A endowed with a binary bilinear operation $\triangleright$ satisfying the left pre-Lie identity

$$(a \triangleright b) \triangleright c - a \triangleright (b \triangleright c) = (b \triangleright a) \triangleright c - b \triangleright (a \triangleright c) \tag{2.1}$$

for $a, b, c \in A$.

One checks easily that antisymmetrizing the operation $\rhd$

$$[a,b] = a \rhd b - b \rhd a$$

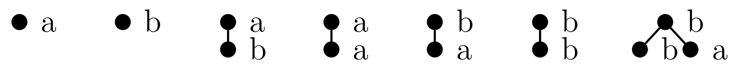
gives A the structure of a Lie algebra.

Example 1. Any associative k–algebra A is a pre-Lie algebra with the pre-Lie structure given by

$$a \rhd b := ab,$$

where the right hand side refers to the associative multiplication in A.

Example 2. One of the most important examples of pre-Lie algebras is given by colored rooted trees. Recall that a *tree* is a graph with no cycles. We denote by $E(t), V(t)$ the edge and vertex sets of t respectively. Let S be a finite set. By a *rooted tree colored by* S we mean a tree with a distinguished vertex $r(t) \in V(t)$ called the *root*, and an assignment of an element of S to each $v \in V(t)$. We adopt the convention that rooted trees are always drawn with the root on top. For example, if $S = \{a,b\}$, then the following are rooted trees colored by S:



Let $\mathbb{T}_S$ denote the set of rooted trees whose vertices are colored by S. Given $t \in \mathbb{T}_S$, and $e \in E(t)$, removing e disconnects t into two colored rooted trees: $R_e(t)$ containing $r(t)$ and $P_e(t)$, whose root is the end of e. Let $\mathcal{T}_S$ be the k–vector space spanned by $\mathbb{T}_S$. We have

$$\mathcal{T}_S = \oplus_{n=0}^{\infty} \mathcal{T}_S[n]$$

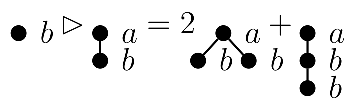
where $\mathcal{T}_S[n]$ is the subspace of $\mathcal{T}_S$ spanned by trees with n vertices. For colored rooted trees $t_1, t_2 \in \mathbb{T}_S$, let

$$t_1 \rhd t_2 := \sum_{s \in \mathbb{T}_S} n(t_1, t_2, s) s$$

where

$$n(t_1, t_2, s) = \#\{e \in E(s) | P_e(s) = t_1, R_e(s) = t_2\}.$$

For example, we have



It is well-known (see for instance [2]) that $\rhd$ defines a pre-Lie structure on $\mathcal{T}_S$. The following theorem is proven in [2]

Theorem 1. $\mathcal{T}_S$ is the free pre-Lie algebra on $|S|$ generators.

Remark 1. In what follows, unless stated otherwise, $k = \mathbb{Q}$.

3. Incidence categories

3.1. **Recollections on posets.** We begin by recalling some basic notions and terminology pertaining to posets (partially ordered sets) following [10, 11].

(1) An *interval* is a poset having unique minimal and maximal elements. For x, y in a poset P, we denote by $[x, y]$ the interval

$$[x, y] := \{z \in P : x \leq z \leq y\}$$

If P is an interval, we will often denote by 0_P and 1_P the minimal and maximal elements.

(2) An *order ideal* in a poset P is a subset $L \subset P$ such that whenever $y \in L$ and $x \leq y$ in P, then $x \in L$.

(3) A sub-poset Q of P is *convex* if, whenever $x \leq y$ in Q and $z \in P$ satisfies $x \leq z \leq y$, then $z \in Q$. Equivalently, Q is convex if $Q = L \backslash I$ for order ideals $I \subset L$ in P.

(4) Given two posets P_1, P_2, their disjoint union is naturally a poset, which we denote by $P_1 + P_2$. In $P_1 + P_2$, $x \leq y$ if both lie in either P_1 or P_2, and $x \leq y$ there.

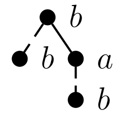
(5) A poset which is not the union of two non-empty posets is said to be *connected*.

(6) The cartesian product $P_1 \times P_2$ is a poset where $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$.

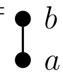
(7) A *distributive lattice* is a poset P equipped with two operations $\wedge$, $\vee$ that satisfy the following properties:
   (a) $\wedge, \vee$ are commutative and associative
   (b) $\wedge, \vee$ are idempotent - i.e. $x \wedge x = x$, $x \vee x = x$
   (c) $x \wedge (x \vee y) = x = x \vee (x \wedge y)$
   (d) $x \wedge y = x \iff x \vee y = y \iff x \leq y$
   (e) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
   (f) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

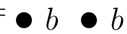
(8) For a poset P, denote by J_P the poset of order ideals of P, ordered by inclusion. J_P forms a distributive lattice with $I_1 \vee I_2 := I_1 \cup I_2$ and $I_1 \wedge I_2 := I_1 \cap I_2$ for $I_1, I_2 \in J_P$. If P_1, P_2 are posets, we have $J_{P_1 + P_2} = J_{P_1} \times J_{P_2}$, and if $I, L \in J_P$, and $I \subset L$, then $[I, L]$ is naturally isomorphic to the lattice of order ideals $J_{L \backslash I}$.

Remark 2. Suppose that the Hasse diagram of a poset P is a rooted tree - that is, P has a unique maximal element $r(P)$, and the Hasse diagram contains no cycles. It is then easy to see that order ideals $I \subset P$ correspond to *admissible cuts* of P, where the latter is a collection of edges $C \subset E(P)$, having the property that at most one edge of C is encountered along any path from root to leaf. For instance, the dotted edges of the poset T below yield an admissible cut:



Each admissible cut $C \subset E(P)$ divides the tree into a rooted connected tree $R_C(P)$ containing $r(P)$, and a rooted forest (a disjoint union of rooted trees) $P_C(P)$. The notation is clearly an extension of that used in example 2. In the last example, we have

$R_C(T) =$ 

$P_C(T) =$ 

3.2. **From posets to categories.** Let $\mathcal{F}$ be a family of colored posets which is closed under the formation of disjoint unions and the operation of taking convex subposets, and let $\mathcal{P}(\mathcal{F}) = \{J_P : P \in \mathcal{F}\}$ be the corresponding family of distributive lattices of order ideals. For each pair $P_1, P_2 \in \mathcal{F}$, let $\mathrm{M}(P_1, P_2)$ denote the set of colored poset isomorphisms $P_1 \to P_2$. It follows that $\mathrm{M}(P, P)$ forms a group, which we denote $\mathrm{Aut}_{\mathrm{M}}(P)$.

3.2.1. *The category $\mathcal{C}_{\mathcal{F}}$.* We proceed to define a category $\mathcal{C}_{\mathcal{F}}$, called the *incidence category of $\mathcal{F}$* as follows. Let

$$\mathrm{Ob}(\mathcal{C}_{\mathcal{F}}) := \mathcal{F} = \{P \in \mathcal{F}\}$$

and

$$\mathrm{Hom}(P_1, P_2) := \{(I_1, I_2, f) : I_i \in J_{P_i}, f \in \mathrm{M}(P_1 \backslash I_1, I_2)\} \;\; i = 1, 2$$

We need to define the composition of morphisms

$$\mathrm{Hom}(P_1, P_2) \times \mathrm{Hom}(P_2, P_3) \to \mathrm{Hom}(P_1, P_3)$$

Suppose that $(I_1, I_2, f) \in \mathrm{Hom}(P_1, P_2)$ and $(I_2', I_3', g) \in \mathrm{Hom}(P_2, P_3)$. Their composition is the morphism (K_1, K_3, h) defined as follows.

- We have $I_2 \wedge I_2' \subset I_2$, and since $f : P_1 \backslash I_1 \to I_2$ is an isomorphism, $f^{-1}(I_2 \wedge I_2')$ is an order ideal of $P_1 \backslash I_1$. Since in J_{P_1}, $[I_1, P] \simeq J_{P_1 \backslash I_1}$, we have that $f^{-1}(I_2 \wedge I_2')$ corresponds to an order ideal $K_1 \in J_{P_1}$ such that $I_1 \subset K_1$.
- We have $I_2' \subset I_2 \vee I_2'$, and since $[I_2', P_2] \simeq J_{P_2 \backslash I_2'}$, $I_2 \vee I_2'$ corresponds to an order ideal $L_2 \in J_{P_2 \backslash I_2'}$. Since $g : P_2 \backslash I_2' \to I_3'$ is an isomorphism, $g(L_2) \subset J_{I_3'}$, and since $J_{I_3'} \subset J_{P_3}$, $g(L_2)$ corresponds to an order ideal $K_3 \in J_{P_3}$ contained in I_3'.
- The isomorphism $f : P_1 \backslash I_1 \to I_2$ restricts to an isomorphism $\bar{f} : P_1 \backslash K_1 \to I_2 \backslash I_2 \wedge I_2' = I_2 \backslash I_2'$, and the isomorphism $g : P_2 \backslash I_2'$ restricts to an isomorphism $\bar{g} : I_2 \vee I_2' \backslash I_2' = I_2 \backslash I_2' \to K_3$. Thus, $g \circ f : P_1 \backslash K_1 \to K_3$ is an isomorphism and $g \circ f \in \mathrm{M}(P_1 \backslash K_1, K_3)$ by the property (4) above.

As shown in [12], the composition of morphisms is associative.

Remark 3.

- We refer to I_2 as the *image* of the morphism $(I_1, I_2, f) : P_1 \to P_2$.
- We denote by $\mathrm{Iso}(\mathcal{C}_{\mathcal{F}})$ the collection of isomorphism classes of objects in $\mathcal{C}_{\mathcal{F}}$, and by $[P]$ the isomorphism class of $P \in \mathcal{C}_{\mathcal{F}}$.

3.3. **Properties of the categories $\mathcal{C}_{\mathcal{F}}$.** We now enumerate some of the properties of the categories $\mathcal{C}_{\mathcal{F}}$.

(1) The empty poset $\emptyset$ is an initial, terminal, and therefore null object. We will sometimes denote it by $\emptyset$.

(2) We can equip $\mathcal{C}_{\mathcal{F}}$ with a symmetric monoidal structure by defining
$$P_1 \oplus P_2 := P_1 + P_2.$$

(3) The indecomposable objects of $\mathcal{C}_{\mathcal{F}}$ are the P with P a connected poset in $\mathcal{F}$.

(4) The simple objects of $\mathcal{C}_{\mathcal{F}}$ are the P where P is a one-element poset.

(5) Every morphism
$$(I_1, I_2, f) : P_1 \to P_2 \tag{3.1}$$
has a kernel
$$(\emptyset, I_1, id) : I_1 \to P_1$$

(6) Similarly, every morphism 3.1 possesses a cokernel
$$(I_2, P_2 \backslash I_2, id) : P_2 \to P_2 \backslash I_2$$
We will use the notation P_2/P_1 for $coker((I_1, I_2, f))$.

Note: Properties 5 and 6 imply that the notion of exact sequence makes sense in $\mathcal{C}_{\mathcal{F}}$.

(7) All monomorphisms are of the form
$$(\emptyset, I, f) : Q \to P$$
where $I \in J_P$, and $f : Q \to I \in \mathrm{M}(Q, I)$. Monomorphisms $Q \to P$ with a fixed image I form a torsor over $\mathrm{Aut}_{\mathrm{M}}(I)$. All epimorphisms are of the form
$$(I, \emptyset, g) : P \to Q$$

where $I \in J_P$ and $g : P\backslash I \to Q \in \mathrm{M}(P\backslash I, Q)$. Epimorphisms with fixed kernel I form a torsor over $\mathrm{Aut}_{\mathrm{M}}(P\backslash I)$

(8) Sequences of the form

$$\emptyset \overset{(\emptyset,\emptyset,id)}{\to} I \overset{(\emptyset,I,id)}{\longrightarrow} P \overset{(I,\emptyset,id)}{\longrightarrow} P\backslash I \overset{(P\backslash I,\emptyset,id)}{\to} \emptyset \tag{3.2}$$

with $I \in J_P$ are short exact, and all other short exact sequences with P in the middle arise by composing with isomorphisms $I \to I'$ and $P\backslash I \to Q$ on the left and right.

(9) Given an object P and a subobject $I, I \in J_P$, the isomorphism $J_{P\backslash I} \simeq [I, P]$ translates into the statement that there is a bijection between subobjects of P/I and order ideals $J \in J_P$ such that $I \subset J \subset P$. The bijection is compatible with quotients, in the sense that $(P/I)/(J/I) \simeq J/I$.

(10) Since the posets in $\mathcal{F}$ are finite, $\mathrm{Hom}(P_1, P_2)$ is a finite set.

(11) We may define Yoneda $\mathrm{Ext}^n(P_1, P_2)$ as the equivalence class of n–step exact sequences with P_1, P_2 on the right and left respectively. $\mathrm{Ext}^n(P_1, P_2)$ is a finite set. Concatenation of exact sequences makes

$$\mathbb{E}xt^* := \cup_{A,B \in I(\mathcal{C}_{\mathcal{F}}),n} \mathrm{Ext}^n(A, B)$$

into a monoid.

(12) We may define the Grothendieck group of $\mathcal{C}_{\mathcal{F}}$, $K_0(\mathcal{C}_{\mathcal{F}})$, as

$$K(\mathcal{C}_{\mathcal{F}}) = \bigoplus_{A \in \mathcal{C}_{\mathcal{F}}} \mathbb{Z}[A]/\sim$$

where $\sim$ is generated by $A + B - C$ for short exact sequences

$$\emptyset \to A \to C \to B \to \emptyset$$

We denote by $k(A)$ the class of an object in $K_0(\mathcal{C}_{\mathcal{F}})$.

4. Ringel-Hall algebras

For an introduction to Ringel-Hall algebras in the context of abelian categories, see [8]. We define the Ringel-Hall algebra of $\mathcal{C}_{\mathcal{F}}$, denoted $\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$, to be the $\mathbb{Q}$–vector space of finitely supported functions on isomorphism classes of $\mathcal{C}_{\mathcal{F}}$. I.e.

$$\mathrm{H}_{\mathcal{C}_{\mathcal{F}}} := \{f : \mathrm{Iso}(\mathcal{C}_{\mathcal{F}}) \to \mathbb{Q} || supp(f)| < \infty\}$$

As a $\mathbb{Q}$–vector space it is spanned by the delta functions $\delta_A, A \in \mathrm{Iso}(\mathcal{C}_{\mathcal{F}})$. The algebra structure on $\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$ is given by the convolution product:

$$f \star g(M) = \sum_{A \subset M} f(A) g(M/A) \tag{4.1}$$

for $M \in \mathrm{Iso}(\mathcal{C}_{\mathcal{F}})$. In what follows, it will be conceptually useful to choose a representative in each isomorphism class. For $M, N, Q \in \mathrm{Iso}(\mathcal{C}_{\mathcal{F}})$, let $F^{Q}_{M,N}$ be the number of exact sequences

$$\emptyset \rightarrow M \xrightarrow{i} Q \xrightarrow{\pi} N \rightarrow \emptyset$$

where (i, π) and (i', π') are considered equivalent iff $i = i'$ and $\pi = \pi'$ (this makes sense, since we have fixed a representative in each isomorphism class). It follows from the definition 4.1 that

$$\delta_M \star \delta_N = \sum_{Q \in \mathrm{Iso}(\mathcal{C}_{\mathcal{F}})} \frac{F^{Q}_{M,N}}{|\,\mathrm{Aut}(M)||\,\mathrm{Aut}(N)|} \delta_Q,$$

from which it is apparent that $\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$ encodes the structure of extensions in $\mathcal{C}_{\mathcal{F}}$.

$\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$ possesses a co-commutative co-product given by

$$\Delta(f)(M, N) = f(M \oplus N) \tag{4.2}$$

as well as a natural $K^{+}_{0}(\mathcal{C}_{\mathcal{F}})$–grading in which δ_A has degree $k(A) \in K^{+}_{0}(\mathcal{C}_{\mathcal{F}})$. If $\mathcal{F}$ is colored by the set S, it is easy to see that $K^{+}_{0}(\mathcal{C}_{\mathcal{F}}) \simeq \mathbb{N}^{|S|}$.

The subobjects of $P \in \mathcal{C}_{\mathcal{F}}$ are exactly $I \in J_P$, and the product 4.1 becomes

$$f \star g([P]) = \sum_{I \in J_P} f([I]) g([P \backslash I]).$$

It is shown in [8] that the product is associative, the co-product co-associative and co-commutative, and that the two are compatible, making $\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$ into a co-commutative bialgebra. Recall that a bialgebra A over a field k is *connected* if it possesses a $\mathbb{Z}_{\geq 0}$–grading such that $A_0 = k$. In addition to the $K^{+}_{0}(\mathcal{C}_{\mathcal{F}})$–grading, $\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$ possesses a grading by the order of the poset - i.e. we may assign $\deg(\delta_P) = |P|$. This gives it the structure of graded connected bialgebra, and hence Hopf algebra. The Milnor-Moore theorem implies that $\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$ is the enveloping algebra of the Lie algebra of its primitive elements, which we denote by $\mathfrak{n}_{\mathcal{F}}$ - i.e. $\mathrm{H}_{\mathcal{C}_{F}} \simeq U(\mathfrak{n}_{\mathcal{F}})$. It follows from 4.2 that $f \in \mathfrak{n}_{\mathcal{F}}$ is primitive if it is supported on the isomorphism classes of connected posets. Thus, we have that

$$\mathfrak{n}_{\mathcal{F}} = \mathrm{span}\{\delta_P | P \in \mathcal{F}, P \text{ connected }\}$$

We will use the notation $\mathcal{F}^{conn} \subset \mathcal{F}$ to denote the sub-collection of $\mathcal{F}$ consisting of connected posets. We have thus established the following:

Theorem 2. The Ringel-Hall algebra of the category $\mathcal{C}_{\mathcal{F}}$ is a co-commutative graded connected Hopf algebra, isomorphic to $U(\mathfrak{n}_{\mathcal{F}})$, where $\mathfrak{n}_{\mathcal{F}}$ denotes the graded Lie algebra of its primitive elements. $\mathfrak{n}_{\mathcal{F}} = \mathrm{span}\{\delta_P | P \in \mathcal{F}^{conn}\}$.

Remark 4. $\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$ is a special case of an *incidence Hopf algebra* introduced by Schmitt in [10, 9].

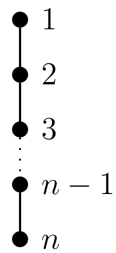
5. A Pre-Lie structure on $\mathfrak{n}_{\mathcal{F}}$

We assume now that the collection $\mathcal{F}$ consists of colored posets whose underlying Hasse diagrams are rooted trees. Recall that $\mathcal{F}$ was assumed to be:

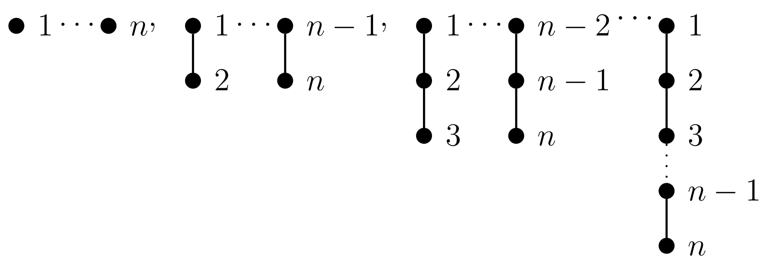
- closed under the operation of taking convex sub-posets
- closed under disjoint unions

It is immediate that to produce an $\mathcal{F}$ satisfying these two requirements, one may start with an arbitrary collection $\mathcal{F}'$ of colored posets, and close it with respect to each operation - i.e. adjoin to $\mathcal{F}'$ all convex sub-posets and all disjoint unions of these. If $\mathcal{F}$ arises in this way as the closure of $\mathcal{F}'$, we will write $\mathcal{F} = \overline{\mathcal{F}'}$.

Example 3. Suppose that $\mathcal{F}'$ consists of a single poset, whose Hasse diagram is an n–vertex ladder colored by the set $S = \{1, \cdots, n\}$.



Let us adopt the notation $L(a_1, a_2, ..., a_k)$ for a k–vertex ladder Hasse diagram labeled by $a_1, a_2, ..., a_k$ root-to-leaf ($\mathcal{F}'$ thus consisting of $L(1, 2, \cdots, n)$). To close $\mathcal{F}'$ with respect to convex subsets, we must adjoin to it $L(r, r+1, r+2, \cdots, r+m)$, where $1 \leq r \leq r + m \leq n$.



Finally, closing with respect to disjoint unions, we can identify elements of $\mathcal{F} = \overline{\mathcal{F}'}$ with Young diagrams having at most n rows, each of whose columns is labeled by $k, k + 1, \cdots, k + m$. For instance

2	1	3	4
3	2		
4	3		
5			

is identified with the poset

$$L(2,3,4,5)+L(1,2,3)+L(3)+L(4).$$

We proceed to equip $\mathfrak{n}_{\mathcal{F}}$ with a pre-Lie structure. For $a,b \in \mathcal{F}^{conn}$, we define

$$\delta_a \rhd \delta_b = \delta_a \star \delta_b - \delta_{a\oplus b} \tag{5.1}$$

and extend the product $\rhd$ to all of $\mathfrak{n}_{\mathcal{F}}$ by linearity. The subtraction of the term $\delta_{a\oplus b}$ in 5.1 has the effect of removing the delta-function supported on the one split extension of b by a, and so the right-hand side of 5.1 does indeed lie in $\mathfrak{n}_{\mathcal{F}}$. It follows easily that we may re-write the definition 5.1 as:

$$\delta_a \rhd \delta_b = \sum_{t\in\mathcal{F}} n(a,b,t)\delta_t \tag{5.2}$$

where $n(a,b,t)$ is defined as in example 2.

Theorem 3. Let $\mathcal{F}$ be a collection of colored posets closed with respect to taking convex sub-posets and disjoint unions. If the Hasse diagrams of posets in $\mathcal{F}$ are rooted trees, then $\rhd$ equips $\mathfrak{n}_{\mathcal{F}}$ with the structure of a pre-Lie algebra.

Proof. A two-sided pre-Lie ideal in a pre-Lie algebra A is a subspace $I \subset A$ such that if $x \in I$, then $a \rhd x \in I$ and $x \rhd a \in I$ $\forall a \in A$. One checks easily that the quotient A/I inherits a pre-Lie structure. Let $\mathcal{F}$ be a collection of colored rooted forests colored by S, closed under the operations of disjoint union and convex sub-poset, and $\mathcal{F}^{conn} \subset \mathcal{F}$ the connected ones (i.e. the rooted trees). $\mathcal{F}^{conn}$ is closed under taking convex sub-posets. I claim that $J = \mathcal{T}_S \backslash \mathcal{F}^{conn}$ is a two-sided pre-Lie ideal in $\mathcal{T}_S$. Let $u \in \mathcal{T}_S$ and $s \in J$. We have

$$\delta_u \rhd \delta_s = \sum_{t\in\mathcal{T}_S} n(u,s,t)\delta_t$$

Suppose that $n(u,s,t) \neq 0$ and $t \in \mathcal{T}_S \backslash J = \mathcal{F}^{conn}$. t has an edge e such that $P_e(t) = u$ and $R_e(t) = s$, and since both are convex sub-posets of the poset $t \in \mathcal{F}^{conn}$, $u, s \in \mathcal{F}^{conn}$, contradicting the fact that $s \in J$. It follows that $\delta_u \rhd \delta_s \in J$. The same argument shows that $\delta_s \rhd \delta_u \in J$. The quotient $\mathcal{T}_S/J$ is canonically identified with $\mathfrak{n}_{\mathcal{F}}$ with the bracket 5.2. □

We give a second proof, very close to the one for $\mathcal{F} = \mathbb{T}_S$ given in [2].

Proof. We need to verify the identity 2.1. It follows from 5.2 that for $a,b,c \in \mathcal{F}^{conn}$,

$$
\begin{aligned}
(\delta_a \rhd \delta_b) \rhd \delta_c &= (\sum_{t\in\mathcal{F}^{conn}} n(a,b,t)t) \rhd c \\
&= \sum_{s,t\in\mathcal{F}^{conn}} n(a,b,t)n(t,c,s)s \\
&\text{and} \\
\delta_a \rhd (\delta_b \rhd \delta_c) &= \sum_{s,t\in\mathcal{F}^{conn}} a \rhd (\sum_{t\in\mathcal{F}^{conn}} n(b,c,t)t) \\
&= \sum_{s,t\in\mathcal{F}^{conn}} n(b,c,t)n(a,t,s)s
\end{aligned}
$$

Because $\mathcal{F}$ is closed under taking convex sub-posets, $P_e(t) \in \mathcal{F}^{conn}$ and $R_e(t) \in \mathcal{F}^{conn}$, $\forall t \in \mathcal{F}^{conn}$. The sum $\sum_{t\in\mathcal{F}^{conn}} n(a,b,t)n(t,c,s)$ may be identified with the number of pairs of edges $\pi = \{e_1, e_2\} \subset E(s)$, such that the resulting cut is NOT admissible (i.e. both edges lie along a single path from root to leaf in s), and the three connected components when π is removed, are, top-to-bottom, c, b and a. Similarly, the sum $\sum_{t\in\mathcal{F}^{conn}} n(b,c,t)n(a,t,s)$ may identified with the number of pairs $\pi' = \{e_1, e_2\} \subset E(s)$ such that the corresponding cut of s results in three components a, b, c, with $r(s) \in c$, and no element of a greater than an element of b. The coefficient of δ_s in

$$\delta_a \rhd (\delta_b \rhd \delta_c) - (\delta_a \rhd \delta_b) \rhd \delta_c$$

therefore counts the number of *admissible* two-edge cuts of s such that the connected component containing $r(s)$ is isomorphic to c, and the remaining two to a, b respectively.

Applying the same analysis to the right-hand-side of 2.1 proves the equality.

□

Remark 5. It follows from 5.2 that $\mathfrak{n}_{\mathcal{F}}$ is defined over $\mathbb{Z}$, and that the structure constants are non-negative.

6. Examples

In this section, we consider different examples of families $\mathcal{F}$, and the resulting pre-Lie algebras $\mathfrak{n}_{\mathcal{F}}$. Recall that since $\mathfrak{n}_{\mathcal{F}}$ is graded by $\mathbb{N}$, the Lie algebra $\mathfrak{n}_{\mathcal{F}}$ is pro-nilpotent (nilpotent if $\mathfrak{n}_{\mathcal{F}}$ is finite-dimensional).

Example 4. Let S be a finite set, and $\mathcal{F} = \overline{\mathcal{T}_S}$, the set of rooted forests colored by S. We then obtain the pre-Lie algebra structure on S–labeled rooted trees described in example 2.

Example 5. Suppose S consists of a single element, and let $\mathcal{F} = \overline{\mathcal{F}'}$, where $\mathcal{F}'$ is the collection of all ladders:

(since there is only one color, we suppress the labeling). Denote by L_n the n-vertex ladder. We have

$$\delta_{L_n} \triangleright \delta_{L_m} = \delta_{L_{m+n}}.$$

so the Lie algebra $\mathfrak{n}_{\mathcal{F}}$ is abelian. In the Ringel-Hall algebra $\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$ we have

$$\delta_{L_n} \star \delta_{L_m} = \delta_{L_{m+n}} + \delta_{L_m \oplus L_n}$$

and

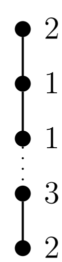
$$\Delta(L_m) = L_m \otimes 1 + 1 \otimes L_m$$

It is well-known (see eg. [6]) that the Hopf algebra $\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$ is isomorphic to the Hopf algebra of symmetric functions, with L_m corresponding to the mth power sum.

Example 6. Let $S = \{1, 2, \cdots, n\}$, and let $\mathcal{F} = \overline{\mathcal{F}'}$, where $\mathcal{F}'$ consists of singleton vertices colored by S. $\mathcal{F}$ is thus the collection of all finite sets colored by S, with trivial partial order. Denote by $X(m_1, m_2, \cdots, m_n)$ the set of $m_1 + m_2 + \cdots + m_n$ elements, with m_i colored i, $1 \leq i \leq n$. $\mathfrak{n}_{\mathcal{F}}$ is therefore spanned by the $\delta_{X(0,\cdots,\underset{i}{1},\cdots,0)}$. The operation $\triangleright$ is identically 0, so the Lie algebra $\mathfrak{n}_{\mathcal{F}}$ is abelian. In $\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$ we have

$$\delta_{X(m_1,\cdots,m_n)} \star \delta_{X(m'_1,\cdots,m'_n)} = \left(\prod_{i}^{n} \binom{m_i + m'_i}{m_i} \right) \delta_{(m_1+m'_1,\cdots,m_n+m'_n)}$$

Example 7. Let $S = \{1, 2, \cdots, n\}$, and let $\mathcal{F} = \overline{\mathcal{F}'}$, where $\mathcal{F}'$ consists of all S–colored ladder trees



Denote by $L(a_1, \cdots, a_k)$ the k-vertex ladder whose ith vertex counting from the *leaf* is colored a_i. We have

$$\delta_{L(a_1,\cdots,a_n)} \rhd \delta_{L(b_1,\cdots,b_m)} = \delta_{L(a_1,\cdots,a_n,b_1,\cdots,b_m)} \tag{6.1}$$

Let $\mathbb{Q} < X_1, \cdots, X_s >$ denote the free associative algebra on S viewed as a Lie algebra. There is a linear isomorphism

$$\rho : \mathfrak{n}_{\mathcal{F}} \to \mathbb{Q} < X_1, \cdots, X_s >$$
$$\rho(L(a_1, \cdots, a_k)) = X_{a_1} X_{a_2} \cdots X_{a_k}$$

It follows from 6.1 that ρ is a Lie algebra isomorphism.

Example 8. Consider the collection $\mathcal{F}$ from example 3, where $\mathcal{F} = \overline{L(1, 2, \cdots, n)}'$. Here $\mathfrak{n}_{\mathcal{F}} = \text{span}\{\delta_{L(k,\cdots,k+m)}\}$, $1 \le k \le k + m \le n$. We have

$$\delta_{L(p,\cdots,p+r)} \rhd \delta_{L(k,\cdots,k+m)} = \begin{cases} \delta_{L(k,\cdots,p+r)} & \text{if } k + m + 1 = p \\ 0 & \text{otherwise} \end{cases}$$

so that in the Lie algebra $\mathfrak{n}_{\mathcal{F}}$,

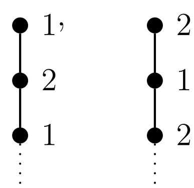
$$[\delta_{L(p,\cdots,p+r)}, \delta_{L(k,\cdots,k+m)}] = \begin{cases} \delta_{L(k,\cdots,p+r)} & \text{if } k + m + 1 = p \\ 0 & \text{otherwise} \end{cases} \tag{6.2}$$

Let $E_{i,j}$ denote the $(n + 1) \times (n + 1)$ matrix with a 1 in entry (i, j) and zeros everywhere else. Then the commutation relations 6.2 imply that the map

$$\phi : \mathfrak{n}_{\mathcal{F}} \to \text{Mat}_{n+1}$$
$$\phi(\delta_{L(k,\cdots,k+m)}) = -E_{k,k+m+1}$$

is an isomorphism of $\mathfrak{n}_{\mathcal{F}}$ onto the Lie algebra of upper-triangular $(n+1) \times (n+1)$ matrices.

Example 9. Let $S = \{1, 2\}$, and let $\mathcal{F} = \overline{\mathcal{F}'}$, where $\mathcal{F}'$ consists of all S–colored ladders where the colors alternate.



Let us denote by $L(i, n)$, $i \in S, n \ge 1$ the alternating ladder with n vertices, whose root is colored i. Then $\mathfrak{n}_{\mathcal{F}} = \text{span}\{L(i, n)\}$, $i \in S, n \ge 1$. We have

$$\delta_{L(i,n)} \rhd \delta_{L(i,m)} = \begin{cases} \delta_{L(i,n+m)} & \text{if } m \equiv 0 \; mod \; 2, \; i \in S \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_{L(i,n)} \rhd \delta_{L(j,m)} = \begin{cases} \delta_{L(j,n+m)} & \text{if } m \equiv 1 \; mod \; 2, \; i \neq j \in S \\ 0 & \text{otherwise} \end{cases}$$

It follows that

$$
\begin{aligned}
[\delta_{L(i,2k)}, \delta_{L(j,2l)}] &= 0 \\
[\delta_{L(i,2k)}, \delta_{L(j,2l+1)}] &= \begin{cases} -\delta_{L(j,2(k+l)+1)} & \text{if } i = j \\ \delta_{L(j,2(k+l)+1)} & \text{if } i \neq j \end{cases} \\
[\delta_{L(i,2k+1)}, \delta_{L(j,2l+1)}] &= \delta_{L(j,2(k+l+1))} - \delta_{L(i,2(k+l+1))}
\end{aligned} \tag{6.3}
$$

Recall that $\mathfrak{gl}_2 = \mathrm{Mat}_2 = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where

$$\mathfrak{n}_- = \mathrm{span}\{f\}, \mathfrak{n}_+ = \mathrm{span}\{e\}, \mathfrak{h} = \mathrm{span}\{h_1, h_2\}$$

and

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Let $L\mathfrak{gl}_2 = \mathfrak{gl}_2 \otimes \mathbb{Q}[t, t^{-1}]$ be the loop algebra of $\mathfrak{gl}_2$, with bracket

$$[X \otimes t^m, Y \otimes t^n] = [X, Y] \otimes t^{n+m}$$

$L\mathfrak{gl}_2$ also has a triangular decomposition $L\mathfrak{gl}_2 = L\mathfrak{gl}_2^+ \oplus \mathfrak{h} \oplus L\mathfrak{gl}_2^-$, where

$$L\mathfrak{gl}_2^+ = \mathfrak{n}_+ \oplus \mathfrak{gl}_2 \otimes t\mathbb{Q}[t] \qquad L\mathfrak{gl}_2^- = \mathfrak{n}_- \oplus \mathfrak{gl}_2 \otimes t^{-1}\mathbb{Q}[t^{-1}]$$

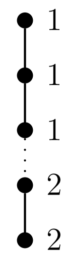
Let

$$
\begin{aligned}
\phi : \mathfrak{n}_{\mathcal{F}} &\to L\mathfrak{gl}_2^+ \\
\phi(\delta_{L(1,2k+1)}) &= e \otimes t^k \\
\phi(\delta_{L(2,2k+1)}) &= f \otimes t^{k+1} \\
\phi(\delta_{L(1,2k)}) &= -h_1 \otimes t^k \\
\phi(\delta_{L(2,2k)}) &= -h_2 \otimes t^k
\end{aligned}
$$

It follows from 6.3 that ϕ is an isomorphism. It follows that $U(L\mathfrak{gl}_2^+)$ has an integral basis which may be identified with Young diagrams whose columns are colored by alternating strings of 1's and 2's.

Example 10. A straightforward generalization of the previous example, with $S = \{1, \cdots, n\}$ and $\mathcal{F}'$ consisting of ladders periodically colored by $1, \cdots, n$ yields $\mathfrak{n}_{\mathcal{F}} \simeq L\mathfrak{gl}_n^+$.

Example 11. Let $S = \{1, 2\}$, and let $\mathcal{F} = \overline{\mathcal{F}'}$, where $\mathcal{F}'$ is the set of all ladders colored by a sequence of 1's followed by a sequence of 2's.



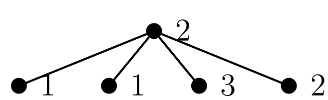
Denote by $L(i, j)$ the ladder with i 1's followed by j 2's. We have

$$\begin{aligned}
\delta_{L(i,j)} \rhd \delta_{L(m,n)} &= 0 \text{ if } ij > 0 \text{ and } mn > 0\\
\delta_{L(i,0)} \rhd \delta_{L(m,n)} &= \begin{cases} \delta_{L(i+m,0)} & \text{if } n = 0\\ 0 & \text{otherwise}\end{cases}\\
\delta_{L(0,j)} \rhd \delta_{L(m,n)} &= \delta_{L(m,n+j)}\\
\delta_{L(i,j)} \rhd \delta_{L(m,0)} &= \delta_{L(i+m,j)}\\
\delta_{L(i,j)} \rhd \delta_{L(0,n)} &= \begin{cases} \delta_{L(0,j+n)} & \text{if } i = 0\\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

so that we obtain the following non-zero commutation relations (i.e. all other commutators are 0):

$$\begin{aligned}
[\delta_{L(i,0)}, \delta_{L(0,n)}] &= -\delta_{L(i,n)}\\
[\delta_{L(i,0)}, \delta_{L(m,n)}] &= -\delta_{L(m+i,n)} \text{ if } n > 0\\
[\delta_{L(0,j)}, \delta_{L(m,n)}] &= \delta_{L(m,n+j)} \text{ if } m > 0
\end{aligned}$$

Example 12. Let $S = \{1, 2, \cdots, n\}$, and let $\mathcal{F} = \overline{\mathcal{F}'}$, where $\mathcal{F}'$ consists of all S–colored corollas (rooted trees where all leaves are connected directly to the root)



Closing $\mathcal{F}'$ with respect to convex sub-posets means adjoining singleton colored trees. Denote by $X(i)$ the singleton tree colored by $1 \le i \le n$, and by $Y(i, a_1, \cdots, a_n)$ the corolla whose root is colored i and which has $a_1 + a_2 \cdots + a_n$

leaves, with a_1 colored 1, a_2 colored 2 etc. In $\mathfrak{n}_{\mathcal{F}}$ we have

$$\begin{aligned}
\delta_{X(i)} \rhd \delta_{X(j)} &= \delta_{Y(j,0,\cdots,\underset{i}{1},\cdots,0)} \\
\delta_{X(i)} \rhd \delta_{Y(j,a_1,\cdots,a_n)} &= \delta_{Y(j,a_1,\cdots,a_i+1,\cdots,a_n)} \\
\delta_{Y(j,a_1,\cdots,a_n)} \rhd \delta_{X(i)} &= 0 \\
\delta_{Y(j,a_1,\cdots,a_n)} \rhd \delta_{Y(j,b_1,\cdots,b_n)} &= 0
\end{aligned}$$

which leads to the following commutation relations:

$$\begin{aligned}
[\delta_{X(i)}, \delta_{X(j)}] &= \delta_{Y(j,0,\cdots,\underset{i}{1},\cdots,0)} - \delta_{Y(i,0,\cdots,\underset{j}{1},\cdots,0)} \\
[\delta_{X(i)}, \delta_{Y(j,a_1,\cdots,a_n)}] &= \delta_{Y(j,a_1,\cdots,a_i+1,\cdots,a_n)} \\
[\delta_{Y(j,a_1,\cdots,a_n)}, \delta_{Y(j,b_1,\cdots,b_n)}] &= 0
\end{aligned}$$

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