THE HOPF ALGEBRA OF SKEW SHAPES, TORSION SHEAVES ON $\mathbb{A}^n_{/\mathbb{F}_1}$, AND IDEALS IN HALL ALGEBRAS OF MONOID REPRESENTATIONS

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ABSTRACT. We study ideals in Hall algebras of monoid representations on pointed sets corresponding to certain conditions on the representations. These conditions include the property that the monoid act via partial permutations, that the representation possess a compatible grading, and conditions on the support of the module. Quotients by these ideals lead to combinatorial Hopf algebras which can be interpreted as Hall algebras of certain sub-categories of modules. In the case of the free commutative monoid on n generators, we obtain a co-commutative Hopf algebra structure on n-dimensional skew shapes, whose underlying associative product amounts to a "stacking" operation on the skew shapes. The primitive elements of this Hopf algebra correspond to connected skew shapes, and form a graded Lie algebra by anti-symmetrizing the associative product. We interpret this Hopf algebra as the Hall algebra of a certain category of coherent torsion sheaves on $\mathbb{A}^n_{/\mathbb{F}_1}$ supported at the origin, where $\mathbb{F}_1$ denotes the field of one element. This Hopf algebra may be viewed as an n-dimensional generalization of the Hopf algebra of symmetric functions, which corresponds to the case $n = 1$.

1. INTRODUCTION

This paper introduces a Lie algebra structure $\mathfrak{sl}_n$ on n-dimensional connected skew shapes. The enveloping algebra $\mathbb{U}(\mathfrak{sl}_n)$ is constructed as the Hall algebra of a sub-category of torsion sheaves on $\mathbb{A}^n_{/\mathbb{F}_1}$ supported at the origin, where $\mathbb{F}_1$ denotes the "field of one element". The connected skew shapes correspond to the indecomposable objects of this category. In the introduction below we explain how the Hall algebra construction, applied in non-additive contexts such as that of algebraic geometry over $\mathbb{F}_1$ produces combinatorial Hopf algebras of a representation theoretic nature which can be viewed as degenerations of quantum-group like objects over $\mathbb{F}_q$.

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1.1. **Hall algebras of Abelian categories.** The study of Hall algebras is by now a well-established area with several applications in representation theory and algebraic geometry (see [24] for a very nice overview). We briefly recall the generic features of the most basic version of this construction. Given an abelian category C, let

$$Fl_i(C) := \{A_0 \subset A_1 \subset \cdots \subset A_i | A_k \in \mathrm{Ob}(C)\}$$

denote the stack parametrizing flags of objects in C of length $i + 1$ (viewed here simply as a set). Thus

$$Fl_0(C) = \mathrm{Iso}(C),$$

the moduli stack of isomorphism classes of objects of C, and

$$Fl_1(C) = \{A_0 \subset A_1 | A_0, A_1 \in \mathrm{Ob}(C)\}$$

is the usual Hecke correspondence. We have maps

(1) $$\pi_i : Fl_1(C) \to Fl_0(C), \ i = 1, 2, 3$$

where

$$\pi_1(A_0 \subset A_1) = A_0$$
$$\pi_2(A_0 \subset A_1) = A_1$$
$$\pi_3(A_0 \subset A_1) = A_1/A_0$$

We may then attempt to define the Hall algebra of C as the space of $\mathbb{Q}$-valued functions on $Fl_0(C)$ with finite support, i.e.

$$\mathrm{H}_C = \mathbb{Q}_c[Fl_0(C)]$$

with the convolution product defined for $f, g \in \mathrm{H}_C$

$$f \star g := \pi_{2*}(\pi_3^*(f)\pi_1^*(g)),$$

where π_i^* denotes the usual pullback of functions and π_{i*} denotes integration along the fiber. To make this work, one has to impose certain finiteness conditions on C. The simplest, and most restrictive such condition is that C is *finitary*, which means that $\mathrm{Hom}(M, M')$ and $\mathrm{Ext}^1(M, M')$ are finite sets for any pair of objects $M, M' \in C$. Important examples of categories C having these properties are $C = \mathrm{Rep}(Q, \mathbb{F}_q)$ - the category of representations of a quiver Q over a finite field $\mathbb{F}_q$, and $C = Coh(X/\mathbb{F}_q)$ - the category of coherent sheaves on a smooth projective

variety X over $\mathbb{F}_q$. In these examples, the structure constants of H_C depend on the parameter q, and H_C recovers (parts of) quantum groups and their higher-loop generalizations. The basic recipe for constructing H_C sketched here has a number of far-reaching generalizations that extend well beyond the case of C finitary (see [24]).

The case when $C = Coh(X_{/\mathbb{F}_q})$ is especially rich, and reasonably explicit results about the structure of H_C are only available when $dim(X) = 1$. In this special case, the stacks $Fl_i(C)$ are naturally the domains of definition of automorphic forms for the function field $\mathbb{F}_q(X)$, and so "live in" the Hall algebra H_C. Here, the theory makes contact with the Langlands program over function fields (for more on this, see the beautiful papers [1, 2, 13, 14]). When $dim(X) \geq 2$, the detailed structure of H_C remains quite mysterious, and the technical problems are considerable.

1.2. **Hall algebras in a non-additive setting.** A closer examination of the basic construction of H_C outlined above shows that the assumption that C be Abelian is unnecessary. All that is needed to make sense of the Hecke correspondence (1) used to define H_C is a category with a reasonably well-behaved notions of kernels/co-kernels and exact sequences. For instance, the paper [9] describes a class of "proto-exact" categories, which need not be additive, and for which the basic setup above can be made to work. Working in a non-additive context is sometimes given the catch-all slogan of "working over $\mathbb{F}_1$", where $\mathbb{F}_1$ stands for the mythical "field of one element". The terminology and notation stem from the observation that functions enumerating various "linear" objects over $\mathbb{F}_q$ often have a limit $q \to 1$ which enumerates some analogous combinatorial objects. This is best illustrated with the example of the Grassmannian, where the number of $\mathbb{F}_q$-points is given by a q-binomial coefficient:

$$\# \mathrm{Gr}(k, n)/\mathbb{F}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}$$

where

$$[n]_q! = [n]_q[n-1]_q \ldots [2]_q$$

and

$$[n]_q = 1 + q + q^2 + \ldots + q^{n-1}$$

In the limit $q \to 1$ this reduces to the binomial coefficient $\binom{n}{k}$, counting k-element subsets of an n-element set. This suggests that a vector space over $\mathbb{F}_1$ should be a (pointed) set, with cardinality replacing dimension.

Many examples of non-additive categories $\mathcal{C}$ where $H_{\mathcal{C}}$ can be constructed thus come from combinatorics. Here, $\mathrm{Ob}(\mathcal{C})$ typically consist of combinatorial structures equipped an operation of "collapsing" a sub-structure, which corresponds to forming a quotient in $\mathcal{C}$. Examples of such $\mathcal{C}$ include trees, graphs, posets, matroids, semigroup representations on pointed sets, quiver representations in pointed sets etc. (see [18, 26–29]). The product in $H_{\mathcal{C}}$, which counts all extensions between two objects, thus amounts to enumerating all combinatorial structures that can be assembled from the two. In this case, $H_{\mathcal{C}}$ is (dual to) a combinatorial Hopf algebra in the sense of [19]. Many combinatorial Hopf algebras arise via this mechanism.

1.3. **Hall algebras over $\mathbb{F}_1$ as degenerations of Hall algebras over $\mathbb{F}_q$.** One may ask how the $q \to 1$ limit behaves at the level of Hall algebras. Let us illustrate this via an example. Given a Dynkin quiver Q, we may consider both the Abelian category $\mathrm{Rep}(Q, \mathbb{F}_q)$ and the non-additive category $\mathrm{Rep}(Q, \mathbb{F}_1)$. The latter is defined by replacing ordinary vector spaces at each vertex by pointed sets (i.e. "vector spaces over $\mathbb{F}_1$" - see [28] for details). The Lie algebra $\mathfrak{g}_Q$ with Dynkin diagram (the underlying unoriented graph) Q has a triangular decomposition

$$\mathfrak{g}_Q = \mathfrak{n}_{Q,-} \oplus \mathfrak{h} \oplus \mathfrak{n}_{Q,+}.$$

By a celebrated result of Ringel and Green, one obtains that

$$H_{\mathrm{Rep}(Q,\mathbb{F}_q)} \simeq \mathbb{U}_{\sqrt{q}}(\mathfrak{n}_{Q,+}),$$

where the right hand side denotes the quantized enveloping algebra of $\mathfrak{n}_{Q,+}$. On the other hand, the author has shown that

$$H_{\mathrm{Rep}(Q,\mathbb{F}_1)} \simeq \mathbb{U}(\mathfrak{n}_{Q,+})/\mathcal{I}$$

where on the right hand side we now have a quotient of the *ordinary* enveloping algebra by a certain ideal $\mathcal{I}$, which in type A is trivial. This suggests the following (unfortunately, at this stage, very vague) principle: given a finitary abelian category $\mathcal{C}_q$ linear over $\mathbb{F}_q$ and $\mathcal{C}_1$ a candidate for its "limit over $\mathbb{F}_1$", $H_{\mathcal{C}_1}$ is a (possibly degenerate) limit of $H_{\mathcal{C}_q}$.

This principle is borne out in another, more geometric example, when $C_q = Coh(\mathbb{P}^1_{/\mathbb{F}_q})$. A result of Kapranov and Baumann-Kassel shows that in this case, there is an embedding

$$\Psi : \mathbf{U}_{\sqrt{q}}(L\mathfrak{sl}_2^+) \to H_{C_q},$$

where $L\mathfrak{sl}_2^+$ denotes a non-standard Borel of the loop algebra of $\mathfrak{sl}_2$. The category $C_1 = Coh(\mathbb{P}^1_{/\mathbb{F}_1})$ is constructed using a theory of schemes over $\mathbb{F}_1$ and coherent sheaves on them developed by Deitmar and others (see [4,6,7]). In this simplest version of algebraic geometry over $\mathbb{F}_1$, schemes are glued out of prime spectra of commutative monoids rather than rings, and coherent sheaves are defined as sheaves of set-theoretic modules over the monoid structure sheaf. In [27], the author shows that

$$H_{C_1} \simeq \mathbf{U}(L\mathfrak{gl}_2^+ \oplus \kappa)$$

where $L\mathfrak{gl}_2^+$ denotes a Borel in the loop algebra of $\mathfrak{gl}_2$ and κ is a certain abelian cyclotomic factor. Moreover, this calculation is much easier than the corresponding one over $\mathbb{F}_q$.

When $dim(X) \geq 2$ and X possesses a model over $\mathbb{F}_1$, one might then hope to use the Hall algebra of $C_1 = Coh(X_{/\mathbb{F}_1})$ to learn something about the more complicated Hall algebra of $Coh(X_{/\mathbb{F}_q})$. Naively, we may hope that the latter is a q-deformation of the former. This paper begins this program by focussing on the simplest piece of the category C_1, that of torsion sheaves supported at a point. We recall that when $Y_{/\mathbb{F}_q}$ is a smooth curve and $y \in Y$ is a closed point, the category $Tor_y \subset Coh(Y)$ of coherent sheaves supported at y is closed under extensions, and yields a Hopf sub-algebra of $H_{Coh(Y)}$ isomorphic to the Hopf algebra Λ of symmetric functions (see [24]). It is shown in [27] that this result holds over $\mathbb{F}_1$ as well, provided one restricts to a certain sub-category of sheaves which in this paper are called "type α", and which in [27] are called "normal". From this perspective, this paper deals with the n-dimensional generalization of the Hopf algebra of symmetric functions.

The monoid scheme $\mathbb{A}^n_{/\mathbb{F}_1}$ has a single closed point - the origin, and coherent sheaves supported there form a sub-category $Tor_0 \subset Coh(\mathbb{A}^n_{/\mathbb{F}_1})$ closed under extensions. Such sheaves correspond to pointed sets $(M, 0_M)$ with a nilpotent action of the free commutative monoid $A = \mathbb{F}_1\langle x_1, \cdots, x_n \rangle$ on n generators $x_1, \cdots, x_n$, and form a category A-mod$_0$ for which we may define a Hall algebra $H_{0,A}$. We consider a sub-category A-mod$_0^{\alpha, gr} \subset$ A-mod of A-modules supported at

0, which are furthermore of type α and admit a $\mathbb{Z}^n$-grading. The type α condition is imposed to recover the Hopf algebra of symmetric functions when $n = 1$, and the grading, which is automatic in the case $n = 1$ is imposed to dispose of certain pathological examples. We show

Theorem (5.4.2). The indecomposable objects of A-mod$_0^{\alpha, gr}$ correspond bijectively to n-dimensional connected skew shapes.

Here by a skew shape we mean a finite sub-poset of $\mathbb{Z}^n$ under the partial order where $(x_1, \cdots, x_n) \leq (y_1, \cdots, y_n)$ iff $x_i \leq y_i$ for $i = 1, \cdots, n$, considered up to translation. Each skew shape determines a $\mathbb{F}_1\langle x_1, \cdots, x_n \rangle$-module where x_i acts by moving 1 box in the positive ith direction. For example, when $n = 2$, the skew shape
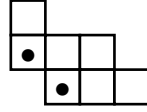
determines a $\mathbb{F}_1\langle x_1, x_2 \rangle$-module on two generators (indicated by black dots), where x_1 moves one box to the right, and x_2 one box up, until the edge of the diagram is reached, and 0 beyond that.

We may now define the Hall algebra $\mathbf{SK}_n$ of A-mod$_0^{\alpha, gr}$ by counting *admissible short exact sequences* in this category. This is a quotient of $H_{0,A}$ above. Paraphrased, our main result says:

Theorem (6.0.1). The Hall algebra $\mathbf{SK}_n$ of A-mod$_0^{\alpha, gr}$ is isomorphic to the enveloping algebra $\mathbb{U}(\mathfrak{sk}_n)$ of a graded Lie algebra $\mathfrak{sk}_n$. $\mathfrak{sk}_n$ has a basis corresponding to connected n-dimensional skew shapes. Furthermore, $\mathfrak{sk}_n$ and $\mathbf{SK}_n$ carries an action of the symmetric group S_n by Lie (resp. Hopf) algebra automorphisms.

The associative product in $\mathbf{SK}_n$ corresponds to a "stacking" operation on skew shapes.

1.4. **Future directions.** By a result of Deitmar, It is known that every smooth proper toric variety $X_{/\mathbb{Z}}$ has a model over $\mathbb{F}_1$ ([7]) I.e. there exists a monoid scheme $X'_{/\mathbb{F}_1}$ such that

$$X \simeq X' \otimes_{\mathbb{F}_1} \mathbb{C}$$

The approach taken here to compute the Hall algebra of normal point sheaves may be extended to construct the full Hall algebra of normal coherent sheaves on X purely in terms of the combinatorics of the fan of X and n-dimensional partitions. For instance, a coherent sheaf on $\mathbb{A}^3$ which is type α, graded, and supported on the union of the coordinate axes may be visualized in terms of a 3-D asymptotic partition like:
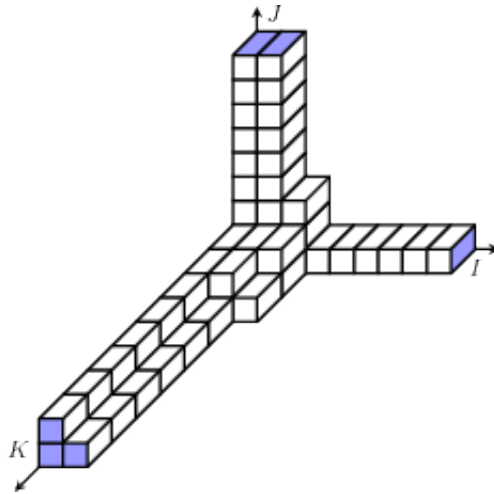
This Hall algebra is once again a co-commutative Hopf algebra, and contains several copies of the Hopf algebra $\mathbf{SK}_n$ above, one for each closed point of $X'_{/\mathbb{F}_1}$ (here $n = dim(X)$), as well as larger Hopf subalgebras corresponding to various co-dimension subvarieties of $X'/\mathbb{F}_1$. This theme will be taken up in future papers [30, 31].

1.5. **Outline.** This paper is structured as follows:

(1) In Section 2 we recall basic properties of the category A-mod of (set-theoretic) finite modules over a monoid A. We discuss what is meant by exact sequences in this non-additive category, normal morphisms, as well as additional conditions which lead to sub-categories of A-mod - type α, graded, and modules with prescribed support.

(2) In Section 3 we briefly sketch the theory of schemes over $\mathbb{F}_1$ via monoid schemes as introduced by Deitmar. This is done to allow a geometric interpretation of our Hall algebra in terms of coherent sheaves on $\mathbb{A}^n_{/\mathbb{F}_1}$ supported at the origin.

(3) In Section 4 we recall the construction of the Hall algebra H_A of A-mod, defined by counting admissible short exact sequences. We show that imposing one or several of the the conditions discussed in 2 (type α, admitting a grading, prescribed support), yields Hopf ideals in H_A. H_A and its various quotients by these ideals are isomorphic to enveloping algebras of Lie algebras with basis the corresponding indecomposable modules.

(4) In Section 5, we explain the bijection between n-dimensional skew shapes and type α modules over $\mathbb{F}_1\langle x_1, \cdots, x_n \rangle$ supported at 0 and admitting a $\mathbb{Z}^n$-grading.

(5) In Section 6 we make explicit the Lie and Hopf structures on skew shapes which emerge from the Hall algebra construction.

(6) In Section 7 we outline structure of the Hopf duals of the Hall algebras introduced previously.

2. Monoids and their modules

A *monoid* A will be an associative semigroup with identity 1_A and zero 0_A (i.e. the absorbing element). We require

$$1_A \cdot a = a \cdot 1_A = a \qquad 0_A \cdot a = a \cdot 0_A = 0_A \qquad \forall a \in A$$

Monoid homomorphisms are required to respect the multiplication as well as the special elements $1_A, 0_A$.

Example 1. Let $\mathbb{F}_1 = \{0, 1\}$ with

$$0 \cdot 1 = 1 \cdot 0 = 0 \cdot 0 = 0 \text{ and } 1 \cdot 1 = 1.$$

We call $\mathbb{F}_1$ *the field with one element*.

Example 2. Let

$$\mathbb{F}_1\langle x_1, \cdots, x_n \rangle := \{x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n} | (e_1, e_2, \cdots, e_n) \in \mathbb{Z}_{\geq 0}^n\} \cup \{0\},$$

the set of monomials in $x_1, \cdots, x_n$, with usual multiplication of monomials. We will often write elements of $\mathbb{F}_1\langle x_1, \cdots, x_n \rangle$ in multiindex notation as $x^e, e \in \mathbb{Z}_{\geq 0}^n$, in

which case the multiplication is written as

$$x^e \cdot x^f = x^{e+f}.$$

We identify x^0 with 1.

$\mathbb{F}_1$ and $\mathbb{F}_1\langle x_1, \cdots, x_n \rangle$ are both commutative monoids.

Definition 2.0.1. Let A be a monoid. An A–*module* is a pointed set $(M, 0_M)$ (with $0_M \in M$ denoting the basepoint), equipped with an action of A. More explicitly, an A–module structure on $(M, 0_M)$ is given by a map

$$A \times M \to M$$
$$(a, m) \to a \cdot m$$

satisfying

$$(a \cdot b) \cdot m = a \cdot (b \cdot m), \quad 1 \cdot m = m, \quad 0 \cdot m = 0_M, \quad a \cdot 0_M = 0_M, \quad \forall a, b, \in A, \ m \in M$$

A *morphism* of A–modules is given by a pointed map $f : M \to N$ compatible with the action of A, i.e. $f(a \cdot m) = a \cdot f(m)$. The A–module M is said to be *finite* if M is a finite set, in which case we define its *dimension* to be $dim(M) = |M| - 1$ (we do not count the basepoint, since it is the analogue of 0). We say that $N \subset M$ is an A–*submodule* if it is a (necessarily pointed) subset of M preserved by the action of A. A always posses the trivial module {0}, which will be referred to as the *zero module*.

Note: This structure is called an A-*act* in [17] and an A-*set* in [5].

We denote by A-mod the category of finite A–modules. It is the $\mathbb{F}_1$ analogue of the category of a finite-dimensional representations of an algebra. Note that for $M \in$ A-mod, $\text{End}_{A\text{-mod}}(M) := \text{Hom}_{A\text{-mod}}(M, M)$ is a monoid (in general non-commutative). An $\mathbb{F}_1$–module is simply a pointed set, and will be referred to as a vector space over $\mathbb{F}_1$. Thus, an A–module structure on $M \in \mathbb{F}_1$-mod amounts to a monoid homomorphism $A \to \text{End}_{\mathbb{F}_1\text{-mod}}(M)$.

Given a morphism $f : M \to N$ in A-mod, we define the *image* of f to be

$$Im(f) := \{n \in N | \exists m \in M, f(m) = n\}.$$

For $M \in$ A-mod and an A–submodule $N \subset M$, the *quotient* of M by N, denoted M/N is the A–module

$$M/N := M \backslash N \cup \{0\},$$

i.e. the pointed set obtained by identifying all elements of N with the base-point, equipped with the induced A–action.

We recall some properties of A-mod, following [5, 17, 29], where we refer the reader for details:

(1) For $M, N \in$ A-mod, $|Hom_{\text{A-mod}}(M, N)| < \infty$
(2) The trivial A–module 0 is an initial, terminal, and hence zero object of A-mod.
(3) Every morphism $f : M \to N$ in C_A has a kernel $Ker(f) := f^{-1}(0_N)$.
(4) Every morphism $f : M \to N$ in C_A has a cokernel $Coker(f) := M/Im(f)$.
(5) The co-product of a finite collection $\{M_i\}, i \in I$ in A-mod exists, and is given by the wedge product

$$\bigvee_{i \in I} M_i = \coprod M_i / \sim$$

where $\sim$ is the equivalence relation identifying the basepoints. We will denote the co-product of $\{M_i\}$ by

$$\oplus_{i \in I} M_i$$

(6) The product of a finite collection $\{M_i\}, i \in I$ in A-mod exists, and is given by the Cartesian product $\prod M_i$, equipped with the diagonal A–action. It is clearly associative. it is however not compatible with the coproduct in the sense that $M \times (N \oplus L) \not\simeq M \times N \oplus M \times L$.
(7) The category A-mod possesses a reduced version $M \wedge N$ of the Cartesian product $M \times N$, called the smash product. $M \wedge N := M \times N / M \vee N$, where M and N are identified with the A–submodules $\{(m, 0_N)\}$ and $\{(0_M, n)\}$ of $M \times N$ respectively. The smash product inherits the associativity from the Cartesian product, and is compatible with the co-product - i.e.

$$M \wedge (N \oplus L) \simeq M \wedge N \oplus M \wedge L.$$

(8) A-mod possesses small limits and co-limits.

(9) If A is commutative, A-mod acquires a monoidal structure called the *tensor product*, denoted $M \otimes_A N$, and defined by

$$M \otimes_A N := M \times N / \sim_\otimes$$

where $\sim_\otimes$ is the equivalence relation generated by $(a \cdot m, n) \sim_\otimes (m, a \cdot n)$ for all $a \in A, m \in M, n \in N$. Note that $(0_M, n) = (0 \cdot 0_M, n) \sim_\otimes (0_M, 0 \cdot n) = (0_M, 0_N)$, and likewise $(m, 0_N) \sim_\otimes (0_M, 0_N)$. This allows us to rewrite the tensor product as $M \otimes_A N = M \wedge N / \sim_{\otimes'}$, where $\sim_{\otimes'}$ denotes the equivalence relation induced on $M \wedge N$ by $\sim_\otimes$. We have

$$M \otimes_A N \simeq N \otimes_A M,$$
$$(M \otimes_A N) \otimes_A L \simeq M \otimes_A (N \otimes_A L),$$
$$M \otimes_A (L \oplus N) \simeq (M \otimes_A L) \oplus (M \otimes_A N).$$

(10) Given M in A-mod and $N \subset M$, there is an inclusion-preserving correspondence between flags $N \subset L \subset M$ in A-mod and A–submodules of M/N given by sending L to L/N. The inverse correspondence is given by sending $K \subset M/N$ to $\pi^{-1}(K)$, where $\pi : M \to M/N$ is the canonical projection. This correspondence has the property that if $N \subset L \subset L' \subset M$, then $(L'/N)/(L/N) \simeq L'/L$.

These properties suggest that A-mod has many of the properties of an abelian category, without being additive. It is an example of a *quasi-exact* and *belian* category in the sense of Deitmar [8] and a *proto-exact* category in the sense of Dyckerhof-Kapranov [9]. Let Iso(A-mod) denote the set of isomorphism classes in A-mod, and by $\overline{M}$ the isomorphism class of $M \in$ A-mod.

Definition 2.0.2.   (1) We say that $M \in$ A-mod is *indecomposable* if it cannot be written as $M = N \oplus L$ for non-zero $N, L \in$ A-mod.

(2) We say $M \in$ A-mod is *irreducible* or *simple* if it contains no proper sub-modules (i.e those different from 0 and M).

It is clear that every irreducible module is indecomposable. We have the following analogue of the Krull-Schmidt theorem ([29]):

Proposition 2.0.1. *Every $M \in$ A-mod can be uniquely decomposed (up to reordering) as a direct sum of indecomposable A–modules.*

Remark 2.0.3. Suppose $M = \oplus_{i=1}^{k} M_i$ is the decomposition of an A–module into indecomposables, and $N \subset M$ is a submodule. It then immediately follows that $N = \oplus(N \cap M_i)$.

Given a ring R, denote by $R[A]$ the semigroup ring of A with coefficients in R. We obtain a base-change functor:

$$(2) \qquad\qquad\qquad \otimes_{\mathbb{F}_1} R : \text{A-mod} \to R[A] - mod$$

to the category of $R[A]$–modules defined by setting

$$M \otimes_{\mathbb{F}_1} R := \bigoplus_{m \in M, m \neq 0_M} R \cdot m$$

i.e. the free R–module on the non-zero elements of M, with the $R[A]$–action induced from the A–action on M.

2.1. **Exact Sequences.** A diagram $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ in A-mod is said to be *exact at* M_2 if $Ker(g) = Im(f)$. A sequence of the form

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a *short exact sequence* if it is exact at M_1, M_2 and M_3.

One key respect in which A-mod differs from an abelian category is he fact that given a morphism $f : M \to N$, the induced morphism $M/Ker(f) \to Im(f)$ need not be an isomorphism. This defect also manifests itself in the fact that the base change functor $\otimes_{\mathbb{F}_1} R : \text{A-mod} \to R[A] - mod$ fails to be exact (i.e. the base change of a short exact sequence is in general no longer short exact). A-mod does however contain a (non-full) subcategory which is well-behaved in this sense, and which we proceed to describe.

Definition 2.1.1. A morphism $f : M \to N$ is *normal* if every fibre of f contains at most one element, except for the fibre $f^{-1}(0_N)$ of the basepoint $0_N \in N$.

It is straightforward to verify that this condition is equivalent to the requirement that the canonical morphism $M/Ker(f) \to Im(f)$ be an isomorphism, and that the composition of normal morphisms is normal.

Definition 2.1.2. Let A-mod$_n$ denote the subcategory of A-mod with the same objects as A-mod, but whose morphisms are restricted to the normal morphisms of A-mod. A short exact sequence in A-mod$_n$ is called *admissible*.

Remark 2.1.3. In contrast to A-mod, A-mod$_n$ is typically neither (small) complete nor co-complete. However, $\otimes_{\mathbb{F}_1} R$ is exact on A-mod$_n$ for any ring R, in the sense that this functor takes admissible short exact sequences in A-mod to short exact sequences in the Abelian category $R[A] - mod$. Note that $\mathrm{Iso}(\text{A-mod}) = \mathrm{Iso}(\text{A-mod}_n)$, since all isomorphisms are normal.

The following simple result is proved in [29]:

Proposition 2.1.1. *Let A be a monoid and A-mod$_n$ as above.*

Suppose A is finitely generated. For $M, N \in$ A-mod$_n$, there are finitely many admissible short exact sequences

$$(3) \qquad\qquad 0 \to M \xrightarrow{f} L \xrightarrow{g} N \to 0$$

up to isomorphism.

Definition 2.1.4. Let A be a monoid. The Grothendieck Group of A is

$$K_0(A) := \mathbb{Z}[\overline{M}]/J, \quad \overline{M} \in \mathrm{Iso}(\text{A-mod}_n),$$

where J is the subgroup generated by $\overline{L} - \overline{M} - \overline{N}$ for all admissible short exact sequences (3).

2.2. **Modules of type α and graded modules.**

2.2.1. *Modules of type α.* Given a monoid A and $M \in$ A-mod, we may for each $a \in A$ consider the morphism

$$\phi_a : M \to M$$

$$\phi_a(m) = a \cdot m$$

in $\mathbb{F}_1$-mod (note that ϕ_a is not a morphism in A-mod unless A is commutative).

Definition 2.2.1. We say that M is of *type α* if $\forall a \in A$, the morphism $\phi_a : M \to M$ is a normal morphism in $\mathbb{F}_1$-mod.

Equivalently, if M is type α, then

$$a \cdot m_1 = a \cdot m_2 \iff m_1 = m_2 \text{ OR } a \cdot m_1 = a \cdot m_2 = 0$$

for $a \in A, m_1, m_2 \in M$.

Example 3. Let $A = \mathbb{F}_1\langle t \rangle$. An A-module is simply a vector space M over $\mathbb{F}_1$ with an endomorphism $t \in \mathrm{End}_{\mathbb{F}_1\text{-mod}}(M)$. M is thus type α as an A–module iff t is a normal morphism, i.e. if $t|_{M \setminus t^{-1}(0_m)}$ is injective. Thus, if $M = \{0, a, b, c\}$, and the action of t is defined by

$$t(a) = 0, \quad t(b) = a, \quad t(c) = b,$$

then M is type α. If however, we define the action of t by

$$t(a) = b = t(b), \quad t(c) = 0,$$

then M is not of type α.

Remark 2.2.2. Let M be A-module, and $a \in A$, and ϕ_a as above. Let $M \otimes_{\mathbb{F}_1} \mathbb{Z}$ be the $\mathbb{Z}[A]$-module obtained via base change. The action of ϕ_a on M extends to an endomorphism $\phi_a^{\mathbb{Z}}$ of $M \otimes_{\mathbb{F}_1} \mathbb{Z}$ which can be represented as a square $dim(M) \times dim(M)$ matrix, having all entries either 0 or 1, and such that each column has at most one 1. The condition that M is type α implies that each row also contains at most one 1 (i.e. that ϕ_a is a partial permutation matrix). Note that we always have the inclusion

$$Ker(\phi_a) \otimes_{\mathbb{F}_1} \mathbb{Z} \subset Ker(\phi_a^{\mathbb{Z}}) \; \forall a \in A.$$

M being type α is equivalent to the requirement that it be an isomorphism $\forall a \in A$.

The proof of the following proposition is straightforward:

Proposition 2.2.1. *Let* A *be monoid.*

(1) *If M is an A-module of type α and $N \subset M$ is a submodule, then both N and M/N are type α.*
(2) *If M_1, M_2 are A-modules of type α, then so are $M_1 \oplus M_2$ and $M_1 \wedge M_2$.*
(3) *If A is commutative, and M_1, M_2 are A-modules of type α, then so is $M_1 \otimes_A M_2$.*

Remark 2.2.3. In general, an extension of type α modules need not be type α. Consider the $\mathbb{F}_1\langle t \rangle$-module $M = \{0, a, b, c, d\}$, with t acting by

$$t(a) = t(b) = c, \quad t(c) = d, \quad t(d) = 0,$$

and the sub-module $N = \{0, a, c, d\}$. Then the short exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$$

is admissible (i.e. all morphisms are normal), and $N, M/N$ are type α, but M is not.

Definition 2.2.4. Let A-mod$_n^\alpha$ denote the full subcategory of A-mod$_n$ consisting of type α A-modules and normal morphisms.

We thus have a chain of sub-categories

$$\text{A-mod}_n^\alpha \subset \text{A-mod}_n \subset \text{A-mod}$$

2.2.2. *Graded modules and modules admitting a grading.* Let Γ is a commutative monoid written additively, and A a Γ-graded monoid. By this we mean that A can be written as

(4)
$$A = \bigoplus_{\gamma \in \Gamma} A_\gamma$$

as $\mathbb{F}_1$ vector spaces, with $1_A, 0_A \in A_0$, and $A_\gamma \cdot A_\delta \subset A_{\gamma+\delta}$.

Definition 2.2.5. A *graded* A-module is an A-module M with a decomposition

(5)
$$M = \bigoplus_{\gamma \in \Gamma} M_\gamma$$

such that $0 \in M_0$ and $A_\gamma \cdot M_\delta \subset M_{\gamma+\delta}$.

We note that a sub-module of a graded module is automatically homogenous. The following lemma is obvious:

Lemma 2.2.6. *Let A be a Γ-graded monoid.*

(1) *If M is a graded A-module and $N \subset M$ is a submodule, then both N and M/N inherit a canonical grading.*
(2) *If M_1, M_2 are graded A-modules, then so are $M_1 \oplus M_2$ and $M_1 \wedge M_2$.*
(3) *If A is commutative, and M_1, M_2 are graded, then $M_1 \otimes_A M_2$ carries a canonical grading with $deg(m_1 \otimes m_2) = deg(m_1) + deg(m_2)$.*

Given a graded A-module M and $\rho \in \Gamma$, we denote by $M[\rho]$ the graded module with $M[\rho]_\gamma := M_{\gamma+\rho}$. We say that a module M *admits a grading* if it has the structure (5) for some grading. We denote by A-mod$_n^{gr}$ the full sub-category of A-mod$_n$ of modules admitting a grading. We do not a priori require the morphisms to be compatible with these. By 2.2.6, every morphism in A-mod$_n^{gr}$ has a kernel and co-kernel. We have a chain of inclusions

$$\text{A-mod}_n^{gr} \subset \text{A-mod}_n \subset \text{A-mod}.$$

3. Schemes and coherent sheaves over $\mathbb{F}_1$

In this section, we briefly review the notions of monoid schemes and coherent sheaves on them. These form the simplest version of a theory of "algebraic geometry over $\mathbb{F}_1$", which allows for a geometric interpretation of the main results of the paper. In the interest of brevity, we do not aim for this section to be self-contained, and refer the interested reader to [4, 6, 23] for details.

Just as an ordinary scheme is obtained by gluing prime spectra of commutative rings, a monoid scheme is obtained by gluing prime spectra of commutative monoids. Given a commutative monoid A, an *ideal* of A is a subset $\mathfrak{a} \subset A$ such that $\mathfrak{a} \cdot A \subset \mathfrak{a}$. A proper ideal $\mathfrak{p} \subset A$ is *prime* if $xy \in \mathfrak{p}$ implies either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. One defines the topological space MSpec A to be the set

$$\text{MSpec}\,A := \{\mathfrak{p} | \mathfrak{p} \subset A \text{ is a prime ideal }\},$$

with the closed sets of the form

$$V(\mathfrak{a}) := \{\mathfrak{p} | \mathfrak{a} \subset \mathfrak{p}, \mathfrak{p} \text{ prime }\},$$

for ideals $\mathfrak{a} \in A$, together with the empty set.

Given a multiplicatively closed subset $S \subset A$, the *localization of A by S*, denoted $S^{-1}A$, is defined to be the monoid consisting of symbols $\{\frac{a}{s} | a \in A, s \in S\}$, with the equivalence relation

$$\frac{a}{s} = \frac{a'}{s'} \iff \exists\, s'' \in S \text{ such that } as's'' = a'ss'',$$

and multiplication is given by $\frac{a}{s} \times \frac{a'}{s'} = \frac{aa'}{ss'}$.

For $f \in A$, let S_f denote the multiplicatively closed subset $\{1, f, f^2, f^3, \cdots, \}$. We denote by A_f the localization $S_f^{-1}A$, and by $D(f)$ the open set MSpec $A \setminus V(f) \simeq$ MSpec A_f, where $V(f) := \{\mathfrak{p} \in \text{MSpec}\,A | f \in \mathfrak{p}\}$. The open sets $D(f)$ cover MSpec A. MSpec A is equipped with a *structure sheaf* of monoids $\mathcal{O}_A$, satisfying the property $\Gamma(D(f), \mathcal{O}_A) = A_f$. Its stalk at $\mathfrak{p} \in \text{MSpec}\,A$ is $A_\mathfrak{p} := S_\mathfrak{p}^{-1}A$, where $S_\mathfrak{p} = A \setminus \mathfrak{p}$. Just as in the classical setting, the category of affine monoid schemes is equivalent to the opposite of the category of commutative monoids. From here, one can proceed to construct general monoid schemes by gluing the affines (MSpec $A, \mathcal{O}_A$).

One develops the theory of quasicoherent sheaves on monoid schemes just as in the classical case, and obtains an equivalence between the category of A-modules and the category of quasicoherent sheaves on MSpec A. An A-module M gives rise to a quasicoherent sheaf $\widetilde{M}$ on MSpec A whose space of sections over a basic open affine $D(f)$ is

$$\widetilde{M}(D(f)) := M_f$$

(where M_f is defined in the obvious way), and all quasicoherent sheaves arise this way. For an element $m \in M$, define

$$\mathrm{Ann}_A(m) := \{a \in A \,|\, a \cdot m = 0_M\}.$$

Obviously, $0_A \subset \mathrm{Ann}_A(m) \; \forall m \in M$. Define

$$\mathrm{Ann}_A(M) = \bigcap_{m \in M} \mathrm{Ann}_A(m).$$

$\mathrm{Ann}_A(M)$ is an ideal in A, and when M is finitely generated defines the scheme-theoretic support of M, i.e.:

$$\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{MSpec\,A} \,|\, M_{\mathfrak{p}} \neq 0\} = V(\mathrm{Ann}_A(M)).$$

We will need the following result, which is proved the same way as for commutative rings and their modules

Lemma 3.0.1. *Suppose that* A *is a commutative monoid, and that*

$$0 \to M \to L \to N \to 0$$

is an admissible exact sequence. Then $\mathrm{Supp}(L) = \mathrm{Supp}(M) \cup \mathrm{Supp}(N)$

This shows that A-modules with support contained in a closed subset $Z \subset$ MSpec A are closed under admissible extensions.

4. THE HALL ALGEBRA OF A-mod_n AND ITS QUOTIENTS

Let A be a finitely generated monoid. We begin this section by reviewing the construction of the Hall algebra H_A of the category A-mod_n following [29]. We then discuss Hopf ideals in H_A corresponding to the conditions introduced in Section 2.2 and prescribed support in Section 3. For more on Hall algebras in the classical setting of Abelian categories, we refer the reader to [24].

As a vector space:

$$H_A := \{f : \mathrm{Iso}(A\text{-mod}_n) \to \mathbb{Q} \mid \#\,\mathrm{supp}(f) < \infty\}.$$

We equip H_A with the convolution product

$$f \star g(\overline{M}) = \sum_{N \subset M} f(\overline{M/N}) g(\overline{N}),$$

where the sum is over all A sub-modules N of M (in what follows, it is conceptually helpful to fix a representative of each isomorphism class). Note that Lemma 2.1.1 and the finiteness of the support of f, g ensures that the sum in (4) is finite, and that $f \star g$ is again finitely supported.

H_A is spanned by δ-functions $\delta_{\overline{M}} \in H_A$ supported on individual isomorphism classes, and so it is useful to make explicit the multiplication of two such elements. We have

(6)
$$\delta_{\overline{M}} \star \delta_{\overline{N}} = \sum_{\overline{R} \in \mathrm{Iso}(A\text{-mod}_n)} \mathbf{P}^R_{M,N} \delta_{\overline{R}}$$

where

$$\mathbf{P}^R_{M,N} := \#|\{L \subset R, L \simeq N, R/L \simeq M\}|$$

The product

$$\mathbf{P}^R_{M,N} |\,\mathrm{Aut}(M)\| \,\mathrm{Aut}(N)|$$

counts the isomorphism classes of admissible short exact sequences of the form

(7)
$$0 \to N \to R \to M \to 0,$$

where $\mathrm{Aut}(M)$ is the automorphism group of M.

H_A is equipped with a coproduct

$$\Delta : H_A \to H_A \otimes H_A$$

given by

(8)
$$\Delta(f)(\overline{M}, \overline{N}) := f(\overline{M \oplus N}).$$

which is clearly co-commutative, as well as a natural grading by $\mathbb{Z}_{\geq 0}$ corresponding to the dimension of $M \in A\text{-mod}_n$, i.e. if $dim(M) = n$, then $deg(\delta_{\overline{M}}) = n$.

Remark 4.0.1. H_A also carries a grading by $K_0(A\text{-mod}_n)$ with $deg(\delta_{\overline{M}}) = [M] \in K_0(A\text{-mod}_n)$

The following theorem is proved in [29]:

Theorem 4.0.2. $(H_A, \star, \Delta)$ *is a graded, connected, co-commutative bialgebra. It is called the* Hall algebra of A–modules.

By the Milnor-Moore theorem, H_A is a Hopf algebra isomorphic to $\mathbb{U}(\mathfrak{n}_A)$ - the universal enveloping algebra of $\mathfrak{n}_A$, where the latter is the Lie algebra of its primitive elements. The definition of the co-product implies that $\mathfrak{n}_A$ is spanned by $\delta_{\overline{M}}$ for isomorphism classes $\overline{M}$ of indecomposable A–modules, with bracket

$$[\delta_{\overline{M}}, \delta_{\overline{N}}] = \delta_{\overline{M}} \star \delta_{\overline{N}} - \delta_{\overline{N}} \star \delta_{\overline{M}}.$$

4.1. **Ideals and quotients of Hall algebras.** The product (6) in H_A counts all admissible A-module extensions of M by N. We now proceed to define Hopf ideals in H_A yielding quotients which may be interpreted as Hall algebras of certain sub-categories of A-mod$_n$.

4.1.1. H_A^α. Let
$$\mathcal{J}^\alpha = \mathrm{span}\{\delta_{\overline{M}}| M \text{ is not of type } \alpha\} \subset H_A .$$

Theorem 4.1.1. $\mathcal{J}^\alpha \subset H_A$ *is a Hopf ideal.*

Proof. $\mathcal{J}^\alpha$ is a graded subspace contained in the augmentation ideal of H_A. By the Milnor-Moore theorem, It suffices to show that $\mathcal{J}^\alpha$ is a bialgebra ideal, since then $H_A / \mathcal{J}^\alpha$ will again be a graded, connected, co-commutative bialgebra. It follows from part (1) of Proposition 2.2.1 that $\mathcal{J}^\alpha$ is two-sided ideal for the product $\star$. To show that it is a co-ideal, it suffices to show that if M is not of type α, then

(9) $$\Delta(\delta_{\overline{M}}) \subset H_A \otimes \mathcal{J}^\alpha + \mathcal{J}^\alpha \otimes H_A.$$

From the definition of the co-product (8), $\Delta(\delta_{\overline{M}})$ is a linear combination of terms of the form $\delta_{\overline{M'}} \otimes \delta_{\overline{M''}}$ where $M = M' \oplus M''$. It follows from part (2) of Proposition 2.2.1 that $\delta_{\overline{M'}} \in \mathcal{J}^\alpha$ or $\delta_{\overline{M''}} \in \mathcal{J}^\alpha$. $\square$

Definition 4.1.2. The *Hall algebra of* A–*modules of type* α is the Hopf algebra
$$H_A^\alpha := H_A / \mathcal{J}^\alpha.$$

It is immediate that $H_A^\alpha \simeq \mathbb{U}(\mathfrak{n}_A^\alpha)$, where the Lie algebra
$$\mathfrak{n}_A^\alpha = \mathrm{span}\{\delta_{\overline{M}}| M \text{ is indecomposable of type } \alpha\},$$

and where

$$(10) \qquad \delta_{\overline{M}} \star \delta_{\overline{N}} = \sum_{\overline{R} \in \mathrm{Iso}(\text{A-mod}_n^\alpha)} \mathbf{P}_{M,N}^R \delta_{\overline{R}}$$

4.1.2. H_A^{gr}. Suppose that A is a Γ-graded monoid as in Section 2.2.2. Let

$$\mathcal{J}^{gr} = \mathrm{span}\{\delta_{\overline{M}} |\ M \text{ does not admit a grading}\} \subset \mathrm{H}_A$$

Theorem 4.1.3. $\mathcal{J}^{gr} \subset \mathrm{H}_A$ *is a Hopf ideal.*

Proof. The proof is identical to that of Theorem 4.1.1 except Lemma 2.2.6 is used rather than Proposition 2.2.1 $\hfill\square$

Definition 4.1.4. The *Hall algebra of modules admitting a grading* is the Hopf algebra

$$\mathrm{H}_A^{gr} := \mathrm{H}_A / \mathcal{J}^{gr}.$$

We have an isomorphism $\mathrm{H}_A^{gr} \simeq \mathbb{U}(\mathfrak{n}_A^{gr})$ where the Lie algebra

$$\mathfrak{n}_A^{gr} = \mathrm{span}\{\delta_{\overline{M}} | M \text{ is indecomposable and admits a grading }\},$$

and where

$$(11) \qquad \delta_{\overline{M}} \star \delta_{\overline{N}} = \sum_{\overline{R} \in \mathrm{Iso}(\text{A-mod}_n^{gr})} \mathbf{P}_{M,N}^R \delta_{\overline{R}}$$

Remark 4.1.5. If A is a graded monoid, we may combine the two previous constructions, as $\mathcal{J}^\alpha + \mathcal{J}^{gr}$ is also a Hopf ideal, to obtain the quotient

$$\mathrm{H}_A^{\alpha,gr} := \mathrm{H}_A / (\mathcal{J}^\alpha + \mathcal{J}^{gr})$$

We then have the following diagram of Hopf algebras

$$(12) \qquad \begin{array}{ccc} \mathrm{H}_A & \longrightarrow & \mathrm{H}_A^\alpha \\ \downarrow & & \downarrow \\ \mathrm{H}_A^{gr} & \longrightarrow & \mathrm{H}_A^{\alpha,gr} \end{array}$$

4.2. **Hall algebras of modules with prescribed support.** Suppose now that A is a finitely generated commutative monoid, and let $Z \subset \text{MSpec } A$ be a closed subset. By Lemma 3.0.1, the A-modules with support contained in Z form a sub-category A-mod$_{Z,n} \subset$ A-mod$_n$ closed under extensions. Likewise, we obtain a sub-category A-mod$_{Z,n}^{\alpha} \subset$ A-mod$_n^{\alpha}$ by restricting to modules of type α with support in Z. Let

$$\text{(13)} \qquad H_{Z,A} := \text{span}\{\delta_{\overline{M}} \in H_A \,|\, \text{supp}(M) \subset Z\}$$

$$\text{(14)} \qquad H_{Z,A}^{\alpha} := \text{span}\{\delta_{\overline{M}} \in H_A^{\alpha} \,|\, \text{supp}(M) \subset Z\}$$

If A is graded, we can also form

$$\text{(15)} \qquad H_{Z,A}^{gr} := \text{span}\{\delta_{\overline{M}} \in H_A^{gr} \,|\, \text{supp}(M) \subset Z\}$$

as well as $H_{Z,A}^{\alpha,gr}$ etc. The following is immediate:

Proposition 4.2.1. *Let A be a finitely generated commutative monoid, and $Z \subset \text{MSpec } A$ a closed subset. Then $H_{Z,A}, H_{Z,A}^{\alpha}, H_{Z,A}^{gr}$, $H_{Z,A}^{\alpha,gr}$ defined as in 13, 14, 15 form graded, connected, co-commutative Hopf subalgebras of H_A, H_A^{α}, $H_{Z,A}^{gr}$, $H_A^{\alpha,gr}$ respectively.*

Example 4. Let $A = \mathbb{F}_1\langle t \rangle$. An A-module M may be described in terms of a directed graph with an edge from m to $t \cdot m$ if $t \cdot m \neq 0$. The possible directed graphs arising this way from indecomposable A–modules were classified in [29]. These are either rooted trees (Figure 1), or a cycle with rooted trees attached (Figure 2). We will refer to these as type 1 and 2 respectively. If M is of type 1, then $\text{Ann}_A(M) = (t)$ (these modules are nilpotent), so the corresponding coherent sheaf is supported at the origin, whereas if M is of type 2, $\text{Ann}_A(M) = 0$, so the support is $\mathbb{A}^1_{/\mathbb{F}_1}$.

(1) The indecomposable modules of type α are those whose graphs have the property that every vertex has at most one incoming edge. These are the vertical ladders (type 1), and the directed polygons (type 2).

(2) If we give A the grading where t has degree 1, only type 1 modules admit a grading.

As shown in [28,29] the corresponding Hall algebras can be described as follows

(1) We have that $H_{0,A} = H_A^{gr} = H_{0,A}^{gr}$. This Hopf algebra is dual to the Connes-Kreimer Hopf algebra of rooted forests.

(2) H_A^*, the Hopf dual of the full Hall algebra, is an extension of the Connes-Kreimer algebra by cycles of type 2.

(3) $H_{0,A}^\alpha = H_A^{\alpha,gr} = H_{0,A}^{\alpha,gr}$ is the Hopf algebra of symmetric functions, with the n-vertex ladder corresponding to the n-th power sum.
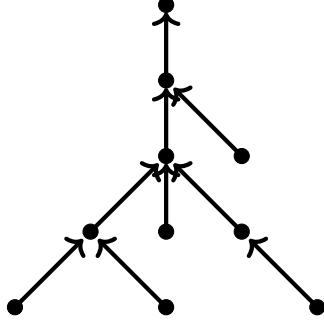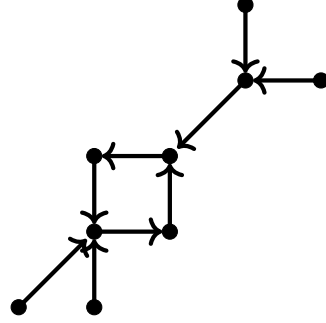

Figure 1   Figure 2

5. Skew shapes and torsion sheaves on $\mathbb{A}^n_{/\mathbb{F}_1}$

In this section, we will be exclusively focussed on the monoid $\mathbb{F}_1\langle x_1, \cdots, x_n \rangle$. It is naturally graded by $\mathbb{Z}^n_{\geq 0}$, with $deg(x_i) = e_i$, where e_i denotes the ith standard basis vector in $\mathbb{Z}^n$. We have

$$\mathbb{A}^n_{/\mathbb{F}_1} = \mathrm{MSpec}\, \mathbb{F}_1\langle x_1, \cdots, x_n \rangle.$$

5.1. **n-dimensional skew shapes.** We begin by introducing a natural partial order on $\mathbb{Z}^n$, where for

$$x = (x_1, \cdots, x_n) \in \mathbb{Z}^n \text{ and } y = (y_1, \cdots, y_n) \in \mathbb{Z}^n,$$

$$x \leq y \iff x_i \leq y_i \text{ for } i = 1, \cdots, n.$$

Definition 5.1.1. An *n-dimensional skew shape* is a finite convex sub-poset $S \subset \mathbb{Z}^n$. S is *connected* iff the corresponding poset is. We consider two skew shapes S, S' to be equivalent iff S' is a translation of S, i.e. if there exists $a \in \mathbb{Z}^n$ such that $S' = a + S$.

The condition that S is connected is easily seen to be equivalent to the condition that any two elements of S can be connected via a lattice path lying in S. The name *skew shape* is motivated by the fact that for $n = 2$, a connected skew shape

in the above sense corresponds (non-uniquely) to a difference $\lambda \backslash \mu$ of two Young diagrams in French notation.

Example 5. Let $n = 2$, and

$$S \subset \mathbb{Z}^2 = \{(1,0), (2,0), (3,0), (0,1), (1,1), (0,2)\}$$

(up to translation by $a \in \mathbb{Z}^2$). Then S corresponds to the connected skew Young diagram
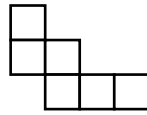
5.2. **Skew shapes as modules.** Let $S \subset \mathbb{Z}^n$ be a skew shape. We may attach to S a $\mathbb{F}_1\langle x_1, \cdots, x_n\rangle$-module M_S with underlying set

$$M_S = S \sqcup \{0\},$$

and action of $\mathbb{F}_1\langle x_1, \cdots, x_n\rangle$ defined by

$$x^e \cdot s = \begin{cases} s + e, & \text{if } s + e \in S \\ 0 & \text{otherwise} \end{cases}$$

$$e \in \mathbb{Z}^n_{\geq 0}, s \in S.$$

It is clear that M_S is a $\mathbb{F}_1\langle x_1, \cdots, x_n\rangle$-module of type α, and that $\mathfrak{m}^k \cdot M_S = 0$ for k sufficiently large, where $\mathfrak{m} = (x_1, \cdots, x_n)$ is the maximal ideal.

Example 6. Let S as in Example 5. x_1 (resp. x_2) act on the $\mathbb{F}_1\langle x_1, x_2\rangle$-module M_S by moving one box to the right (resp. one box up) until reaching the edge of the diagram, and 0 beyond that. A minimal set of generators for M_S is indicated by the black dots:
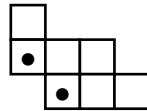
The proof of the following lemma is straightforward:

Lemma 5.2.1. *Let $S \subset \mathbb{Z}^n$ be a skew shape, and M_S the corresponding $\mathbb{F}_1\langle x_1, \cdots, x_n\rangle$-module. Then the following hold:*

(1) $\operatorname{Supp} M_S = 0 \subset \mathbb{A}^n$.

(2) M_S admits a grading, unique up to translation by $\mathbb{Z}^n$.

(3) M_S is indecomposable iff S is connected.

(4) The decomposition of S into connected components

$$S = S_1 \sqcup S_2 \cdots \sqcup S_k$$

corresponds to the decomposition of M_S into indecomposable factors

$$M_S = M_{S_1} \oplus M_{S_2} \oplus \cdots \oplus M_{S_k}.$$

(5) The minimal elements of S form a unique minimal set of generators for M_S.

(6) M_S is cyclic (i.e. generated by a single element) iff S corresponds to an ordinary n-dimensional Young diagram.

(7) If $T \subset S$ is a sub-poset, then M_T is a sub-module of M_S iff $S \backslash T$ is an order ideal.

(8) If $M_T \subset M_S$ is a sub-module corresponding to the sub-poset $T \subset S$, then the quotient $M_S / M_T \simeq M_{S \backslash T}$.

Remark 5.2.2. It follows from part (6) of Lemma 5.2.1 that sub-modules of M_S correspond to (not necessarily connected) skew shapes aligned along the outer edge of S.

Remark 5.2.3. Cyclic $\mathbb{F}_1\langle x_1, \cdots, x_n \rangle$-modules can be thought of as the Hilbert Scheme of points on $\mathbb{A}^n / \mathbb{F}_1$, and correspond to ordinary n-dimensional Young diagrams. This is consistent with the fact that these parametrize the torus fixed-points of $(\mathbb{C}^*)^n$ on the Hilbert scheme of points on $\mathbb{A}^n / \mathbb{C}$, which correspond to monomial ideals in $\mathbb{C}[x_1, \cdots, x_n]$.

Example 7. Let S be the skew shape below, with $T \subset S$ corresponding to the sub-shape whose boxes contain t's.
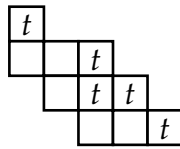
We have

$$M_T = \square \oplus \square \oplus \begin{array}{c}\square \\ \square\,\square\end{array}$$

and

$$M_S / M_T = \begin{array}{c}\square\,\square \\ \square\end{array} \oplus \square\,\square$$

5.3. Tableaux and filtrations.

When $n = 2$, standard and semi-standard skew tableaux may be interpreted as filtrations on $\mathbb{F}_1\langle x_1, x_2 \rangle$-modules.

Definition 5.3.1. let S be a skew shape with m boxes.

(1) A *standard tableau* with shape S is a filling of S with $1, 2, \cdots, m$ such that the entries are strictly increasing left to right in each row and bottom to top in each column.

(2) A *semistandard tableau* with shape S is a filling of S with positive integers such that the entries are weakly increasing left to right in each row and strictly increasing bottom to top in each column.

For $s \in S$, we denote by $f(s) \in \mathbb{Z}$ the integer assigned to that box, and refer to a tableau by the pair (S, f). Let $S_{\geq k} = \{s \in S | f(s) \geq k\}$. When S is standard or semi-standard, each $S_{\geq k}$ is easily seen to be a skew sub-shape corresponding to a sub-module $M_{S_{\geq k}} \subset M_S$. We have a (necessarily finite) filtration

$$\cdots M_{S_{\geq r}} \subset M_{S_{\geq r-1}} \subset M_{S_{\geq r-2}} \subset \cdots$$

The requirement that (S, f) be standard is easily seen to be equivalent to the requirement that

$$M_{S_r} / M_{S_{r-1}} \simeq \square$$

whereas semi-standard (S, f) correspond to filtrations with

$$M_{S_{\geq r-1}} / M_{S_{\geq r}} \simeq \boxed{\square\square\cdots\cdots\square} \oplus \boxed{\square\square\cdots\cdots\square} \oplus \cdots$$

i.e. the associated graded consists of direct sums of modules with underlying shapes horizontal strips. Geometrically, these correspond to coherent sheaves supported on the x_1-axis in $\mathbb{A}^2_{/\mathbb{F}_1}$.

5.4. Type α modules admitting a $\mathbb{Z}^n$ grading are connected skew shapes.

In this section we show that indecomposable type α modules over $\mathbb{F}_1\langle x_1, \cdots, x_n \rangle$ admitting a $\mathbb{Z}^n$ grading and supported at $0 \subset \mathbb{A}^n_{/\mathbb{F}_1}$ arise from connected skew shapes. We begin with a lemma:

Lemma 5.4.1. *Let M be a finite $\mathbb{F}_1\langle x_1, \cdots, x_n \rangle$-module supported at $0 \subset \mathbb{A}^n_{/\mathbb{F}_1}$. Then M has a finite filtration*

$$M \supset \mathfrak{m} \cdot M \supset \mathfrak{m}^2 \cdot M \supset \cdots \supset \mathfrak{m}^r \cdot M = 0_M$$

where $r \leq \dim(M)$.

Proof. The hypothesis implies that the radical of $\text{Ann}(M)$ is $\mathfrak{m}$. Since $\mathfrak{m}$ is finitely generated, we have that $\mathfrak{m}^s \subset \text{Ann}(M)$ for some $s \in \mathbb{N}$, which implies that $\mathfrak{m}^s \cdot M = 0_M$ for sufficiently large s. The bound on r follows from the fact that the filtration is strictly decreasing. $\qquad\square$

Theorem 5.4.2. *Let M be a finite, $\mathbb{Z}^n$-graded, indecomposable, type α $\mathbb{F}_1\langle x_1, \cdots, x_n \rangle$-module such that $\text{Supp}\, M = 0$. Then $M \simeq M_S$ for a connected skew shape $S \subset \mathbb{Z}^n$.*

Proof. We proceed by induction on $d = \dim(M)$. The result is obvious for $d = 0, 1$. Suppose now that it holds for $d < p = \dim(M)$. Write the decomposition into graded components as

$$M = \bigoplus_{d \in \mathbb{Z}^n} M_d.$$

By Lemma 5.4.1, we can find $v \in M$ such that $\mathfrak{m} \cdot v = 0$. This implies that v generates a one-dimensional $\mathbb{F}_1\langle x_1, \cdots, x_n \rangle$-submodule $\langle v \rangle$ of M. Denote by

$$\pi : M \to M/\langle v \rangle$$

the quotient map, and let

(16) $$M/\langle v \rangle = M_1 \oplus M_2 \oplus \cdots \oplus M_k$$

be the decomposition of $M/\langle v \rangle$ into non-zero indecomposable $\mathbb{F}_1\langle x_1, \cdots, x_n \rangle$-modules. For each i, let $\widetilde{M}_i = \pi^{-1}(M_i)$. We have $M = \cup_i \widetilde{M}_i$, $\widetilde{M}_i \cap \widetilde{M}_j = \{0, v\}$ for $i \neq j$, and $\widetilde{M}_i/\langle v \rangle = M_i$. π yields a bijection between M_i and the subset $\sigma(M_i) := \widetilde{M}_i \backslash \{v\} \subset M$. By part (1) of Lemma 2.2.6, each M_i is graded and satisfies condition α, so by the induction hypothesis, $M_i \simeq M_{S_i}$, $i = 1, \cdots, k$ for connected skew shapes $S_i \subset \mathbb{Z}^n$.

We make the following two observations:

(1) $v \in \mathfrak{m} \cdot \sigma(M_i)$ for each i. If not, and say $v \notin \mathfrak{m} \cdot \sigma(M_1)$, then $\sigma(M_1)$ is a proper direct summand of M, contradicting the hypothesis that M is indecomposable.

(2) By property α, there is at most one $v_i \in M_{deg(v)-e_i}$ such that $x_i \cdot v_i = v$, and since M is graded, the v_i's are pairwise distinct.

These observations combined show that $k \leq n$, and that each $\widetilde{M}_i$ contains a non-empty set $\{v_{i,j_1}, \cdots, v_{i,j_r}\}$ of elements such that $x_{j_p} \cdot v_{i,j_p} = v$.

We consider the case $n = 2$. There are two possibilities.

(1) $M/\langle v \rangle$ is indecomposable (i.e. $k = 1$). Let $S = S_1 \sqcup deg(v) \subset \mathbb{Z}^2$. There are three possibilities:

(a) $deg(v) - e_1 \in S$ but $deg(v) - e_2 \notin S$. This means that $deg(v)$ is located on the right end of the bottom row of S.

(b) $deg(v) - e_2 \in S$ but $deg(v) - e_1 \notin S$. This means that $deg(v)$ is located at the top of the leftmost column of S.

(c) $deg(v) - e_1 \in S$ and $deg(v) - e_2 \in S$. This means that $deg(v)$ is located at outer boundary of S with boxes immediately to the left and below $deg(v)$. Since S_1 was assumed to be a connected skew shape, $deg(v) - e_1 - e_2 \in S$ as well.

In all three cases, S is a connected skew shape, and it is clear that $M \simeq M_S$.

(2) $M/\langle v \rangle \simeq M_1 \oplus M_2$ (i.e. $k = 2$). Let $S = S_1 \sqcup S_2 \sqcup deg(v)$. We may assume, after switching the labels M_1 and M_2 if necessary, that there exist $v_1 \in M_{S_1}, v_2 \in M_{S_2}$ (necessarily unique) such that $x_1 \cdot v_1 = v$, $x_2 \cdot v_2 = v$. This means that $deg(v) - e_1 \in S_1, deg(v) - e_2 \in S_2$. $deg(v)$ is therefore located at the right end of the bottom row of S_1 and the top of the leftmost column of S_2. S is therefore a connected skew shape (since S_1, S_2 are assumed connected skew), and $M \simeq M_S$.

The case of general n can be dealt with using the same approach.

$\square$

5.5. **Action of S_n.** S_n - the symmetric group on n letters, acts by automorphisms on $A = \mathbb{F}_1\langle x_1, \cdots, x_n \rangle$ by $\sigma \cdot x_i = x_{\sigma(i)}$, $\sigma \in S_n$. It is easily seen to be the full automorphism group of A, or equivalently, of $\mathbb{A}^n_{/\mathbb{F}_1}$. This action induces an action on the categories A-mod$_n$, A-mod$_n^\alpha$, and A-mod$_n^{gr}$ sending a module M to the module M^σ with the same underlying pointed set, and where A acts on M^σ by

(17) $$a \cdot m := \sigma(a) \cdot m, \ m \in M^\sigma$$

This action preserves the property of a module being indecomposable, and so by Theorem 5.4.2, induces an action on skew shapes, which is easily seen to coincide with the "classical" action. For $n = 2$, this amounts to transposition of the skew shape.

Example 8. Let $n = 2$, and $\sigma = (1\ 2) \in S_2$ the only non-identity element. If

$$S = \ \begin{array}{c}\square \\ \square\square\square\end{array}$$

then

$$S^\sigma = \ \begin{array}{c}\square \\ \square \\ \square\square\end{array}$$

6. Hopf and Lie structures on skew shapes

Let $A = \mathbb{F}_1\langle x_1, \cdots, x_n\rangle$, and

$$Z = 0 = V((x_1, \cdots, x_n)) \subset \mathbb{A}^n_{/\mathbb{F}_1} = \mathrm{MSpec}\,\mathbb{F}_1\langle x_1, \cdots, x_n\rangle$$

the origin. By Proposition 4.2.1, the Hall algebra $H_{0,A}^{\alpha,gr}$ has the structure of a graded connected co-commutative Hopf algebra, which we denote by $\mathbf{SK}_n$. By the Milnor-Moore theorem,

$$\mathbf{SK}_n \simeq \mathbb{U}(\mathfrak{sk}_n)$$

where the Lie algebra

$$\mathfrak{sk}_n = \mathrm{span}\{\delta_{\overline{M}} | M \text{ type } \alpha, \text{ indecomposable, admits a grading, Supp}\,M = 0\}.$$

By Theorem 5.4.2, $\mathfrak{sk}_n$ has a basis which can be identified with the set of connected n-dimensional skew shapes. The S_n action on skew shapes 17 is easily seen to induce an action on $\mathbf{SK}_n$ by Hopf algebra automorphisms. We thus have

Theorem 6.0.1. *The Hall algebra* $\mathbf{SK}_n = H_{0,A}^{\alpha,gr}$ *is isomorphic to the enveloping algebra* $\mathbb{U}(\mathfrak{sk}_n)$*. The Lie algebra* $\mathfrak{sk}_n$ *may be identified with*

$$\mathfrak{sk}_n = \{\delta_{M_S} | S \text{ a connected } n - \text{ dimensional skew shape}\}$$

with Lie bracket

$$[\delta_{\overline{M_S}}, \delta_{\overline{M_T}}] = \delta_{\overline{M_S}} \star \delta_{\overline{M_T}} - \delta_{\overline{M_T}} \star \delta_{\overline{M_S}}$$

The symmetric group S_n *acts on* $\mathbf{SK}_n$ *(resp.* $\mathfrak{sk}_n$*) by Hopf (resp. Lie) algebra automorphisms.*

We recall from Section 4 that the product in $\mathbf{SK}_n$ is

$$(18) \qquad\qquad \delta_{\overline{M_S}} \star \delta_{\overline{M_T}} = \sum_{R \text{ a skew shape}} \mathbf{P}_{S,T}^R \delta_{\overline{M_R}}$$

where

$$\mathbf{P}^R_{S,T} := \#|\{M_L \subset M_R, M_L \simeq M_T, M_R/M_L \simeq M_S\}|$$

Note that the skew shapes R being summed over need not be connected. In fact, when S and T are connected, the product (18) will contain precisely one disconnected skew shape $S \sqcup T$ corresponding to the split extension $M_S \oplus M_T$.

Example 9. Let $n = 2$. By abuse of notation, we identify the skew shape S with the delta-function $\delta_{\overline{M_S}} \in \mathcal{S}_2$. Let

$$S = \begin{array}{c}\square \\ \square\end{array} \qquad T = \square\,\square$$

We have

$$S \star T = \boxed{\begin{array}{l} s \\ \hline s\,s\,t\,t \end{array}} \;+\; \boxed{\begin{array}{l} t\,t \\ s \\ s\,s \end{array}} \;+\; \boxed{\begin{array}{l} s \\ s\,s \end{array}} \oplus \boxed{t\,t}$$

$$T \star S = \boxed{\begin{array}{l} s \\ s\,s \\ t\,t \end{array}} \;+\; \boxed{\begin{array}{l} s \\ s\,s \\ t\,t \end{array}} \;+\; \boxed{\begin{array}{l} t\,t\,s \\ s\,s \end{array}} \;+\; \boxed{\begin{array}{l} s \\ s\,s \end{array}} \oplus \boxed{t\,t}$$

and

$$[S,T] = \boxed{\begin{array}{l} s \\ s\,s\,t\,t \end{array}} \;+\; \boxed{\begin{array}{l} t\,t \\ s \\ s\,s \end{array}} \;-\; \boxed{\begin{array}{l} s \\ s\,s \\ t\,t \end{array}} \;-\; \boxed{\begin{array}{l} s \\ s\,s \\ t\,t \end{array}} \;-\; \boxed{\begin{array}{l} t\,t\,s \\ s\,s \end{array}}$$

where for each skew shape we have indicated which boxes correspond to S and T.

As the above example shows, for connected skew shapes S, T, the product $S \star T$ in the Hall algebra involves all ways of "stacking" the shape T onto that of S to achieve a skew shape, as well as one disconnected shape (which may be drawn in a number of ways).

Remark 6.0.2. it is easy to see that the structure constants of the Lie algebra $\mathfrak{sl}_n$ in the basis of skew shapes are all $-1, 0$, or 1.

7. Hopf algebra duals of Hall algebras

In this section A will denote an arbitrary finitely generated monoid. The Hall algebras constructed in this paper are all isomorphic to enveloping algebras, where the combinatorially interesting information resides in the product.

Dualizing, we obtain commutative (but in general not co-commutative) Hopf algebras whose co-product now records the interesting combinatorics. The underlying associative algebra of these duals is always a polynomial ring on a collection of indecomposable A-modules. We begin with the dual of H_A.

Let $H_A^* = \mathbb{Q}[x_{\overline{M}}]$, where $\overline{M}$ runs over the distinct isomorphism classes of indecomposable A-modules. As an associative algebra, H_A^* is therefore a polynomial ring. We adopt the convention that if $M \in$ A-mod is written in terms of indecomposable modules as $M = M_1 \oplus M_2 \cdots \oplus M_k$, then

$$x_{\overline{M}} := x_{\overline{M_1}} \cdot x_{\overline{M_2}} \cdots x_{\overline{M_{k-1}}} \cdot x_{\overline{M_k}}$$

Let

$$\Delta : H_A^* \to H_A^* \otimes H_A^*$$

be defined on $x_{\overline{M}}$, M indecomposable, by

(19)
$$\Delta(x_{\overline{M}}) = \sum_{N \subset M} x_{\overline{M/N}} \otimes x_{\overline{N}}$$

and extended by the requirement that it be an algebra morphism (i.e. multiplicative). Let

$$\langle, \rangle : H_A^* \otimes H_A \to \mathbb{Q}$$

denote the pairing defined by

$$\langle x_{\overline{M}}, \delta_{\overline{N}} \rangle = \begin{cases} 0 & M \not\simeq N \\ 1 & M \simeq N \end{cases}$$

One can readily verify that H_A^* with its commutative product and the co-product Δ is a graded connected commutative bialgebra, hence a Hopf algebra, and that $\langle, \rangle$ is a Hopf pairing.

If A is graded, we may dualize the diagram (12) to obtain:

(20)
$$\begin{array}{ccc} H_A^* & \longleftarrow & (H_A^\alpha)^* \\ \uparrow & & \uparrow \\ (H_A^{gr})^* & \longleftarrow & (H_A^{\alpha,gr})^* \end{array}$$

Each of the Hopf duals here may be given a description analogous to H_A^*. For instance, $(H_A^\alpha)_* = \mathbb{Q}[x_{\overline{M}}]$, where $\overline{M}$ runs over the isomorphism classes of indecomposable type α modules. The formula (19) remains the same. When A is commutative, and $Z \subset \mathrm{MSpec}\, A$ is a closed subset, we may incorporate support conditions in the obvious way.

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