INCIDENCE CATEGORIES

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ABSTRACT. Given a family $\mathcal{F}$ of posets closed under disjoint unions and the operation of taking convex subposets, we construct a category $\mathcal{C}_{\mathcal{F}}$ called the *incidence category of* $\mathcal{F}$. This category is "nearly abelian" in the sense that all morphisms have kernels/cokernels, and possesses a symmetric monoidal structure akin to direct sum. The Ringel-Hall algebra of $\mathcal{C}_{\mathcal{F}}$ is isomorphic to the incidence Hopf algebra of the collection $\mathcal{P}(\mathcal{F})$ of order ideals of posets in $\mathcal{F}$. This construction generalizes the categories introduced by K. Kremnizer and the author In the case when $\mathcal{F}$ is the collection of posets coming from rooted forests or Feynman graphs.

1. INTRODUCTION

The notion of the *incidence algebra* of an interval-closed family $\mathcal{P}$ of posets was introduced by G.-C. Rota in [9]. The work of W. Schmitt demonstrated that incidence algebras frequently possess important additional structure - namely that of a Hopf algebra. The seminal papers [11, 12] established various key structural and combinatorial properties of incidence Hopf algebras.

In this paper, we show that a certain class of incidence Hopf algebras can be "categorified". Given a family $\mathcal{F}$ of posets which are closed under disjoint unions and the operation of taking convex subposets, we construct a category $\mathcal{C}_{\mathcal{F}}$, whose objects are in one-to-one correspondence with the posets in $\mathcal{F}$. While $\mathcal{C}_{\mathcal{F}}$ is not abelian (morphisms only form a set), it is "nearly" so, in the sense that all morphisms possess kernels and cokernels, it has a null object, and symmetric monoidal structure akin to direct sum. We can therefore talk about exact sequences in $\mathcal{C}_{\mathcal{F}}$, its Grothendieck group, and Yoneda Ext's.

In particular, we can define the Ringel-Hall algebra $\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$ of $\mathcal{C}_{\mathcal{F}}$. $\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$ is the $\mathbb{Q}$–vector space of finitely supported functions on isomorphism classes of $\mathcal{C}_{\mathcal{F}}$:

$$\mathrm{H}_{\mathcal{C}_{\mathcal{F}}} := \{f : \mathrm{Iso}(\mathcal{C}_{\mathcal{F}}) \rightarrow \mathbb{Q} \,|\, |supp(f)| < \infty\}$$

with product given by convolution:

$$f \star g(M) = \sum_{A \subset M} f(A) g(M/A). \tag{1}$$

The author is supported by an NSA grant.

$\mathrm{H}_{\mathcal{C}_\mathcal{F}}$ possesses a co-commutative co-product given by

$$\Delta(f)(M,N) = f(M \oplus N) \tag{2}$$

We show that $\mathrm{H}_{\mathcal{C}_\mathcal{F}}$ is isomorphic to the incidence Hopf algebra of the family $\mathcal{P}(\mathcal{F})$ of order ideals of posets in $\mathcal{F}$. This is the sense in which $\mathcal{C}_\mathcal{F}$ is a categorification. The resulting Hopf algebra $\mathrm{H}_{\mathcal{C}_\mathcal{F}}$ is graded connected and co-commutative, and so by the Milnor-Moore theorem isomorphic to $U(\mathfrak{n}_\mathcal{F})$, where $\mathfrak{n}_\mathcal{F}$ is the Lie algebra of its primitive elements.

When $\mathcal{F}$ is the family of posets coming from rooted forests or Feynman graphs, $\mathcal{C}_\mathcal{F}$ coincides with the categories introduced by K. Kremnizer and the author in [8].

Acknowledgements: I would like to thank Muriel Livernet for valuable conversations and in particular for suggesting that there ought to be an interpretation of the categories introduced in [8] in terms of incidence algebras. I would also like to thank Dirk Kreimer, Valerio Toledano Laredo, and Kobi Kremnizer for valuable conversations.

2. Recollections on posets

We begin by recalling some basic notions and terminology pertaining to posets (partially ordered sets) following [12, 13].

(1) An *interval* is a poset having unique minimal and maximal elements. For x, y in a poset P, we denote by $[x, y]$ the interval

$$[x, y] := \{z \in P : x \leq z \leq y\}$$

If P is an interval, we will often denote by 0_P and 1_P the minimal and maximal elements.

(2) An *order ideal* in a poset P is a subset $L \subset P$ such that whenever $y \in L$ and $x \leq y$ in P, then $x \in L$.

(3) A subposet Q of P is *convex* if, whenever $x \leq y$ in Q and $z \in P$ satisfies $x \leq z \leq y$, then $z \in Q$. Equivalently, Q is convex if $Q = L \backslash I$ for order ideals $I \subset L$ in P.

(4) Given two posets P_1, P_2, their disjoint union is naturally a poset, which we denote by $P_1 + P_2$. In $P_1 + P_2$, $x \leq y$ if both lie in either P_1 or P_2, and $x \leq y$ there.

(5) A poset which is not the union of two non-empty posets is said to be *connected*.

(6) The cartesian product $P_1 \times P_2$ is a poset where $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$.

(7) A *distributive lattice* is a poset P equipped with two operations $\wedge, \vee$ that satisfy the following properties:

   (a) $\wedge, \vee$ are commutative and associative

(b) $\wedge, \vee$ are idempotent - i.e. $x \wedge x = x$, $x \vee x = x$
(c) $x \wedge (x \vee y) = x = x \vee (x \wedge y)$
(d) $x \wedge y = x \iff x \vee y = y \iff x \leq y$
(e) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
(f) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(8) For a poset P, denote by J_P the poset of order ideals of P, ordered by inclusion. J_P forms a distributive lattice with $I_1 \vee I_2 := I_1 \cup I_2$ and $I_1 \wedge I_2 := I_1 \cap I_2$ for $I_1, I_2 \in J_P$. If P_1, P_2 are posets, we have $J_{P_1+P_2} = J_{P_1} \times J_{P_2}$, and if $I, L \in J_P$, and $I \subset L$, then $[I, L]$ is naturally isomorphic to the lattice of order ideals $J_{L \backslash I}$.

3. From posets to categories

Let $\mathcal{F}$ be a family of posets which is closed under the formation of disjoint unions and the operation of taking convex subposets, and let $\mathcal{P}(\mathcal{F}) = \{J_P : P \in \mathcal{F}\}$ be the corresponding family of distributive lattices of order ideals. For each pair $P_1, P_2 \in \mathcal{F}$, let $\mathrm{M}(P_1, P_2)$ denote a set of maps $P_1 \rightarrow P_2$ such that the collections $\mathrm{M}(P_1, P_2)$ satisfy the following properties:

(1) for each $f \in \mathrm{M}(P_1, P_2)$, $f : P_1 \rightarrow P_2$ is a poset isomorphism
(2) $M(P, P)$ contains the identity map
(3) If $f \in \mathrm{M}(P_1, P_2)$ then $f^{-1} \in \mathrm{M}(P_2, P_1)$.
(4) if $f \in \mathrm{M}(P_1, P_2)$ and $g \in \mathrm{M}(P_2, P_3)$, then $g \circ f \in \mathrm{M}(P_1, P_3)$
(5) If $f \in \mathrm{M}(P_1, P_2)$ and $g \in \mathrm{M}(Q_1, Q_2)$ then $f \cup g \in \mathrm{M}(P_1 + Q_1, P_2 + Q_2)$, where $f \cup g$ denotes the map induced on the disjoint union.

Typically, the collection $\mathrm{M}(P_1, P_2)$ will consist of poset isomorphisms respecting some additional structure (such as for instance a coloring). It follows from the above properties that $\mathrm{M}(P, P)$ forms a group, which we denote $\mathrm{Aut}_{\mathrm{M}}(P)$.

3.1. **The category $\mathcal{C}_{\mathcal{F}}$.** We proceed to define a category $\mathcal{C}_{\mathcal{F}}$, called the *incidence category of $\mathcal{F}$* as follows. Let

$$\mathrm{Ob}(\mathcal{C}_{\mathcal{F}}) := \mathcal{F} = \{X_P : P \in \mathcal{F}\}$$

and

$$\mathrm{Hom}(X_{P_1}, X_{P_2}) := \{(I_1, I_2, f) : I_i \in J_{P_i}, f \in \mathrm{M}(P_1 \backslash I_1, I_2)\} \quad i = 1, 2$$

We need to define the composition of morphisms

$$\mathrm{Hom}(X_{P_1}, X_{P_2}) \times \mathrm{Hom}(X_{P_2}, X_{P_3}) \rightarrow \mathrm{Hom}(X_{P_1}, X_{P_3})$$

Suppose that $(I_1, I_2, f) \in \mathrm{Hom}(X_{P_1}, X_{P_2})$ and $(I_2', I_3', g) \in \mathrm{Hom}(X_{P_2}, X_{P_3})$. Their composition is the morphism (K_1, K_3, h) defined as follows.

- We have $I_2 \wedge I_2' \subset I_2$, and since $f : P_1 \backslash I_1 \rightarrow I_2$ is an isomorphism, $f^{-1}(I_2 \wedge I_2')$ is an order ideal of $P_1 \backslash I_1$. Since in J_{P_1}, $[I_1, P] \simeq J_{P_1 \backslash I_1}$, we

have that $f^{-1}(I_2 \wedge I_2')$ corresponds to an order ideal $K_1 \in J_{P_1}$ such that $I_1 \subset K_1$.
- We have $I_2' \subset I_2 \vee I_2'$, and since $[I_2', P_2] \simeq J_{P_2 \backslash I_2'}$, $I_2 \vee I_2'$ corresponds to an order ideal $L_2 \in J_{P_2 \backslash I_2'}$. Since $g : P_2 \backslash I_2' \to I_3'$ is an isomorphism, $g(L_2) \subset J_{I_3'}$, and since $J_{I_3'} \subset J_{P_3}$, $g(L_2)$ corresponds to an order ideal $K_3 \in J_{P_3}$ contained in I_3'.
- The isomorphism $f : P_1 \backslash I_1 \to I_2$ restricts to an isomorphism $\bar{f} : P_1 \backslash K_1 \to I_2 \backslash I_2 \wedge I_2' = I_2 \backslash I_2'$, and the isomorphism $g : P_2 \backslash I_2'$ restricts to an isomorphism $\bar{g} : I_2 \vee I_2' \backslash I_2' = I_2 \backslash I_2' \to K_3$. Thus, $g \circ f : P_1 \backslash K_1 \to K_3$ is an isomorphism and $g \circ f \in \mathrm{M}(P_1 \backslash K_1, K_3)$ by the property (4) above.

lemma 1. *Composition of morphisms is associative.*

Proof. Suppose that $P_1, P_2, P_3, P_4 \in \mathcal{F}$, and that we have three morphisms as follows:

$$X_{P_1} \stackrel{(A_1,A_2,f)}{\longrightarrow} X_{P_2} \stackrel{(B_2,B_3,g)}{\longrightarrow} X_{P_3} \stackrel{(C_3,C_4,h)}{\longrightarrow} X_{P_4}$$

Given a poset P and subsets $S_1, \cdots, S_k$ of P, denote by $[S_1, \cdots, S_k]$ the smallest order ideal containing $\cup_{i=1}^k S_i$. We have :

$$\begin{aligned}(C_3, C_4, h)\circ&((B_2, B_3, g) \circ (A_1, A_2, f)) \\ &= (C_3, C_4, h) \circ ([A_1, f^{-1}(B_2)], g(A_2 \backslash B_2), g \circ f) \\ &= ([A_1, f^{-1}(B_2), (g \circ f)^{-1}(C_3)], h(g(A_2 \backslash B_2) \backslash C_3), h \circ g \circ f)\end{aligned}$$

whereas

$$\begin{aligned}((C_3, C_4, h)\circ&(B_2, B_3, g)) \circ (A_1, A_2, f) \\ &= ([B_2, g^{-1}(C_3)], h(B_3 \backslash C_3), h \circ g) \circ (A_1, A_2, f) \\ &= ([A_1, f^{-1}([B_2, g^{-1}(C_3)])], h \circ g(A_2 \backslash [B_2, g^{-1}(C_3)]), h \circ g \circ f)\end{aligned}$$

We have

$$f^{-1}([B_2, g^{-1}(C_3)]) = f^{-1}(B_2 \cup g^{-1}(C_3)) = f^{-1}(B_2) \cup (g \circ f)^{-1}(C_3)$$

which implies that

$$[A_1, f^{-1}(B_2), (g \circ f)^{-1}(C_3)] = [A_1, f^{-1}([B_2, g^{-1}(C_3)])]$$

and

$$h(g(A_2 \backslash B_2) \backslash C_3) = h \circ g(A_2 \backslash (B_2 \cup g^{-1}(C_3))) = h \circ g(A_2 \backslash [B_2, g^{-1}(C_3)])$$

This proves the two compositions are equal. □

Finally,

- We refer to X_{I_2} as the *image* of the morphism $(I_1, I_2, f) : X_{P_1} \to X_{P_2}$.

- We denote by $\mathrm{Iso}(\mathcal{C}_{\mathcal{F}})$ the collection of isomorphism classes of objects in $\mathcal{C}_{\mathcal{F}}$, and by $[X_P]$ the isomorphism class of $X_P \in \mathcal{C}_{\mathcal{F}}$.

4. Properties of the categories $\mathcal{C}_{\mathcal{F}}$

We now enumerate some of the properties of the categories $\mathcal{C}_{\mathcal{F}}$.

(1) The empty poset $\emptyset$ is an initial, terminal, and therefore null object. We will sometimes denote it by $X_{\emptyset}$.

(2) We can equip $\mathcal{C}_{\mathcal{F}}$ with a symmetric monoidal structure by defining

$$X_{P_1} \oplus X_{P_2} := X_{P_1+P_2}.$$

(3) The indecomposable objects of $\mathcal{C}_{\mathcal{F}}$ are the X_P with P a connected poset in $\mathcal{F}$.

(4) The irreducible objects of $\mathcal{C}_{\mathcal{F}}$ are the X_P where P is a one-element poset.

(5) Every morphism

$$(I_1, I_2, f) : X_{P_1} \to X_{P_2} \tag{3}$$

has a kernel

$$(\emptyset, I_1, id) : X_{I_1} \to X_{P_1}$$

(6) Similarly, every morphism 3 possesses a cokernel

$$(I_2, P_2 \backslash I_2, id) : X_{P_2} \to X_{P_2 \backslash I_2}$$

We will frequently use the notation X_{P_2}/X_{P_1} for $coker((I_1, I_2, f))$.

Note: Properties 5 and 6 imply that the notion of exact sequence makes sense in $\mathcal{C}_{\mathcal{F}}$.

(7) All monomorphisms are of the form

$$(\emptyset, I, f) : X_Q \to X_P$$

where $I \in J_P$, and $f : Q \to I \in \mathrm{M}(Q, I)$. Monomorphisms $X_Q \to X_P$ with a fixed image X_I form a torsor over $\mathrm{Aut}_{\mathrm{M}}(I)$. All epimorphisms are of the form

$$(I, \emptyset, g) : X_P \to X_Q$$

where $I \in J_P$ and $g : P \backslash I \to Q \in \mathrm{M}(P \backslash I, Q)$. Epimorphisms with fixed kernel X_I form a torsor over $\mathrm{Aut}_{\mathrm{M}}(P \backslash I)$

(8) Sequences of the form

$$X_\emptyset \overset{(\emptyset,\emptyset,id)}{\rightarrow} X_I \overset{(\emptyset,I,id)}{\longrightarrow} X_P \overset{(I,\emptyset,id)}{\longrightarrow} X_{P\setminus I} \overset{(P\setminus I,\emptyset,id)}{\rightarrow} X_\emptyset \tag{4}$$

with $I \in J_P$ are short exact, and all other short exact sequences with X_P in the middle arise by composing with isomorphisms $X_I \to X_{I'}$ and $X_{P\setminus I} \to X_Q$ on the left and right.

(9) Given an object X_P and a subobject X_I, $I \in J_P$, the isomorphism $J_{P\setminus I} \simeq [I, P]$ translates into the statement that there is a bijection between subobjects of X_P/X_I and order ideals $J \in J_P$ such that $I \subset J \subset P$ via $X_J \leftrightarrow J$. The bijection is compatible with quotients, in the sense that $(X_P/X_I)/(X_J/X_I) \simeq X_J/X_I$.

(10) Since the posets in $\mathcal{F}$ are finite, $\mathrm{Hom}(X_{P_1}, X_{P_2})$ is a finite set.

(11) We may define Yoneda $\mathrm{Ext}^n(X_{P_1}, X_{P_2})$ as the equivalence class of n–step exact sequences with X_{P_1}, X_{P_2} on the right and left respectively. $\mathrm{Ext}^n(X_{P_1}, X_{P_2})$ is a finite set. Concatenation of exact sequences makes

$$\mathbb{E}xt^* := \cup_{A,B \in I(\mathcal{C}_\mathcal{F}),n} \mathrm{Ext}^n(A, B)$$

into a monoid.

(12) We may define the Grothendieck group of $\mathcal{C}_\mathcal{F}$, $K_0(\mathcal{C}_\mathcal{F})$, as

$$K(\mathcal{C}_\mathcal{F}) = \bigoplus_{A \in \mathcal{C}_\mathcal{F}} \mathbb{Z}[A]/\sim$$

where $\sim$ is generated by $A + B - C$ for short exact sequences

$$X_\emptyset \to A \to C \to B \to X_\emptyset$$

We denote by $k(A)$ the class of an object in $K_0(\mathcal{C}_\mathcal{F})$.

5. RINGEL-HALL ALGEBRAS AND INCIDENCE ALGEBRAS

5.1. **Incidence Hopf algebras.** We begin by recalling the definition of the incidence Hopf algebra of a hereditary interval-closed family of posets introduced in [12]. Incidence algebras of posets were originally introduced in [9].

A family $\mathcal{P}$ of finite intervals is said to be *interval closed*, if it is non-empty, and for all $P \in \mathcal{P}$ and $x \leq y \in P$, the interval $[x, y]$ belongs to $\mathcal{P}$. An *order compatible relation* on an interval closed family $\mathcal{P}$ is an equivalence relation $\sim$ such that whenever $P \sim Q$ in $\mathcal{P}$, there exists a bijection $\phi : P \to Q$ such that $[0_P, x] \sim [0_Q, \phi(x)]$ and $[x, 1_P] \sim [\phi(x), 1_Q]$ for all $x \in P$. Typical examples of order compatible relations are poset isomorphism, or isomorphism preserving some additional structure (such as a coloring). We denote by $\tilde{\mathcal{P}}$ the set equivalence classes of $\mathcal{P}$ under $\sim$, i.e. $\tilde{\mathcal{P}} = \mathcal{P}/\sim$, and by $[P]$ the equivalence class of a poset P in $\tilde{\mathcal{P}}$. The *incidence algebra* of the family $(\mathcal{P}, \sim)$, denoted $\mathrm{H}_{\mathcal{P},\sim}$, is

$$\mathrm{H}_{\mathcal{P},\sim} := \{f : \tilde{\mathcal{P}} \to \mathbb{Q} : |supp(f)| < \infty\}$$

(note that the finiteness of the support of f is not standard). $\mathrm{H}_{\mathcal{P},\sim}$ is naturally a $\mathbb{Q}$–vector space, and becomes an associative $\mathbb{Q}$–algebra under the convolution product

$$f \bullet g([P]) := \sum_{x \in P} f([0_P, x]) g([x, 1_P]) \tag{5}$$

We would now like to equip $\mathrm{H}_{\mathcal{P},\sim}$ with a Hopf algebra structure. To do this, the family $\mathcal{P}$ and the relation $\sim$ must satisfy some additional properties. A *hereditary family* is an interval closed family $\mathcal{P}$ of posets which is closed under the formation of direct products. We will assume that $\sim$ satisfies the following two properties:

- whenever $P \sim Q$ in $\mathcal{P}$, then $P \times R \sim Q \times R$ and $R \times P \sim R \times Q$ for all $R \in \mathcal{P}$
- if $P, Q \in \mathcal{P}$ and $|Q| = 1$, then $P \times Q \sim Q \times P \sim P$

An order compatible relation $\sim$ on $\mathcal{P}$ satisfying these additional properties is called a *Hopf relation*. In this case, $\tilde{\mathcal{P}}$ is a monoid under the operation $[P][Q] = [P \times Q]$ with unit the class of any one-element poset. We may now introduce a coproduct on $\mathrm{H}_{\mathcal{P},\sim}$

$$\Delta(f)([M], [N]) := f([M \times N]) \tag{6}$$

It is shown in [12] that Δ equips $\mathrm{H}_{\mathcal{P},\sim}$ with the structure of a bialgebra, and furthermore, that an antipode exists, making it into a Hopf algebra, which we call the *incidence Hopf algebra* of $(\mathcal{P}, \sim)$.

Note: the original definition of incidence Hopf algebra given in [12] is dual to the one used here.

Suppose that $\mathcal{F}$ is a collection of finite posets which is closed under the operation of taking convex subposets and disjoint unions. The collection of order ideals

$$\mathcal{P}(\mathcal{F}) := \{J_P : P \in \mathcal{F}\}$$

is then a collection of intervals which is interval-closed and hereditary. If we define $J_P \sim J_Q$ whenever there exists an $f : P \to Q \in \mathrm{M}(P, Q)$, then $\sim$ is a Hopf relation, and so we may consider the Hopf algebra $\mathrm{H}_{\mathcal{P}(\mathcal{F}),\sim}$. Using the fact that for $I, L \in J_P$, $[I, L] \simeq J_{L \setminus I}$ and $J_{P+Q} \simeq J_P \times J_Q$, the product 5 and coproduct 6 become:

$$f \bullet g([J_P]) := \sum_{I \in J_P} f([J_I]) g([J_{P \setminus I}]) \tag{7}$$

$$\Delta(f)([J_P],[J_Q]) := f([J_{P+Q}]) \tag{8}$$

5.2. **Ringel-Hall algebras.** For an introduction to Ringel-Hall algebras in the context of abelian categories, see [10]. We define the Ringel-Hall algebra of $\mathcal{C}_{\mathcal{F}}$, denoted $\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$, to be the $\mathbb{Q}$–vector space of finitely supported functions on isomorphism classes of $\mathcal{C}_{\mathcal{F}}$. I.e.

$$\mathrm{H}_{\mathcal{C}_{\mathcal{F}}} := \{ f : \mathrm{Iso}(\mathcal{C}_{\mathcal{F}}) \to \mathbb{Q} | |supp(f)| < \infty \}$$

As a $\mathbb{Q}$–vector space it is spanned by the delta functions δ_A, $A \in \mathrm{Iso}(\mathcal{C}_{\mathcal{F}})$. The algebra structure on $\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$ is given by the convolution product:

$$f \star g(M) = \sum_{A \subset M} f(A) g(M/A) \tag{9}$$

$\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$ possesses a co-commutative co-product given by

$$\Delta(f)(M,N) = f(M \oplus N) \tag{10}$$

as well as a natural $K_0^+(\mathcal{C}_{\mathcal{F}})$–grading in which δ_A has degree $k(A) \in K_0^+(\mathcal{C}_{\mathcal{F}})$.

The subobjects of $X_P \in \mathcal{C}_{\mathcal{F}}$ are exactly X_I for $I \in J_P$, and the product 9 becomes

$$f \star g([X_P]) = \sum_{I \in J_P} f([X_I]) g([X_{P \setminus I}])$$

while the coproduct becomes

$$\Delta(f)([X_P],[X_Q]) = f([X_{P+Q}])$$

Thus, the map

$$\phi : \mathrm{H}_{\mathcal{C}_{\mathcal{F}}} \to \mathrm{H}_{\mathcal{P}(\mathcal{F}),\sim}$$

determined by

$$\phi(f)([J_P]) := f([X_P])$$

is an isomorphism of Hopf algebras. Recall that a Hopf algebra A over a field k is *connected* if it possesses a $\mathbb{Z}_{\geq 0}$–grading such that $A_0 = k$. In addition to the $K_0^+(\mathcal{C}_{\mathcal{F}})$–grading, $\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$ possesses a grading by the order of the poset - i.e. we may assign $\deg(\delta_{X_P}) = |P|$. This gives it the structure of graded connected Hopf algebra. The Milnor-Moore theorem implies that $\mathrm{H}_{\mathcal{C}_{\mathcal{F}}}$ is the enveloping algebra of the Lie algebra of its primitive elements, which we denote by $\mathfrak{n}_{\mathcal{F}}$ - i.e. $\mathrm{H}_{\mathcal{C}_F} \simeq U(\mathfrak{n}_{\mathcal{F}})$. We have thus established the following:

Theorem 1. *The Ringel-Hall algebra of the category $\mathcal{C}_{\mathcal{F}}$ is isomorphic to the Incidence Hopf algebra of the family $\mathcal{P}(\mathcal{F})$. These are graded connected Hopf algebras, graded by the order of poset, and isomorphic to $U(\mathfrak{n}_{\mathcal{F}})$, where $\mathfrak{n}_{\mathcal{F}}$ denotes the Lie algebra of its primitive elements.*

6. Examples

In this section, we give some examples of families $\mathcal{F}$ of posets closed under disjoint unions and the operation of taking convex subposets.

Example 1: Let $\mathcal{F} = \mathrm{Fin}$ be the collection of all finite posets, and let $M(P, Q)$ consist of poset isomorphisms. Fin is clearly closed under disjoint unions and taking convex subposets. I claim that $K_0(\mathcal{C}_{\mathrm{Fin}}) = \mathbb{Z}$. Let P be a finite poset, and $m \in \mathcal{P}$ a minimal element. Then m is also an order ideal in P, and so we have an exact sequence

$$\emptyset \to X_{\bullet} \to X_P \to X_{P \backslash m} \to \emptyset$$

where $\bullet$ denotes the one-element poset. Repeating this procedure with $P \backslash m$, we see that in $K_0(\mathcal{C}_{\mathrm{Fin}})$ every element can be written as multiple of $X_{\bullet}$, with $X_P \sim |P| X_{\bullet}$, and $K_0(\mathcal{C}_{\mathrm{Fin}}) \simeq \mathbb{Z}$. Thus, the K_0^+–grading on $\mathrm{H}_{\mathcal{C}_{\mathrm{Fin}}}$ coincides with that by order of poset.

$\mathrm{Iso}(\mathcal{C}_{\mathrm{Fin}})$ consists of isomorphism classes of finite posets. The Lie algebra $\mathfrak{n}_{\mathrm{Fin}}$ is spanned by $\delta_{[X_P]}$ for P connected posets. For $P, Q \in \mathrm{Fin}$, both connected, we have

$$\delta_{[X_P]} \star \delta_{[X_Q]} = \sum_{\left\{ \substack{R \in \mathrm{Iso}(\mathrm{Fin}) | P \in J(R), \\ Q \simeq R \backslash P} \right\}} N(P, Q; R) \delta_{[X_R]}$$

where $N(P, Q; R) := |\{ I \in J_R | I \simeq P \}|$, and

$$[\delta_{[X_P]}, \delta_{[X_Q]}] = \delta_{[X_P]} \star \delta_{[X_Q]} - \delta_{[X_Q]} \star \delta_{[X_P]}.$$

Example 2: Let $\mathcal{F} = \mathcal{S}$ denote the collection of all finite sets (including the empty set). A finite set can be viewed as a poset where any two distinct elements are incomparable. $\mathcal{S}$ is clearly closed under disjoint unions and taking convex subposets (which coincide with subsets). Also, the order ideals of $S \in \mathcal{S}$ are exactly the subsets of S. Let $M(P, Q)$ consist of set isomorphisms. By the same argument as in the previous example, we have $K_0(\mathcal{C}_{\mathcal{S}}) \simeq \mathbb{Z}$. $\mathrm{Iso}(\mathcal{C}_S) \simeq \mathbb{Z}_{\geq 0}$, and we denote by $[n]$ the isomorphism class of the set with n elements. We have

$$\delta_{[n]} \star \delta_{[m]} = \binom{m+n}{n} \delta_{[m+n]}$$

$\mathfrak{n}_S$ is abelian, and isomorphic the one-dimensional Lie algebra spanned by $\delta_{[1]}$. $\mathrm{H}_{\mathcal{C}_{\mathcal{S}}}$ is dual to the binomial Hopf algebra in [12].

Example 3:

Let $\mathcal{S}\{k\}$ denote the collection of all finite k–colored sets (including the empty set), and take $M(P,Q)$ to consist of color-preserving set isomorphism. Clearly, the previous example corresponds to $k=1$. We have $K_0(\mathcal{C}_{\mathcal{S}\{k\}}) \simeq \mathbb{Z}^k$ and $\mathrm{Iso}(\mathcal{C}_{\mathcal{S}\{k\}}) \simeq (\mathbb{Z}_{\geq 0})^k$. Denoting by $[n_1, \cdots, n_k]$ the isomorphism class of the set consisting of n_1 elements of color $c_1, \cdots, n_k$ elements of color c_k, we have

$$\delta_{[n_1,\cdots,n_k]} \star \delta_{[m_1,\cdots,m_k]} = \left(\prod_i^k \binom{n_i+m_i}{n_i} \right) \delta_{[n_1+m_1,\cdots,n_k+m_k]}$$

$\mathfrak{n}_{\mathcal{S}\{k\}}$ is a k–dimensional abelian Lie algebra spanned by δ_{e_i}, where e_i denotes the k–tuple $[0,\cdots,1,\cdots,0]$ with 1 in the ith spot.

The following two examples are treated in detail in [8], and we refer the reader there for details. Please see also [12, 4]. The resulting Hopf algebras (or rather their duals) introduced in [6, 2], form the algebraic backbone of the renormalization process in quantum field theory. The corresponding Lie algebras $\mathfrak{n}_{\mathcal{F}}$ are studied in [3].

Example 4:

Recall that a rooted tree t defines a poset whose Hasse diagram is t. Let $\mathcal{F} = \mathrm{RF}$ denote the family of posets defined by rooted forests (i.e. disjoint unions of rooted trees). It is obviously closed under disjoint unions. An order ideal in a rooted forest corresponds to an *admissible cut* in the sense of [2] - that is, a cut having the property that any path from root to leaf encounters at most one cut edge, and so RF is closed under the operation of taking convex posets. We have $K_0(\mathcal{C}_{\mathrm{RF}}) = \mathbb{Z}$, and as shown in [8], $\mathrm{H}_{\mathcal{C}_{\mathrm{RF}}}$ is isomorphic to the dual of the Connes-Kreimer Hopf algebra on forests, or equivalently, to the Grossman-Larson Hopf algebra.

This example also has a colored version, where we consider the family $\mathrm{RF}\{k\}$ of k–colored rooted forests. In this case, $K_0(\mathcal{C}_{\mathrm{RF}\{k\}}) \simeq (\mathbb{Z}_{\geq 0})^k$.

Example 5:

A graph determines a poset of subgraphs under inclusion. We may for instance consider the poset $\mathcal{F} = \mathrm{FG}$ of subgraphs of Feynman graphs of a quantum field theory such as ϕ^3 theory, which is closed under disjoint unions and convex posets. The Ringel-Hall algebra $\mathrm{H}_{\mathcal{C}_{\mathrm{FG}}}$ is isomorphic to the dual of the Connes-Kreimer Hopf algebra on Feynman graphs. Please see [8] for details.

In the case of ϕ^3 theory, it is shown in [7] that $K_0(\mathcal{C}_{\mathrm{FG}}) \simeq \mathbb{Z}[p]$, $p \in \mathfrak{P}$, where $\mathfrak{P}$ is the set of primitively divergent graphs.

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