ON THE STRUCTURE AND REPRESENTATIONS OF THE INSERTION-ELIMINATION LIE ALGEBRA

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ABSTRACT. We examine the structure of the insertion-elimination Lie algebra on rooted trees introduced in [CK]. It possesses a triangular structure $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathbb{C}.d \oplus \mathfrak{n}_-$, like the Heisenberg, Virasoro, and affine algebras. We show in particular that it is simple, which in turn implies that it has no finite-dimensional representations. We consider a category of lowest-weight representations, and show that irreducible representations are uniquely determined by a "lowest weight" $\lambda \in \mathbb{C}$. We show that each irreducible representation is a quotient of a Verma-type object, which is generically irreducible.

1. INTRODUCTION

The insertion-elimination Lie algebra $\mathfrak{g}$ was introduced in [CK] as a means of encoding the combinatorics of inserting and collapsing subgraphs of Feynman graphs, and the ways the two operations interact. A more abstract and universal description of these two operations is given in terms of rooted trees, which encode the hierarchy of subdivergences within a given Feynman graph, and it is this description that we adopt in this paper. More precisely, $\mathfrak{g}$ is generated by two sets of operators $\{D^+_t\}$, and $\{D^-_t\}$, where t runs over the set of all rooted trees, together with a grading operator d. In [CK] $\mathfrak{g}$ was defined in terms of its action on a natural representation $\mathbb{C}\{\mathbb{T}\}$, where the latter denotes the vector space spanned by rooted trees. For $s \in \mathbb{C}\{\mathbb{T}\}$, $D^+_t.s$ is a linear combination of the trees obtained by attaching t to s in all possible ways, whereas $D^-_t.s$ is a linear combination of all the trees obtained by pruning the tree t from branches of s. $\mathfrak{n}_+ = \{D^+_t\}$ and $\mathfrak{n}_- = \{D^-_t\}$ form two isomorphic nilpotent Lie subalgebras, and $\mathfrak{g}$ has a triangular structure

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathbb{C}.d \oplus \mathfrak{n}_-$$

as well as a natural $\mathbb{Z}$–grading by the number of vertices of the tree t. The Hopf algebra $U(\mathfrak{n}_\pm)$ is dual to Kreimer's Hopf algebra of rooted trees [K].

This note aims to estblish a few basic facts regarding the structure and representation theory of $\mathfrak{g}$. We begin by showing that $\mathfrak{g}$ is simple, which together with its infinite-dimensionality implies that it has no non-trivial finite-dimensional representations, and that any non-trivial representation is necessarily faithful. We then proceed to develop a highest-weight theory for $\mathfrak{g}$ along the lines of [K1, K2].

Date: October 2007.

In particular, we show that every irreducible highest-weight representation of $\mathfrak{g}$ is a quotient of a Verma-like module, and that these are generically irreducible.

One can define a larger, "two-parameter" version of the insertion-elimination Lie algebra $\widetilde{\mathfrak{g}}$, where operators are labelled by pairs of trees D_{t_1,t_2} (roughly speaking, in acting on $\mathbb{C}\{\mathbb{T}\}$, this operator replaces occurrences of t_1 by t_2). In the special case of ladder trees, $\widetilde{\mathfrak{g}}$ was studied in [M, KM1, KM2]. The finite-dimensional representations of the nilpotent subalgebras $\mathfrak{n}_{\pm}$ as well as many other aspects of the Hopf algebra $U(\mathfrak{n}_{\pm})$ were studied in [F].

Acknowledgements: The author would like to thank Dirk Kreimer for many illuminating conversations and explanations of renormalization as well as related topics. This work was supported by NSF grant DMS-0401619.

2. The insertion-elimination Lie algebra on rooted trees

In this section, we review the construction of the insertion-elimination Lie algebra introduced in [CK], with some of the notational conventions introduced in [M].

Let $\mathbb{T}$ denote the set of rooted trees. An element $t \in \mathbb{T}$ is a tree (finite, one-dimensional contractible simplicial complex), with a distinguished vertex $r(t)$, called the root of t. Let $V(t)$ and $E(t)$ denote the set of vertices and edges of t, and let

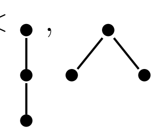
$$|t| = \#V(t)$$

Let $\mathbb{C}\{\mathbb{T}\}$ denote the vector space spanned by rooted trees. It is naturally graded,

$$\mathbb{C}\{\mathbb{T}\} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{C}\{\mathbb{T}\}_n \tag{2.1}$$

where $\mathbb{C}\{\mathbb{T}\}_n = \text{span}\{t \in \mathbb{T} | |t| = n\}$. $\mathbb{C}\{\mathbb{T}\}_0$ is spanned by the empty tree, which we denote by $\mathbf{1}$. We have

$\mathbb{C}\{\mathbb{T}\}_0 =< 1 >$ $\quad$ $\mathbb{C}\{\mathbb{T}\}_1 =< \bullet >$ $\quad$ $\mathbb{C}\{\mathbb{T}\}_2 = <$  $>$

$\mathbb{C}\{\mathbb{T}\}_3 = <$  $>$

where $<, >$ denotes span, and the root is the vertex at the top. If $e \in E(t)$, by a *cut along* e we mean the operation of cutting e from t. This divides t into two components - $R_c(t)$ containing the root, and $P_e(t)$, the remaining one. $R_e(t)$ and $P_e(t)$ are naturally rooted trees, with $r(R_c(t)) = r(t)$ and $r(P_e(t)) =$ (endpoint of e). Note that $V(t) = V(R_e(t)) \cup V(P_e(t))$.

Let $\mathfrak{g}$ denote the Lie algebra with generators D_t^+, D_t^-, d, $t \in \mathbb{T}$, and relations

$$[D_{t_1}^+, D_{t_2}^+] = \sum_{v \in V(t_2)} D^+_{t_2 \cup_v t_1} - \sum_{v \in V(t_1)} D^+_{t_1 \cup_v t_2} \tag{2.2}$$

$$[D_{t_1}^-, D_{t_2}^-] = \sum_{v \in V(t_1)} D^-_{t_1 \cup_v t_2} - \sum_{v \in V(t_2)} D^-_{t_2 \cup_v t_1} \tag{2.3}$$

$$[D_{t_1}^-, D_{t_2}^+] = \sum_{t \in \mathbb{T}} \alpha(t_1, t_2; t) D_t^+ + \sum_{t \in T} \beta(t_1, t_2; t) D_t^- \tag{2.4}$$

$$[D_t^-, D_t^+] = d \tag{2.5}$$

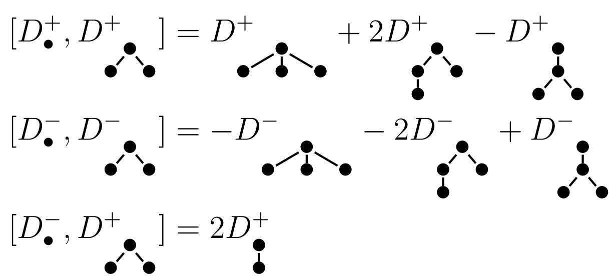
$$[d, D_t^-] = -|t| D_t^- \tag{2.6}$$

$$[d, D_t^+] = |t| D_t^+ \tag{2.7}$$

where for $s, t \in \mathbb{T}$, and $v \in V(s)$ $s \cup_v t$ denotes the rooted tree obtained by joining the root of t to s at the vertex v via a single edge, and

- $\alpha(t_t, t_2; t) = \#\{e \in E(t_2) | R_e(t_2) = t,\ P_e(t_2) = t_1\}$
- $\beta(t_1, t_2; t) = \#\{e \in E(t_1) | R_e(t_1) = t,\ P_e(t_1) = t_2\}$

Thus, for example



$\mathfrak{g}$ acts naturally on $\mathbb{C}\{\mathbb{T}\}$ as follows. If $s \in \mathbb{T}$, viewed as an element of $\mathbb{C}\{\mathbb{T}\}$, and $t \in \mathbb{T}$, then

$$D_t^+(s) = \sum_{v \in V(s)} s \cup_v t$$

$$D_t^-(s) = \sum_{e \in E(s), P_e(s) = t} R_e(s)$$

$$d(s) = |s| s$$

3. Structure of $\mathfrak{g}$

Let $\mathfrak{n}_+$ and $\mathfrak{n}_-$ be the Lie subalgebras s of $\mathfrak{g}$ generated by D_t^+ and D_t^- , $t \in \mathbb{T}$. We have a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathbb{C}.d \oplus \mathfrak{n}_- \tag{3.1}$$

The relations 2.5, 2.6, and 2.7 imply that for every $t \in \mathbb{T}$

$$\mathfrak{g}^t =< D_t^+, D_t^-, d >$$

forms a Lie subalgebra isomorphic to $\mathfrak{sl}_2$. We have that $\mathfrak{g}_t \cap \mathfrak{g}_s = \mathbb{C}.d$ if $s \neq t$. Assigning degree $|t|$ to D_t^+, $-|t|$ to D_t^-, and 0 to d equips $\mathfrak{g}$ with a $\mathbb{Z}$–grading.

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$$

$\mathfrak{g}$ possesses an involution ι, with

$$\iota(D_t^+) = D_t^- \qquad \iota(D_t^-) = D_t^+ \qquad \iota(d) = -d$$

Thus ι is a gradation-reversing Lie algebra automorphism exchanging $\mathfrak{n}_+$ and $\mathfrak{n}_-$.

Theorem 3.1. $\mathfrak{g}$ *is a simple Lie algebra*

Proof. Suppose that $\mathcal{I} \subset \mathfrak{g}$ is a proper Lie ideal. If $x \in \mathcal{I}$, let $x = \sum_i x_i, \;\; x_i \in \mathfrak{g}_i$ be its decomposition into homogenous components. We have

$$[d, x] = \sum_n n x_n$$

which implies that $x_n \in I$ for every n (because the Vandermonde determinant is invertible) i.e. $\mathcal{I} = \oplus_{n \in \mathbb{Z}} (\mathcal{I} \cap \mathfrak{g}_n)$. Suppose now that $x_n \in \mathfrak{g}_n, \; n > 0$. We can write x_n as a linear combination of n–vertex rooted trees

$$x_n = \sum_{t \in \mathbb{T}_n} \alpha_t \cdot t \tag{3.2}$$

We proceed to show that $D_\bullet^+ \in \mathcal{I}$, where $\bullet$ is the rooted tree with one vertex. Let $S(x_n) \subset \mathbb{T}_n$ be the subset of n-vertex trees occurring with a non-zero α_t in 3.2. Given a rooted tree t, let $St(t)$ denote the set of rooted trees obtained by removing all the edges emanating from the root. Let

$$St(x_n) = \bigcup_{s \in S(x_n)} St(s)$$

and let $\xi \in St(x_n)$ be of maximal degree. It is easy to see that $[D_\xi^-, x_n]$ is a non-zero element of $\mathfrak{g}_{n-|\xi|}$. Starting with $x_n \in \mathfrak{n}_+, \; x_n \neq 0$, and repeating this process if necessary, we eventually obtain a non-zero element of $\mathfrak{g}_1 =< D_\bullet^+ >$. Now, $[D_\bullet^-, D_\bullet^+] = d$, and since $[d, \mathfrak{g}] = \mathfrak{g}]$, this implies $\mathcal{I} = \mathfrak{g}$. We have thus shown that if $\mathcal{I}$ is proper, then

$$\mathcal{I} \cap \mathfrak{n}_+ = 0$$

Applying ι shows that $\mathcal{I} \cap \mathfrak{n}\_0$ as well, and it is clear that $\mathcal{I} \cap \mathbb{C}.d = 0$.

□

We can now use this result to deduce a couple of facts about the representation theory of $\mathfrak{g}$.

Corollary 3.1. *If V is a non-trivial representation of $\mathfrak{g}$, then V is faithful.*

Corollary 3.2. *$\mathfrak{g}$ has no non-trivial finite-dimensional representations.*

The latter can also be easily deduced by analyzing the action of the $\mathfrak{sl}_2$ subalgebras $\mathfrak{g}^t$ as follows. Suppose that V is a finite-dimensional representation of $\mathfrak{g}$. To show that V is trivial, it suffices to show that it restricts to a trivial representation of $\mathfrak{g}^t$ for every $t \in \mathbb{T}$. This in turn, will follow if we can show that for a *single* tree $t \in \mathbb{T}$, $\mathfrak{g}^t$ acts trivially, because this implies that d acts trivially, and $\mathbb{C}.d \subset \mathfrak{g}^t$ plays the role of the Cartan subalgebra. Let

$$V = \bigoplus_{i=1\cdots k} V_{\delta_i}$$

be a decomposition of V into d–eigenspaces - i.e. if $v \in V_{\delta_i}$, then $d.v_i = \delta_i v$. Since V is finite-dimensional, the set $\{\delta_i\}$ is bounded, and so lies in a disc of radius R in $\mathbb{C}$. If $v \in V_{\delta_i}$ then $[d, D_t^+] = |t| D_t^+$ implies that $D_t^+.v \in V_{\delta_i + |t|}$. Choosing a $t \in \mathbb{T}$ such that $|t| > 2R$ shows that $D_t^+.v = 0$ for every $v \in V$.

3.1. **Lowest-weight representations of $\mathfrak{g}$.** We begin by examining the "defining" representation $\mathbb{C}\{\mathbb{T}\}$ of $\mathfrak{g}$ introduced in section 2. Its decomposition into d–eigenspaces is given by 2.1. Given a representation V of $\mathfrak{g}$ on which d is diagonalizable, with finite-dimensional eigenspaces, and writing

$$V = \bigoplus_{\delta} V_{\delta}$$

for this decomposition, we define the emphcharacter of V, $char(V, q)$ to be the formal series

$$char(V, q) = \sum_{\delta} dim(V_{\delta}) q^{\delta}$$

The case $V = \mathbb{C}\{\mathbb{T}\}$, where $dim(V_n)$ is the number of rooted trees on n vertices, suggests that representations of $\mathfrak{g}$ may contain interesting combinatorial information. The triangular structure 3.1 of $\mathfrak{g}$ suggests that a theory of highest– or lowest–weight representations may be appropriate.

Definition 3.1. We say that a representation V of $\mathfrak{g}$ is *lowest–weight* if the following properties hold

1. $V = \oplus V_{\delta}$ is a direct sum of finite-dimensional eigenspaces for d.
2. The eigenvalues δ are bounded in the sense that there exists $L \in \mathbb{R}$ such that $Re(\delta) \geq L$.

We call the δ the *weights* of the representation, and category of such representations $\mathcal{O}$. If $V \in \mathcal{O}$, we say $v \in V_\delta$ is a *lowest-weight vector* if $\mathfrak{n}_- v = 0$. Since D_t^- decreases the weight of a vector by $|t|$, and the weights all lie in a half-plane, it is clear that every $V \in \mathcal{O}$ contains a lowest-weight vector.

Recall that a representation V of $\mathfrak{g}$ is *indecomposable* if it cannot be written as $V = V_1 \oplus V_2$ for two non-zero representations. Let $U(\mathfrak{h})$ denote the universal enveloping algebra of a Lie algebra $\mathfrak{h}$.

Lemma 3.1. *If $v \in V_\lambda$ is a lowest-weight vector, then $U(\mathfrak{n}_+).v$ is an indecomposable representation of $\mathfrak{g}$*

Proof. $U(\mathfrak{g}).v$ is clearly the smallest sub-representation of V containing v. The decomposition 3.1 together with the PBW theorem implies that

$$U(\mathfrak{g}) = U(\mathfrak{n}_+) \otimes \mathbb{C}[d] \otimes U(\mathfrak{n}_-)$$

Because v is a lowest-weight vector, $\mathbb{C}[d] \otimes U(\mathfrak{n}_-).v = \mathbb{C}.v$. It follows that $U(\mathfrak{g}).v = U(\mathfrak{n}_+).v$. That the latter is indecomposable follows from the fact that in $U(\mathfrak{n}_+).v$, the weight space corresponding to λ is one-dimensional, and so if $U(\mathfrak{n}_+).v = V_1 \oplus V_2$, then $v \in V_1$ or $v \in V_2$. □

Observe that

$$U(\mathfrak{n}_+).v = \oplus (U(\mathfrak{n}_+).v)_{\lambda+k}, \quad k \in \mathbb{Z}_{\geq 0}$$

where $(U(\mathfrak{n}_+).v)_{\lambda+k}$ is spanned by monomials of the form

$$D^+_{t_1} D^+_{t_2} \cdots D^+_{t_i}.v \tag{3.3}$$

with $|t_1| + \cdots |t_i| = k$.

The category $\mathcal{O}$ contains Verma-like modules. For $\lambda \in \mathbb{C}$, let $\mathbb{C}_\lambda$ denote the one-dimensional representation of $\mathbb{C}.d \oplus \mathfrak{n}_-$ on which $\mathfrak{n}_-$ acts trivially, and d acts by multiplication by λ.

Definition 3.2. The $\mathfrak{g}$–module

$$W(\lambda) = U(\mathfrak{g}) \underset{\mathbb{C}[d] \otimes U(\mathfrak{n}_-)}{\otimes} \mathbb{C}_\lambda$$

will be called *the Verma module* of lowest weight λ.

Choosing an ordering on trees yields a PBW basis for $\mathfrak{n}_+$, and thus also a basis of the form 3.3 for $W(\lambda)$.

Given a representation $V \in \mathcal{O}$, and a lowest weight vector $v \in V_\lambda$, we obtain a map of representations

$$\begin{aligned} W(\lambda) &\mapsto V \\ \mathbf{1} &\mapsto v \end{aligned} \tag{3.4}$$

Lemma 3.2. *If $V \in \mathcal{O}$ is an irreducible representation, then V is the quotient of a Verma module.*

Proof. Since $V \in \mathcal{O}$, V possesses a lowest-weight vector $v \in V_\lambda$ for some $\lambda \in \mathbb{C}$. Since V is irreducible, $V = U(\mathfrak{g}).v = U(\mathfrak{n}_+).v$. The latter is a quotient of $W(\lambda)$. □

We have

$$\begin{aligned} Char(W(\lambda)) &= q^\lambda \sum_{n \in \mathbb{Z}_{\geq 0}} dim(\mathbb{C}\{\mathbb{T}\}_n) q^n \\ &= q^\lambda \prod_{n \in \mathbb{Z}_{\geq 0}} \frac{1}{(1-q^n)^{P(n)}} \end{aligned}$$

where $P(n)$ is the number of primitive elements of degree n in $\mathcal{H}_K$.

3.2. **Irreducibility of $W(\lambda)$.** It is a natural question whether $W(\lambda)$ is irreducible. In this section we prove the following result:

Theorem 3.2. *For λ outside a countable subset of $\mathbb{C}$ containing 0, $W(\lambda)$ is irreducible.*

Proof. Let $v \neq 0$ be a basis for $W(\lambda)_\lambda$. $W(\lambda)$ contains a proper sub-representation if and only if contains a lowest-weight vector w such that $w \notin \mathbb{C}.v$. In $W(0)$, $D^+_\bullet.v \in W(0)_1$ is a lowest-weight vector, since

$$D^-_\bullet D^+_\bullet.v = D^+_\bullet D^-_\bullet.v + d.v = 0$$

and $D^-_t.v = 0$ for all $t \in \mathbb{T}$ with $|t| \geq 2$ by degree considerations. It follows that $W(0)$ is not irreducible.

If $I = (t_1, \cdots, t_k)$ is a k–tuple of trees such that

$$t_1 \preceq t_2 \preceq \cdots \preceq t_k$$

in the chosen order, let $D^+_I.v$ denote the vector

$$D^+_{t_k} \cdots D^+_{t_1}.v \ \in W(\lambda) \tag{3.5}$$

$w \in W(\lambda)_{\lambda+n}$ is a lowest-weight vector if and only if

$$D^-_t.w = 0 \tag{3.6}$$

for all t such that $|t| \leq n$. Writing w in the basis 3.5

$$w = \sum_{|I|=n} \alpha_I D^+_I.v$$

the conditions 3.6 translate into a system of equations for the coefficients α_I. For example, if $w \in W(\lambda)_{\lambda+2}$, then

$$w = \alpha_1 D^+_{\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}}.v + \alpha_2 D^+_\bullet D^+_\bullet.v$$

and conditions $D^-_{\substack{\bullet \\ | \\ \bullet}}.v = 0$, $D^-_{\bullet}.w = 0$ translate into

$$\begin{aligned} \lambda\alpha_1 + \lambda\alpha_2 &= 0 \\ \alpha_1 + (2\lambda + 1)\alpha_2 &= 0 \end{aligned}$$

The determinant of the corresponding matrix is $2\lambda^2$, and so for $\lambda \neq 0$, there is no lowest-weight vector $w \in W(\lambda)_{\lambda+2}$. For a general n, the system can be written in the form

$$(A + \lambda B)[\alpha_I] = 0$$

where A and B are matrices whose entries are non-negative integers. Let

$$f_n(\lambda) = dim(Ker(A + \lambda B))$$

Then for every $r \in \mathbb{N}$

$$S_{n,r} = \{\lambda \in \mathbb{C} | f_n(\lambda) \geq r\}$$

if proper, is a finite subset of $\mathbb{C}$, since the condition is equivalent to the vanishing a finite collection of sub-determinants, each of which is a polynomial in λ. The set of $\lambda \in \mathbb{C}$ for which $W(\lambda)$ is irreducible is therefore

$$\bigcup_{n \in \mathbb{N}} \{\mathbb{C} \backslash S_{n,1}\}$$

The theorem will follow if $S_{n,1}$ is proper for each $n \in \mathbb{N}$. This follows from the following Lemma.

□

Lemma 3.3. $Z(1)$ *is irreducible.*

Proof. We begin by examining the representation $\mathbb{C}\{\mathbb{T}\}$. The degree 0 subspace $\mathbb{C}.1$ is a trivial representation of $\mathfrak{g}$. Let M denote the quotient $\mathbb{C}\{\mathbb{T}\}/\mathbb{C}.1$. It is easily seen that the exact sequence

$$0 \mapsto \mathbb{C} \mapsto \mathbb{C}\{\mathbb{T}\} \mapsto M \mapsto 0$$

is non-split. M has highest weight 1, and the subspace M_1 can be identified with the span of the tree on one vertex $\bullet$. By the universal property of Verma modules, 3.4 we have a map

$$W(1) \mapsto M \tag{3.7}$$

sending the lowest-weight vector of $W(1)$ to $\bullet$. Now, $W(1)_n$ is spanned by all vectors 3.5 such that $|t_1| + \cdots |t_k| = n - 1$, and so can be identified with the set of forests on $n - 1$ vertices, while M_n can be identified with $\mathbb{C}\{\mathbb{T}\}_n$. The operation of adding a root to a forest on $n - 1$ vertices to produce a rooted tree with n vertices yields an isomorphism $W(1)_n \cong M_n$. Thus, if the map 3.7 is a surjection, it is an isomorphism. This in turn, follows from the fact that M is irreducible.

It suffices to show that M_n contains no lowest-weight vectors for $n > 1$. This follows from an argument similar to the one used to prove 3.1. Let $w \in M_n$, and write

$$w = \alpha_1 t_1 + \cdots \alpha_k t_k$$

where $|t_i| = n$ and we may assume that $\alpha_i \neq 0$. In the notation of 3.1, let $\xi \in St(w)$ be of maximal degree. Then

$$D^-_\xi . w \neq 0$$

Thus, M is irreducible, and hence isomorphic to $W(1)$ by the map 3.7. □

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