TWISTED MODULES AND CO-INVARIANTS FOR COMMUTATIVE VERTEX ALGEBRAS OF JET SCHEMES

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ABSTRACT. Let $Z \subset \mathbb{A}^k$ be an affine scheme over $\mathbb{C}$ and $\mathcal{J}Z$ its jet scheme. It is well-known that $\mathbb{C}[\mathcal{J}Z]$, the coordinate ring of $\mathcal{J}Z$, has the structure of a commutative vertex algebra. This paper develops the orbifold theory for $\mathbb{C}[\mathcal{J}Z]$. A finite-order linear automorphism g of Z acts by vertex algebra automorphisms on $\mathbb{C}[\mathcal{J}Z]$. We show that $\mathbb{C}[\mathcal{J}^g Z]$, where $\mathcal{J}^g Z$ is the scheme of g–twisted jets has the structure of a g-twisted $\mathbb{C}[\mathcal{J}Z]$ module. We consider spaces of orbifold coinvariants valued in the modules $\mathbb{C}[\mathcal{J}^g Z]$ on orbicurves $[Y/G]$, with Y a smooth projective curve and G a finite group, and show that these are isomorphic to $\mathbb{C}[Z^G]$.

1. INTRODUCTION

Let $Z \subset \mathbb{A}^k$ be an affine scheme over $\mathbb{C}$, and

$$\mathcal{J}Z := \mathrm{Hom}_{Sch}(\mathrm{Spec}\,\mathbb{C}[[t]], Z)$$

its jet scheme. It is well-known [4, 3] that the coordinate ring $\mathbb{C}[\mathcal{J}Z]$ has the structure of a commutative vertex algebra. Such vertex algebras often arise as quasiclassical limits of noncommutative vertex algebras, and have found a number of applications, such as in the study of chiral differential operators and the invariant theory of vertex algebras [1, 2, 8]. This paper is devoted to the orbifold theory of the commutative vertex algebra $\mathbb{C}[\mathcal{J}Z]$, or more specifically, to the construction of twisted modules for $\mathbb{C}[\mathcal{J}Z]$ and coinvariants valued in such.

Given a linear automorphism $g : Z \to Z$ of finite order m, we obtain an induced action on $\mathcal{J}Z$ and hence on $\mathbb{C}[\mathcal{J}Z]$ by vertex algebra automorphisms. We may also associate to this data the *g–twisted jet scheme*

$$\mathcal{J}^g Z := \{x(t^{1/m}) \in \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[[t^{1/m}]], Z) | x(e^{2\pi i/m} t^{1/m}) = g(x(t^{1/m}))\}$$

of g-equivariant jets. An abbreviated version of our result is the following :

Theorem (4.1)**.** $\mathbb{C}[\mathcal{J}^g Z]$ carries the structure of g–twisted $\mathbb{C}[\mathcal{J}Z]$–module.

Suppose now that Y is a smooth projective curve with an effective action of the group G. We proceed to study the space of coinvariants for the vertex algebra $\mathbb{C}[\mathcal{J}Z]$ on the orbicurve (or stacky curve) $[Y/G]$. We follow the approach of [5], which entails defining coordinate-independent versions of twisted vertex operators as sections of an certain sheaf on $[Y/G]$. More precisely, we use the g–twisted module structure on $\mathbb{C}[\mathcal{J}^g Z]$ to produce an equivariant section $\mathcal{Y}_y$ near the point $[y/\langle g \rangle]$. Using the sections $\mathcal{Y}_y$, we define a space of coinvariants $H_{G,Z}(Y, \widetilde{y}_1, \cdots, \widetilde{y}_s)$ for the vertex algebra $\mathbb{C}[\mathcal{J}Z]$ valued in the twisted modules $\mathbb{C}[\mathcal{J}^g Z]$. Our result is as follows:

Theorem (5.2)**.** $H_{G,Z}(Y, \widetilde{y}_1, \cdots, \widetilde{y}_s)$ is isomorphic to $\mathbb{C}[Z^G]$ - the coordinate ring of the fixed-point set of G on Z,

When G is the trivial group, and there are no twisted module insertions, the space of coinvariants is simply $\mathbb{C}[Z]$, which recovers a result proven in section 9.4.4 of [4].

The outline of the paper is as follows. In section 2 we recall some basics on vertex algebras and their twisted modules. Section 3 reviews the construction of jet schemes. In section 4 we prove Theorem 4.1. Finally, in section 5 we recall the coordinate-independent construction or orbifold coinvariants from [5], and prove Theorem 5.2.

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2. VERTEX ALGEBRAS AND TWISTED MODULES

In this section, we recall some basic definitions regarding vertex algebras and their twisted modules. We refer the reader to [4, 6] for further information regarding vertex algebras.

Definition 2.1. A *vertex algebra* is a vector space V equipped with:

- a linear map

$$Y : V \to End(V)[[z, z^{-1}]]$$
$$a \to Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$$

- a vector $\mathbf{1} \in V$, called the *vacuum vector*,
- a linear operator $T : V \to V$, called the *translation operator*.

which are required to satisfy the following properties:

(1) $Y(Ta, z) = \partial_z Y(a, z)$,
(2) $Y(\mathbf{1}, z) = id_V$,
(3) $Y(a, z)\mathbf{1} \in V[[z]]$ and $a_{(-1)}\mathbf{1} = a$,
(4) For $m, n, k \in \mathbb{Z}$

$$\sum_{j\geq 0} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)}$$
$$= \sum_{j\geq 0} (-1)^j \binom{n}{j} \left(a_{(m+n-j)} b_{(k+j)} - (-1)^n b_{(n+k-j)} a_{(m+j)} \right).$$

Example 2.1 (Commutative Vertex Algebras)**.** Let A be a commutative algebra over $\mathbb{C}$ equipped with a derivation T_A. We may give A the structure of vertex algebra by taking $V = A$, $T = T_A$, $\mathbf{1} = 1_A$, and defining

$$Y(a, z) = e^{zT}(a) = \sum_{n\geq 0} \frac{z^n}{n!} T^n(a)$$

Conversely, given any vertex algebra V such that $Y(a, z) \in V[[z]], \forall a \in V$, the operation $ab := Y(a, z)b|_{z=0}$ makes V into a commutative algebra with derivation T. We note that for commutative vertex algebras, Y is multiplicative, i..e

$$Y(ab, z) = Y(a, z)Y(b, z).$$

□

A *vertex algebra automorphism* consists of a linear map $g : V \to V$ such that

$$Y(g(a), z) = gY(a, z)g^{-1} \ \forall a \in V,$$

or equivalently, such that

$$g(a)_{(n)} g(b) = g(a_{(n)} b) \ \forall a, b \in V.$$

For an automorphism g of V of finite order m, set

$$V^r = \{u \in V \mid gu = \zeta_m^r u\},\ 0 \leq r \leq m-1,$$

where $\zeta_m = \exp(2\pi i/m)$. We recall the definition of g-twisted V-modules:

Definition 2.2. Let g be an automorphism of V of order m. A g-twisted V-module M is a vector space equipped with a linear map

$$\begin{aligned} Y_M : V &\to End(M)[[z^{1/m}, z^{-1/m}]] \\ a &\to Y_M(a,z) = \sum_{n \in \frac{1}{m}\mathbb{Z}} a_{(n)} z^{-n-1} \end{aligned}$$

which satisfies the following conditions:

(1) $Y_M(a, z^{1/m}) = \sum_{i \in r/p + \mathbb{Z}} u_i z^{-i-1}$ for $a \in V^r$.
(2) $Y_M(a, z^{1/m})v \in M((z^{1/m}))$ for $a \in V$ and $v \in M$.
(3) $Y_M(\mathbf{1}, z^{1/m}) = \mathrm{id}_M$.
(4) For $a \in V^r$, $b \in V^s$, $m \in r/T + \mathbb{Z}$, $n \in s/T + \mathbb{Z}$, and $l \in \mathbb{Z}$,

$$\begin{aligned} &\sum_{i=0}^{\infty} \binom{m}{i} (a_{l+i} b)_{m+n-i} \\ &= \sum_{i=0}^{\infty} \binom{l}{i} (-1)^i \left(a_{l+m-i} b_{n+i} + (-1)^{l+1} b_{l+n-i} a_{m+i} \right). \end{aligned}$$

We note for future reference that the property

$$Y_M(Ta, z^{1/m}) = \partial_z Y_M(a, z^{1/m}) \tag{2.1}$$

holds in any twisted modulle M.

Remark 2.1. It follows from property (4) above by taking $l = -1$ that if V is a commutative vertex algebra as in Example 2.1, then Y_M is multiplicative, i.e.

$$Y_M(ab, z^{1/m}) = Y_M(a, z^{1/m}) Y(b, z^{1/m}).$$

3. JET SCHEMES

Let $Z \subset \mathbb{A}^k$ be an affine scheme. We write $Z = \mathrm{Spec}(A)$, where

$$A = \mathbb{C}[x_1, \cdots, x_k]/(P_1, \cdots, P_r)$$

for some polynomials $P_1, \cdots, P_r \in \mathbb{C}[x_1, \cdots, x_k]$. Recall that the *jet scheme* of Z is the scheme $\mathcal{J}Z$ defined by the property

$$\mathrm{Hom}_{Sch}(\mathrm{Spec}\, R, \mathcal{J}Z) = \mathrm{Hom}_{Sch}(\mathrm{Spec}\, R[[t]], Z)$$

for any commutative $\mathbb{C}$–algebra R. $\mathcal{J}Z$ therefore represents the space of maps from the formal disk $D = \mathrm{Spec}\,\mathbb{C}[[t]]$ to Z. Writing a map $D \to \mathbb{A}^k$ as

$$x_i(t) = \sum_{n \le 0} x_{i,n} t^{-n},\ 1 \le i \le k,$$

$\mathcal{J}Z$ may be explicitly described as $\mathrm{Spec}\, A_\infty$, where

$$A_\infty = \mathbb{C}[x_{1,n}, x_{2,n}, \cdots, x_{k,n}]_{n \le 0} / (P_{1,n}, \cdots, P_{r,n}) \tag{3.1}$$

and

$$P_{i,n} = \frac{\partial_t^n}{n!} P_i(x_1(t), x_2(t), \cdots, x_k(t))|_{t=0} \tag{3.2}$$

Identifying the variables x_i with $x_{i,0}$ we obtain a $\mathbb{C}$–algebra homomorphism $A \to A_\infty$ which is dual to the canonical projection

$$\mu : \mathcal{J}Z \to Z$$

that evaluates a jet at $t = 0$.

Suppose now that $g : \mathbb{A}^k \to \mathbb{A}^k$ is a linear automorphism of order m. After a linear change of coordinates we may diagonalize g such that its action is given by

$$g(x_i) := \xi_m^{\alpha_i} x_i \tag{3.3}$$

with $\xi_m = \exp(2\pi i/m)$. Let

$$\mathcal{J}^g \mathbb{A}^k = \{x(t^{1/m}) \in \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[[t^{1/m}]], \mathbb{A}^k) | x(\xi_m t^{1/m}) = g(x(t^{1/m}))\}.$$

We refer to $\mathcal{J}^g \mathbb{A}^k$ as the scheme of g–twisted jets to $\mathbb{A}^k$. It is the closed subscheme of $\mathcal{J}\mathbb{A}^k = \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[[t^{1/m}]], \mathbb{A}^k)$ consisting of fixed points under the action of g by $g(x(t^{1/m})) := g^{-1} x(\xi_m t^{1/m})$. Writing

$$x_i(t) = \sum_{n \in \frac{1}{m}\mathbb{Z}_{\le 0}} x_{i,n} t^{-n}$$

we see that

$$\mathcal{J}^g \mathbb{A}^k = \mathrm{Spec}\,\mathbb{C}[x_{i,n}],\ \ i = 1 \cdots k, n \in \frac{\alpha_i}{m} + \mathbb{Z}, n \le 0.$$

Suppose furthermore that the action of g on $\mathbb{A}^k$ preserves the affine scheme Z above, or, in other words, that g preserves the ideal $(P_1, \cdots, P_r)$. We may then consider the scheme $\mathcal{J}^g Z$ of g–twisted jets to Z, where

$$\mathcal{J}^g Z := \{x(t^{1/m}) \in \mathrm{Hom}(\mathrm{Spec}\, \mathbb{C}[[t^{1/m}]], Z) | x(\xi_m t^{1/m}) = g(x(t^{1/m}))\}$$

We can write $\mathcal{J}^g Z = \mathrm{Spec}\, A^{\sigma}_{\infty}$, where

$$A^g_\infty = \mathbb{C}[x_{1,n}, x_{2,n}, \cdots, x_{k,n}] / (P^g_{1,n}, \cdots, P^g_{k,n}) \quad n \in \frac{\alpha_i}{m} + \mathbb{Z}, n \leq 0. \tag{3.4}$$

and $P^g_{i,n}$ is the coefficient of t^n in $P_i(x_1(t), x_2(t), \cdots, x_k(t))$

4. Twisted modules from twisted jets

Let $Z \subset \mathbb{A}^k$ be an affine scheme. The algebra $A_\infty = \mathbb{C}[\mathcal{J}Z]$ from (3.1) is equipped with a derivation T defined by

$$T \cdot x_{i,n} = -(n-1) x_{i,n-1}$$

We can write the polynomials $P_{i,n}$ in 3.2 as

$$P_{i,n} = \frac{T^n}{n!} P_{i,0}$$

and thereby write

$$A_\infty = \mathbb{C}[\mathcal{J}Z] = \mathbb{C}[\mathcal{J}\mathbb{A}^k] / (T^n P_{i,0}), \ \ 1 \leq i \leq r, n \geq 0. \tag{4.1}$$

As explained in Example 2.1 (A_∞, T) carries a commutative vertex algebra structure

$$Y : A_\infty \to \mathrm{End}(A_\infty)[[z, z^{-1}]]$$
$$Y(a, z) = e^{zT}(a) = \sum_{n \geq 0} \frac{T^n(a)}{n!} z^n$$

Let g be an automorphism of $\mathbb{A}^k$ of finite order m inducing an automorphism of $Z \subset \mathbb{A}^k$. g acts on $\mathcal{J}Z$ by sending $x(t) \in \mathcal{J}Z$ to $gx(t) := g(x(t))$, inducing an algebra automorphism $\widetilde{g}$ of A_∞ determined by $\widetilde{g}(x_{i,n}) := g(x_i)_n$. After diagonalizing g as in 3.3, this action is given by $\widetilde{g}(x_{i,n}) = \xi_m^{\alpha_i} x_{i,n}$. $\widetilde{g}$ commutes with T, inducing a vertex algebra automorphism of A_∞. Let $A^g_\infty = \mathbb{C}[\mathcal{J}^g Z]$ as in 3.4.

Theorem 4.1. *A_∞^g carries the structure of a $\widetilde{g}$–twisted A_∞ vertex algebra module given by the assignment*

$$Y_g : A_\infty \to \mathrm{End}\, A_\infty^g[[z^{1/m}, z^{-1/m}]] \tag{4.2}$$

where

$$Y_g(x_{i,0}, z^{1/m}) = \sum_{n\in \frac{\alpha_i}{m}+\mathbb{Z}, n\leq 0} x_{i,n} z^{-n} \tag{4.3}$$

and

$$Y_g(x_{i_1,n_1}\cdots x_{i_s,n_s}, z^{1/m}) := \prod_{j=1}^{s} \partial_z^{-n_j} Y(x_{i_j,0}, z^{1/m}) \tag{4.4}$$

Proof. It follows from a twisted version of the reconstruction theorem for vertex operators in [7] that $\mathbb{C}[\mathcal{J}^g\mathbb{A}^k]$ has the structure of $\widetilde{g}$-twisted $\mathbb{C}[\mathcal{J}\mathbb{A}^k]$–module with the field assignment 4.3. It then follows from the multiplicativity of Y_g (see Remark 2.1) and the property 2.1 that Y_g is defined by 4.4 on general elements of $\mathbb{C}[\mathcal{J}\mathbb{A}^k]$. It remains to show that the twisted module structure descends to the quotients A_∞, A_∞^g . We have

$$\begin{aligned} Y_g(P_{i,n}, z^{1/m}) = Y_g(\frac{T^n}{n!}P_{i,0}, z^{1/m}) &= \frac{\partial_z^n}{n!} Y_g(P_{i,0}, z^{1/m}) \\ &= \frac{\partial_z^n}{n!} P_i(x_1(z^{1/m}), ..., x_k(z^{1/m})) \end{aligned}$$

It follows that if P lies in the ideal generated by the $P_{i,n}$, then the coefficients of the field $Y_g(P, z^{1/m})$ lie in the ideal generated by the $P^g_{j,m}$, hence that Y_g induces a well-defined structure on A_∞^g as a $\widetilde{g}$–twisted A_∞–module. □

4.1. **Quasi-conformal structure.** We recall the quasi-conformal structure on the vertex algebra A_∞ following [4]. It will be used to define coordinate-independent versions of twisted vertex operators in the next section. Let $\mathrm{Aut}\,\mathcal{O}$ denote the group of algebra automorphisms of $\mathbb{C}[[z]]$. An automorphism $\rho \in \mathrm{Aut}\,\mathcal{O}$ is determined by where it sends the generator z, and can be written as

$$\rho(z) = a_1 z + a_2 z^2 + \cdots$$

where $a_1 \neq 0$. We think of $\mathrm{Aut}\,\mathcal{O}$ as the automorphism group of the formal disk $D = \mathrm{spec}\,\mathbb{C}[[z]]$ preserving the origin. The Lie algebra of $\mathrm{Aut}\,\mathcal{O}$ is spanned by the vector fields $L_n = -z^{n+1}\partial_z,\ n \geq 0$, satisfying the commutation relations

$$[L_m, L_n] = (n-m)L_{n+m}$$

The action of $\operatorname{Aut}\mathcal{O}$ on $\operatorname{Spec}\mathbb{C}[[z]]$ induces an action on the jet scheme $\mathcal{J}Z$ and hence on $A_\infty = \mathbb{C}[\mathcal{J}Z]$. The action of $Lie(\operatorname{Aut}\mathcal{O})$ may be written explicitly as

$$L_m \to \sum_{i=1}^{k} \sum_{n<0} -n x_{i,n} \frac{\partial}{\partial x_{i,n-m}}$$

Consider now the map

$$f_m : \operatorname{Spec}\mathbb{C}[[z^{1/m}]] \to \operatorname{Spec}\mathbb{C}[[z]]$$

induced by the inclusion $\mathbb{C}[[z]] \subset \mathbb{C}[[z^{1/m}]]$. We think of $\operatorname{Spec}\mathbb{C}[[z^{1/m}]]$ as an m-th order ramified cover of $\operatorname{Spec}\mathbb{C}[[z]]$. Let $\operatorname{Aut}^{(m)}\mathcal{O}$ be the group of automorphisms of $\mathbb{C}[[z^{1/m}]]$ preserving the subring $\mathbb{C}[[z]]$. $\psi \in \operatorname{Aut}^{(m)}\mathcal{O}$ may be written as

$$\psi(z^{1/m}) = \sum_{n \geq 0} a_{\frac{1}{m}+n} z^{\frac{1}{m}+n}$$

where $a_{\frac{1}{m}} \neq 0$. $\operatorname{Aut}^{(m)}\mathcal{O}$ can be thought of as the group of m-th roots of coordinate changes on $\operatorname{Spec}\mathbb{C}[[z]]$. There is a short exact sequence of groups

$$1 \to \mathbb{Z}/m\mathbb{Z} \to \operatorname{Aut}^{(m)}\mathcal{O} \stackrel{h_m}{\to} \operatorname{Aut}\mathcal{O} \to 1$$

where $h_m(\psi)(z) = (\psi(z^{1/m}))^m$. The Lie algebra of $\operatorname{Aut}^{(m)}\mathcal{O}$ is spanned by the vector fields $\widetilde{L}_n = -z^{1/m+n}\partial_{z^{1/m}},\ n \geq 0$, and h_m induces a Lie algebra isomomorphism

$$h_{m,*} : Lie(\operatorname{Aut}^{(m)}\mathcal{O}) \to Lie(\operatorname{Aut}\mathcal{O})$$
$$\widetilde{L}_n \to mL_n$$

The action of $\operatorname{Aut}^{(m)}\mathcal{O}$ on $\operatorname{Spec}\mathbb{C}[[z^{1/m}]]$ induces an action on $\mathcal{J}^g Z$ and hence on A^g_∞. The action of $Lie(\operatorname{Aut}^{(m)}\mathcal{O})$ may be written explicitly as

$$\widetilde{L}_r \to \sum_{i=1}^{k} \sum_{n \in \alpha_i/m+\mathbb{Z}, n<0} -nm x_{i,n} \frac{\partial}{\partial x_{i,n-r}}$$

5. Coinvariants and conformal blocks

In this section, we study the spaces of coinvariants and conformal blocks for the vertex algebra A_∞ on a stacky curve (i.e. orbicurve) with values in twisted modules of type A_∞^g. We begin by briefly recalling the definition and construction of these spaces following the approach in [5], where we refer the reader for details.

Let Y be a smooth complex projective curve. Given a point $y \in Y$, we denote by $\mathcal{O}_y$ the local ring at y, $\widehat{\mathcal{O}}_y$ its completion, and $\widehat{m}_y \subset \widehat{\mathcal{O}}_y$ the maximal ideal. We refer to a generator of $\widehat{m}_y$ as a *formal coordinate* at y. Given such a formal coordinate z at y, we may canonically identify $\widehat{\mathcal{O}}_y$ with $\mathbb{C}[[z]]$, and $\widehat{m}_y$ with $z\mathbb{C}[[z]]$. Let

$$\widehat{Y} = \{(y, t_y) | y \in Y, t_y \in \widehat{\mathcal{O}}_y, (t_y) = \widehat{m}_y\}$$

be the bundle of formal coordinates over Y. $\widehat{Y}$ is an $\operatorname{Aut}\mathcal{O}$–principal bundle, where the latter acts by changes of formal coordinates. $\operatorname{Aut}\mathcal{O}$ acts on A_∞ as explained in section 4.1, and we may therefore form the associated bundle

$$\mathcal{A} = \widehat{Y} \underset{\operatorname{Aut}\mathcal{O}}{\times} A_\infty \tag{5.1}$$

Given an open subset $U \subset Y$ equipped with an etale coordinate $f : U \to \mathbb{C}$, we obtain a section, hence trviailization

$$\begin{aligned} j_f : U &\to \widehat{Y}|_U \\ y &\to (y, f - f(y)) \end{aligned}$$

This induces a trivialization of $\mathcal{A}$

$$\begin{aligned} \tilde{j}_f : U \times A_\infty &\to \mathcal{A} \\ (y, p(x_{i,n})) &\to [y, f - f(y), p(x_{i,n})] \end{aligned}$$

where the notation $[y, f - f(y), p(x_{i,n})]$ is used to denote the equivalence class of element of $\mathcal{A}$ whose $\widehat{Y}$ component is $(y, f - f(y))$ and whose A_∞ component is $p(x_{i,n})$.

$\mathcal{A}$ is a bundle of commutative algebras over Y, and denoting by $\mathcal{A}_y$ the fiber of $\mathcal{A}$ at $y \in Y$, and $\operatorname{Spec}\widehat{\mathcal{O}}_y$ by $\mathbb{D}_y$ we have a canonical identification

$$\operatorname{Spec}\mathcal{A}_y = \operatorname{Hom}(\mathbb{D}_y, Z)$$

The relative spectrum Spec $\mathcal{A} \to Y$ is therefore identified with the bundle of jets of sections of $Y \times Z \to Y$, which we denote $\mathcal{J}_{Z\times Y/Y}$. $\mathcal{A}$ carries a canonical flat connection, which in a local coordinate z is given by $\nabla_{\partial_z} = \partial_z - T$. Horizontal sections of $\mathcal{A}$ over $U \subset Y$ are holomorphic maps $U \to Z$, and since Y is projective and Z affine, global horizontal sections are constant maps to Z.

Remark 5.1. Note that the subring $A \subset A_\infty$ generated by the images of $x_{i,0}$ forms a trivial Aut $\mathcal{O}$ sub-representation of A_∞. Forming the associated bundle 5.1 and taking relative spectra, we obtain the canonical map

$$\mu : \mathcal{J}_{Z\times Y/Y} \to Z$$

which associates to the jet of a section $\phi \in \mathrm{Hom}(\mathbb{D}_y, Z)$ its value $\phi(y)$.

Suppose now that G is a finite group acting effectively on Y, and that G acts on $\mathbb{A}^k$ preserving $Z \subset \mathbb{A}^k$. We may then define an action of G on $\mathcal{A}$ by

$$g \cdot [y, t_y, p(x_{i,n})] = [gy, t_y \circ g^{-1}, p(x_{i,n} \circ g^{-1})]$$

which commutes with the flat connection ∇. This G–equivariant structure on $\mathcal{A}$ allows us to descend it to a sheaf of algebras $\mathcal{A}_G$ over the orbicurve (or stacky curve) $[Y/G]$ equipped with a flat connection ∇^G.

Denote by

$$\pi : Y \to [Y/G]$$

the projection to the quotient. Points of $[Y/G]$ will be denoted by $\widetilde{y}_1, \widetilde{y}_2, \cdots$. Given $\widetilde{y} \in [Y/G]$, we may write

$$\widetilde{y} = [y/G_y],$$

where $y \in \pi^{-1}(\widetilde{y})$, and G_y is the stabilizer of y in G. A point $\widetilde{y}$ with non-trivial G_y is called a *stacky point*. It is well-known that G_y cyclic, and we may choose a generator $g \in G_y$ and an etale coordinate $z^{1/m}$ in a neighborhood U centered at y, with $m = |G_y|$, such that $z^{1/m} \circ g^{-1} = \xi_m z^{1/m}$. We call a coordinate possessing this property *special*. The formal neighborhood of $\widetilde{y}$ in $[Y/G]$ can be described as the stack

$$[\mathbb{D}_y/G_y] = [\mathbb{D}_y/\langle g \rangle]$$

The twisted module structure on A^g_∞, $g \in G$ may be used to construct local sections of $\mathcal{A}^*_G$ on $[Y/G]$ as follows. Let

$$\mathcal{J}_{\widetilde{y}} Z := \mathrm{Hom}([\mathbb{D}_y/G_y], [Z/G_y])$$

$\mathcal{J}_{\widetilde{y}} Z$ is isomorphic to

$$\mathrm{Spec}\, \mathbb{C}[[t^{1/m}]] \underset{\mathrm{Aut}^{(m)}\mathcal{O}}{\times} \mathcal{J}^g Z$$

and the choice of special formal coordinate $z^{1/m}$ allows us to identify $\mathcal{J}_{\widetilde{y}} Z$ with the twisted jet scheme $\mathcal{J}^g Z$. We thus obtain an isomorphism

$$\lambda_{z^{1/m}} : \mathcal{A}_{\widetilde{y}} := \mathbb{C}[\mathcal{J}_{\widetilde{y}} Z] \to A^g_\infty.$$

Denote by

$$Y_g : A_\infty \to \mathrm{End}\, A^g_\infty[[z^{1/m}, z^{-1/m}]]$$

the operator associated with the twisted vertex algebra module structure from Theorem 4.1, $\mathcal{K}_y$ the field of fractions of $\widehat{\mathcal{O}}_y$, and $\mathbb{D}^\times_y = \mathrm{Spec}\, \mathcal{K}_y$. We have $\mathcal{K}_y \simeq \mathbb{C}((z^{1/m}))$.

We proceed to define a section

$$\mathcal{Y}_{\widetilde{y}} \in \Gamma([\mathbb{D}^\times_y/G_y], \mathcal{A}^*_G \otimes \mathrm{End}(\mathcal{A}_{\widetilde{y}})).$$

Note that $z = (z^{1/m})^m$ is an etale coordinate on $U \backslash y$, and therefore yields a trivialization

$$\widetilde{j}_z : (U \backslash y) \times A_\infty \to \mathcal{A}|_{U \backslash y}$$

which may be pulled back to $\mathbb{D}^\times_y$. We define $\mathcal{Y}_{\widetilde{y}}$ by the property that

$$\langle \lambda^t_{z^{1/m}}(\phi), \mathcal{Y}_{\widetilde{y}}(\widetilde{j}_z(p)) \cdot \lambda^{-1}_{z^{1/m}}(q) \rangle := \langle \phi, Y_g(p, z^{1/m}) \cdot q \rangle \in \mathbb{C}((z^{1/m})) \tag{5.2}$$

where $\phi \in (A^g_\infty)^*, p \in A_\infty, q \in A^g_\infty$. The following is an immediate consequence of Theorem 5.1 of [5], and provides a coordinate-independent description of the twisted vertex operation Y_g.

Theorem 5.1. *The section $\mathcal{Y}_{\widetilde{y}}$ is independent of the choice of special coordinate $z^{1/m}$ and point $y \in \pi^{-1}(\widetilde{y})$, and thus defines a canonical element*

$$\mathcal{Y}_{\widetilde{y}} \in \Gamma([\mathbb{D}^\times_y/G_y], \mathcal{A}^*_G \otimes \mathrm{End}(\mathcal{A}_{\widetilde{y}})).$$

Remark 5.2. We think of the stack $[\mathbb{D}^\times_y/G_y]$ as a "punctured formal disk" around $\widetilde{y} \in [Y/G]$.

We now define the spaces of coinvariants and conformal blocks for the commutative vertex algebra A_∞ on $[Y/G]$. Let $\{\widetilde{y}_1, \cdots, \widetilde{y}_s\}$ be a non-empty set of points of $[Y/G]$ which includes all the stacky points. Let

$$Y^\circ = Y \backslash \pi^{-1}(\widetilde{y}_1) \cup \cdots \cup \pi^{-1}(\widetilde{y}_s).$$

We have

$$[Y^\circ/G] = Y^\circ/G = [Y/G] \backslash \{\widetilde{y}_1, \cdots, \widetilde{y}_s\}.$$

Let

$$\mathcal{A}_{out} = \Gamma([Y^\circ/G], \mathcal{A}_G \otimes \Omega^1),$$

or equivalently, the G–invariant sections of $\mathcal{A} \otimes \Omega^1$ over Y°. For each $\widetilde{y}_j$, there is a map

$$\alpha_j : \mathcal{A}_{out} \to \mathrm{End}(\mathcal{A}_{\widetilde{y}_j})$$

given by

$$\omega \in \mathcal{A}_{out} \to Res_{\widetilde{y}_j} \langle \omega, \mathcal{Y}_{\widetilde{y}_j} \rangle \in \mathrm{End}(\mathcal{A}_{\widetilde{y}_j})$$

$\omega \in \mathcal{A}_{out}$ thus acts on $\mathcal{A}_{\widetilde{y}_1} \otimes \cdots \otimes \mathcal{A}_{\widetilde{y}_s}$ by

$$s \cdot (a_1 \otimes \cdots \otimes a_s) := \sum_{j=1}^{s} a_1 \otimes \cdots \otimes \alpha_j(s) \cdot a_j \otimes \cdots \otimes a_s$$

Definition 5.1. Let G, Y, Z, $\widetilde{y}_1, \cdots, \widetilde{y}_s \in [Y/G]$ be as above. The *space of coinvariants* for the vertex algebra $\mathcal{J}Z$ on the orbicurve $[Y/G]$ is

$$H_{G,Z}(Y, \widetilde{y}_1, \cdots, \widetilde{y}_s) := \mathcal{A}_{\widetilde{y}_1} \otimes \cdots \otimes \mathcal{A}_{\widetilde{y}_s} / \mathcal{A}_{out} \cdot (\mathcal{A}_{\widetilde{y}_1} \otimes \cdots \otimes A_{\widetilde{y}_s})$$

where $\mathcal{A}_{out} \cdot (\mathcal{A}_{\widetilde{y}_1} \otimes \cdots \otimes A_{\widetilde{y}_s})$ denotes the ideal generated by $\mathcal{A}_{out}$ in the algebra $\mathcal{A}_{\widetilde{y}_1} \otimes \cdots \otimes A_{\widetilde{y}_s})$.

The dual space

$$C_{G,Z}(Y, \widetilde{y}_1, \cdots, \widetilde{y}_s) := \mathrm{Hom}_{\mathbb{C}}(H_{G,Z}(Y, \widetilde{y}_1, \cdots, \widetilde{y}_s), \mathbb{C})$$

is called the *space of conformal blocks* for the vertex algebra $\mathcal{J}Z$ on the orbicurve $[Y/G]$.

Remark 5.3. $H_{G,Z}(Y, \widetilde{y}_1, \cdots, \widetilde{y}_s)$, being the quotient of a commutative algebra by an ideal, has the structure of a $\mathbb{C}$-algebra.

The definition of $C_{G,Z}(Y, \widetilde{y}_1, \cdots, \widetilde{y}_s)$ above is given in terms of the of the orbicurve $[Y/G]$ and the sheaf $\mathcal{A}_G$ on $[Y/G]$. As explained in [5], by using the strong residue theorem, we may restate the definition in terms of curve Y and the G–equivariant structure on $\mathcal{A}$ as follows:

Definition 5.2. Let $G, Y, Z, \widetilde{y}_1, \cdots, \widetilde{y}_s \in [Y/G]$ be as above. Choose $y_j \in \pi^{-1}(\widetilde{y}_j)$ for $j = 1, \cdots, s$. The space of *conformal blocks* for the vertex algebra $\mathcal{J}Z$ on the orbicurve $[Y/G]$ is

$$C_{G,Z}(Y, \widetilde{y}_1, \cdots, \widetilde{y}_s) := \varphi \in (A_{\widetilde{y}_1} \otimes \cdots A_{\widetilde{y}_s})^*$$

such that

$$\forall\, a_1 \in A_{\widetilde{y}_1}, \cdots, a_s \in A_{\widetilde{y}_s}$$

the sections

$$\langle \varphi, a_1 \otimes \cdots \otimes \mathcal{Y}_{\widetilde{y}_j} \cdot a_j \otimes \cdots \otimes a_s \rangle \ \in \Gamma(\mathbb{D}^{\times}_{y_j}, \mathcal{A}^*) \ j = 1, \cdots, s \tag{5.3}$$

extend to a single G–invariant horizontal section $\mathcal{Y}_\varphi$ of $\mathcal{A}^*$ on Y°.

Remark 5.4. As explained in [5], the space $C_{G,Z}(Y, \widetilde{y}_1, \cdots, \widetilde{y}_s)$ is independent of the choice of $y_j \in \pi^{-1}(\widetilde{y}_j)$. This choice is made only to provide a concrete model of the formal punctured neighborhood of $\widetilde{y}_j \in [Y/G]$ as $[\mathbb{D}^{\times}_{y_j}/G_{y_j}]$

We now state our main result regarding the spaces $H_{G,Z}(Y, \widetilde{y}_1, \cdots, \widetilde{y}_s)$. Denote by $Z^G \subset Z$ the closed sub-scheme of G–fixed points, and $\mathbb{C}[Z^G]$ its coordinate ring.

Theorem 5.2. *The space of coinvariants $H_{G,Z}(Y, \widetilde{y}_1, \cdots, \widetilde{y}_s)$ is isomorphic to $\mathbb{C}[Z^G]$.*

Proof. By remark 5.3, $H_{G,Z}(Y, \widetilde{y}_1, \cdots, \widetilde{y}_s)$ has the structure of commutative $\mathbb{C}$–algebra. Let

$$\varphi : H_{G,Z}(Y, \widetilde{y}_1, \cdots, \widetilde{y}_s) \to \mathbb{C}$$

be a closed point of $\operatorname{Spec} H_{G,Z}(Y, \widetilde{y}_1, \cdots, \widetilde{y}_s)$. Then in particular, $\varphi \in C_{G,Z}(Y, \widetilde{y}_1, \cdots, \widetilde{y}_s)$, and we denote by $\omega_\varphi \in \Gamma(Y^\circ, \mathcal{A}^*)$ the G–invariant horizontal section of $\mathcal{A}^*$ which near $y_j \in \pi^{-1}(\widetilde{y}_j)$ agrees with

$$\langle \varphi, 1 \otimes \cdots \otimes \mathcal{Y}_{\widetilde{y}_j} \cdot 1 \otimes \cdots \otimes 1 \rangle \ \in \Gamma(\mathbb{D}^{\times}_{y_j}, \mathcal{A}^*)$$

for $j = 1, \cdots, s$. As explained in Proposition of [4], for each $y \in Y^\circ$ the restriction

$$\omega_\varphi : \mathcal{A}_y \to \mathbb{C}$$

is a ring homomorphism, whose associated point of the jet scheme $\mathrm{Hom}(\mathbb{D}_y, Z)$ is the jet at y of a map

$$h_\varphi : Y^\circ \to Z.$$

The G–invariance of ω_φ ensures that h_φ is G–invariant as well, i.e. satisfies $h_\varphi(g \cdot y) = g \cdot h_\varphi(y)$. We now show that h_φ extends to all of Y. It follows from Remark 5.1 that the composition

$$\mathbb{C}[Z] \to \mathcal{A}_y \overset{\omega_\varphi}{\to} \mathbb{C}$$

is simply the map

$$p \to p(h_\varphi(y))$$

If $y_j \in \pi^{-1}(\widetilde{y}_j)$, then after fixing a special coordinate $z^{1/m}$ near y_j, we have that on $\mathbb{D}^\times_{y_j}$,

$$\omega_\varphi(\widetilde{j_z}(p)) = \langle \varphi, Y(p, z^{1/m}) \cdot 1 \rangle \in \mathbb{C}((z^{1/m})) \tag{5.4}$$

However, note that for $p \in \mathbb{C}[Z] \subset A_\infty$, $Y(p, z^{1/m}) \in \mathbb{C}[[z^{1/m}]]$. Thus, the limit $z^{1/m} \to 0$ is well-defined, showing that $p(h_\varphi(y))$ is well-defined, hence that h_φ extends to y_j. Since Y is projective, and Z affine, h_φ is constant. The G–invariance forces the image to lie in Z^G.

□

Remark 5.5. When G is trivial, and $s = 1$, we recover a result proven in section 9.4.4 of [4] identifying the space of one-point coinvariants of the vertex algebra A_∞ on Y with $\mathbb{C}[Z]$.

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