MA 123 - Practice Final Exam

Name:

Instructions: For some of the questions, you must show all your work as indicated. No calculators, books or notes of any form are allowed.

Note that the questions have different point values. Pace yourself accordingly.

Good luck!

Question	Score	Out of
1		20
2		20
3		20
4		20
5		10
6		30
7		10
8		15
9		20
Total		165

1. (**20 points**). Differentiate the following functions:

a) $f(x) = \sin(x^3 + 1)$

$$f'(x) = 3x^2 \cos(x^3 + 1)$$

b) $f(x) = \sqrt{e^{2x} + 3x}$

$$f'(x) = \frac{1}{2}(e^{2x} + 3)^{-1/2}(2e^{2x} + 3)$$

2. (**20 points**).

Evaluate the following limits:

a) $\lim_{x \to -1} \frac{x^2-1}{x+1}$

$$= \lim_{x \to -1} (x+1) = 0$$

b) $\lim_{x \to -\infty} x^2 e^x$

$$= \lim_{x \to -\infty} \frac{x^2}{e^{-x}} = \lim_{x \to -\infty} \frac{2x}{-e^{-x}} = \lim_{x \to -\infty} \frac{2}{e^{-x}} = 0$$

3. (**20 points**). A kite 100 ft. above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft. of string have been let out ?

Let $H(t)$ be the horizontal distance of the kite from the person holding the string. We have

$$dH/dt = 8ft/s$$

We have

$$\frac{H(t)}{100} = \cot(\theta)$$

Differentiating, we get

$$\frac{H'(t)}{100} = -\csc(\theta)\cot(\theta)\theta'(t)$$

or

$$\theta'(t) = -\frac{8}{100\csc(\theta)\cot(\theta)}$$

When 200 ft. of string have been let out, $\csc(\theta) = 2, \cot(\theta) = \sqrt{3}$

4. (**20 points**). A piece of wire 10 m long is cut into two pieces. Each piece is bent into a square. How should the wire be cut so that the total area enclosed is minimal ?

The pieces have length $x, 10-x$. The total area contained is

$$A(t) = \frac{1}{16}(x^2 + (10-x)^2) = \frac{1}{16}(2x^2 - 20x + 100)$$

This is a parabola opening up. Its minimum occurs at $x = 5$.

5. (**10 points**). Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} e^{1+i\frac{2}{n}} \frac{2}{n}$

This is the limit of Riemann sums for the integral

$$\int_1^3 e^x dx = e^3 - e$$

The limit is therefore $e^3 - e$

6. (**30 points**). Evaluate

a) $\int_1^2 (3x^2 + e^x)dx$

$$= (x^3 + e^x)|_1^2 = (8 + e^2) - (1 + e) = 7 + e^2 - e$$

b) $\int \sqrt{2x+1}dx$

Making the substitution $u = 2x + 1$

$$= \frac{1}{2}\int \sqrt{u}du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(2x+1)^{3/2} + C$$

c) $\int \frac{(ln(x))^2}{x} dx$ Making the substitution $u = ln(x)$

$$= \int u^2 du = \frac{1}{3}u^3 = \frac{1}{3}(ln(x))^3$$

7. (**10 points**). Evaluate

$$\frac{d}{dx}\int_1^x \sin(e^u+1)du$$

By the Fundamental theorem of calculus, this is

$$\sin(e^x+1)$$

8. (**15 points**). The acceleration of a particle at time t seconds is given by

$$a(t) = 3t + 1 \ \ m/s^2$$

Find its velocity and position functions $v(t), s(t)$ if $v(0) = 1m/s$ and $s(0) = 4m$

$$v(t) = 3/2t^2 + t + C$$

and the requirement that $v(0) = 1$ implies $C = 1$

$$s(t) = 1/2t^3 + 1/2t^2 + t + D$$

and the requirement that $s(0) = 4$ implies $D = 4$.

9. (**20 points**). If f is a differentiable function such that $f(x)$ is never 0 and

$$\int_0^x f(t)dt = [f(x)]^2$$

for all x, find $f(x)$.

Differentiating, and using the FTC, we get

$$f(x) = 2f(x)f'(x)$$

or

$$f(x)(1 - 2f'(x)) = 0$$

Since $f(x) \neq 0$, this means $1 - 2f'(x) = 0$, or that $f(x) = 1/2x + C$. Substituting back in the original equation we get

$$\int_0^x (1/2t + C)dt = (1/2x + C)^2$$

or

$$x^2/4 + Cx = x^2/4 + Cx + C^2$$

Thus $C = 0$, and the only such function is $f(x) = x/2$.