MA 123 - Practice Exam #2 Solutions

Name:

Instructions: For some of the questions, you must show all your work as indicated. No calculators, books or notes of any form are allowed.

Note that the questions have different point values. Pace yourself accordingly.

Good luck!

Question	Score	Out of
1		60
2		15
3		15
4		20
5		15
6		40
7		30
8		18
9		20
Total		233

1. (**60 points**). Differentiate the following functions:

a) $f(x) = e^{(x^2+1)}$

$$f' = 2xe^{(x^2+1)}$$

b) $f(x) = \frac{sin(2x)}{e^{2x}-1}$

$$f' = \frac{2e^{2x}(cos(2x) - sin(2x)) - 2cos(2x)}{(e^{2x} - 1)^2}$$

c) $f(x) = x^2 \ln(cos^2(x) - 1)$

$$f' = 2x \ln(cos^2(x) - 1) - \frac{2x^2 cos(x) sin(x)}{cos^2(x) - 1}$$

d) $f(x) = \frac{3^x tan(x)(x+1)^{10}}{(x^2-1)(x+2)}$

$$\ln(f) = x\ln(3) + \ln(tan(x)) + 10\ln(x+1) - \ln(x^2-1) - \ln(x+2)$$

$$\frac{f'}{f} = \ln(3) + \frac{sec^2(x)}{tan(x)} + \frac{10}{x+1} - \frac{2x}{x^2-1} - \frac{1}{x+2}$$

$$f' = f\left(\ln(3) + \frac{sec^2(x)}{tan(x)} + \frac{10}{x+1} - \frac{2x}{x^2-1} - \frac{1}{x+2}\right)$$

e) $f(x) = x^{\sqrt{x}}$

$$\ln(f) = \sqrt{x}\ln(x)$$

$$\frac{f'}{f} = \frac{1}{2\sqrt{x}}\ln(x) + \sqrt{x}\frac{1}{x}$$

$$f' = f(\frac{\ln(x)+2}{2\sqrt{x}}) = x^{\sqrt{x}}(\frac{\ln(x)+2}{2\sqrt{x}})$$

f) $f(x) = arcsin(3x-1)$

$$f' = \frac{3}{\sqrt{1-(3x-1)^2}}$$

2. **15 points**. Find the equation of the tangent line to the curve $x^{2/3} + y^{3/5} = 2$ at the point $(1, 1)$.

$$\frac{2}{3}x^{-1/3} + \frac{3}{5}y^{-2/5}y' = 0$$

$$y' = -\frac{10}{9}x^{-1/3}y^{2/5}$$

At $(1, 1)$,

$$y' = -\frac{10}{9}$$

The equation of the tangent line is therefore

$$y - 1 = -\frac{10}{9}(x - 1)$$

3. (**15 points**) Evaluate

$$\lim_{n\to\infty}(1+\frac{2}{n})^n$$

Recall that

$$\lim_{x\to 0}(1+x)^{1/x}=e$$

Let

$$y=\frac{2}{n}$$

Then the limit can be rewritten as

$$\lim_{y\to 0}(1+y)^{2/y}=(\lim_{y\to 0}(1+y)^{1/y})^2=e^2$$

4. (**20 points**)

A trough is 10 ft. long, and its ends have the shape of isoceles triangles that are 3 ft. across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 12 ft^3/min., how fast is the water level rising when the water is 6 inches deep ?

Let V denote the volume of water in the trough in ft^3, h the depth of the water, and b the "base" of the triangle formed by the water (both in ft). We have

$$V = 1/2bh(10ft) = 5bhft^3$$

By similarity

$$\frac{b}{3} = \frac{h}{1}$$

so

$$V = 15h^2 ft^3$$

Differentiating

$$dV/dt = 30hdh/dt$$

$$dh/dt = \frac{dV/dt}{30h}$$

Substitute $dV/dt = 12ft^3, h = 1/2ft$ (remember that we are working in feet, not inches).

5. (**15 points**)

Find the absolute maximum and absolute minimum of the function

$$f(x) = x - \ln(x)$$

on the interval $[\frac{1}{2}, 2]$.

$$f' = 1 - \frac{1}{x}$$

Critical numbers are $x = 1$. $f(1) = 1$. Checking endpoints, we get

$$f(1/2) = 1/2 - \ln(1/2) = 1/2 + \ln(2) = 1.193..$$

$$f(2) = 2 - \ln(2) = 1.307...$$

Thus the absolute maximum is $f(2) = 1.307..$ and the absolute minimum is $f(1) = 1$. Note that this problem was difficult to do without a calculator. On an actual test, the numbers would be easier.

6. (**40 points**) Let $f(x) = \ln(x^4 + 27)$.

(a) Find the intervals of increase / decrease of $f(x)$.

$$f' = \frac{4x^3}{x^4 + 27}$$

Increasing on $x > 0$, decreasing on $x < 0$.

(b) Find the local minima and maxima.

Derivative only changes sign at $x = 0$. This is a local minimum, with value $ln(27) = 3ln(3)$.

(c) Find the intervals of concavity and the inflection points

$$f'' = \frac{-4x^2(x^4 - 81)}{(x^4 + 27)^2}$$

f'' vanishes at $0, 3, -3$. f'' is : negative when $x < -3$, positive when $-3 < x < 0$, positive when $0 < x < 3$, and negative when $x > 3$. The inflection points are therefore -3 and $+3$.

(d) Sketch the graph.

7. (**30 points**) Evaluate the following limits

a) $\lim_{x\to 1}(\frac{1}{ln(x)} - \frac{1}{x-1})$

$$= \lim_{x\to 1} \frac{x-1-\ln(x)}{\ln(x)(x-1)} = \lim_{x\to 1} \frac{1-1/x}{\ln(x)+1-1/x} = \lim_{x\to 1} \frac{1/x^2}{1/x+1/x^2} = \frac{1}{2}$$

b) $\lim_{x\to 0} \cot(2x)\sin(6x)$

$$= \lim_{x\to 0} \frac{\cos(2x)\sin 6x}{\sin(2x)} = (\lim_{x\to 0} \cos(2x))(\lim_{x\to 0} \frac{\sin 6x}{\sin(2x)}) = 1 \times \lim_{x\to 0} \frac{6\cos 6x}{2\cos(2x)}) = 3$$

c) $\lim_{x\to 0^+} x^{x^2}$

$$y = x^{x^2}$$

$$\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} x^2 \ln(x) = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} = \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+} -1/2x^2 = 0$$

Therefore

$$\lim_{x\to 0^+} x^{x^2} = e^0 = 1$$

8. (**18 points.**)

Determine whether the following statements are true or false. If true, give a complete argument (referring to theorems, properties etc.). If false, give a counterexample.

(a) Suppose that f is a continuous function on the interval $[1, 3]$. A global max./min. of f always occurs at a critical number.

FALSE. The global max./min. may occur at an endpoint. Take for instance the function $f(x) = x$. Its global max./min. occur at the endpoints.

(b) Suppose that f is a continuous function on the open interval $(1, 2)$. Then there exist $1 < c < 2$ and $1 < d < 2$ such that $f(c)$ is the global maximum of f on $(1, 2)$ and $f(d)$ is the global minimum of f on $(1, 2)$.

FALSE. There is no guarantee that a function achieves either a global max. or min. on an open interval. Again, $f(x) = x$ has neither on the open interval in question.

(c) Suppose that f is a differentiable function on $(1,2)$ that has a vertical asymptote at $x = 2$. Then both of the following statements cannot be true:

i. f achieves an absolute minimum on $(1,2)$

ii. f achieves an absolute maximum on $(1,2)$

TRUE. The vertical asymptote at 2 implies that

$$\lim_{x \to 2^-} f(x) = +\infty$$

or

$$\lim_{x \to 2^-} f(x) = -\infty$$

. In the first case, the function will fail to have an absolute max., and in the second an absolute min.

9. (**20 points**)

In an automobile race along a straight road, car A passed car B twice. Prove that at some time during the race their accelerations were equal. (Prove means give a rigorous, concise, complete mathematical argument, referring to theorems etc. as needed).

Let $s_A(t)$ and $s_B(t)$ be the position functions of cars A and B resp. Let

$$F(t) = s_A(t) - s_B(t)$$

The fact that A passes B twice implies that there exist (at least) three times $t_1 < t_2 < t_3$ when their positions are equal. t_1 is the time when A overtakes B the first time, t_2 when B overtakes A, and t_3 the time when A again overtakes B. Thus

$$F(t_1) = F(t_2) = F(t_3) = 0$$

By the mean value theorem, there exists a time u_1, $t_1 < u_1 < t_2$ where $F'(u_1) = 0$ and similarly, a time $t_2 < u_2 < t_3$ such that $F'(u_2) = 0$. Applying the mean value theorem again, this time to the function $F'(t)$, we see that there is a time r, $u_1 < r < u_2$, such that $F''(r) = 0$. Now, $F''(t) = s''_A(t) - s''_B(t)$, so $s''_A(r) = s''_B(r)$ - i.e. the accelerations are equal at r as desired.