MA 242 – Linear Algebra
Final Exam

Name:

Instructions: For each question, to receive full credit you must show all work. Explain your answers fully and clearly. You may refer to theorems in the book or from class unless the question specifically states otherwise. No calculators, books or notes of any form are allowed.

Note that the questions have different point values. Pace yourself accordingly.

Good luck!

Question	Score	Out of
1		10
2		8
3		10
4		15
5		8
6		12
7		10
8		10
9		10
10		28
Total		121

1. **(10 points)**

(a) Find the general solution of the following system of equations.

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 + 4x_3 &= 1 \\ 3x_1 + x_2 + 5x_3 &= 2 \end{aligned}$$

(b) Describe the geometric shape of the collection of all solutions to the above equations considered as a subset of $\mathbf{R}^3$.

2. **(8 points)** Let A be a 3×3 matrix such that the equation

$$A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

has both $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ as solutions. Find another solution to this equation. Explain.

3. **(10 points)**

Let

$$\mathcal{B} = \{1 + 2x, x - x^2, x + x^2\}$$

(a) Show that $\mathcal{B}$ is a basis for $\mathbb{P}_2$

(b) Let $p(x) = 1 + 3x + x^2$. Compute $[p(x)]_{\mathcal{B}}$.

(c) Let $T : \mathbb{P}_2 \mapsto \mathbb{P}_2$ be the transformation $T(p(t)) = p^{'}(t) - p(t)$. Write down the matrix of T with respect to the basis $\mathcal{B}$.

4. **(20 points)**

Let $W \subset \mathbb{R}^4$ be the subspace of vectors (x_1, x_2, x_3, x_4) satisfying

$$2x_1 - x_3 + x_4 = 0$$

Find an orthonormal basis for W.

5. **(8 points)**

Let A be a 2×2 matrix such that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for A with eigenvalue 2 and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector for A with eigenvalue 1.

If $\mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, compute $A^3\mathbf{v}$.

6. **(12 points)**

Let $W \subset \mathbb{R}^3$ be the plane with equation $x_1 - x_2 + x_3 = 0$. Let $T : \mathbb{R}^3 \mapsto W$ be the orthogonal projection to W.

(a) Find the standard matrix associated to the linear map T.

7. **(10 points)** Let A be an $n \times n$-matrix and let

$$W = \{\mathbf{v} \in \mathbf{R}^n \text{ such that } A\mathbf{v} = \lambda \mathbf{v}\},$$

that is, the λ-eigenspace for A. Prove that W is a subspace of $\mathbf{R}^n$

8. **(10 points)** Explain (algebraically) why **one or the two** facts is true. (Your choice.)

   - Let A be an $n \times n$ matrix.
     If λ is an eigenvalue of A, then $\det(A - \lambda I_n) = 0$.
   - If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set, then these three vectors are linearly independent.

9. **(10 points)** Let A be the matrix

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

Determine if A is diagonalizable, and if so, diagonalize it.

10. **(28 points)** In each question circle either True of False. No justification is needed. Answer 7 out of 9.

(a) Let A and B be $m \times n$ matrices. If A is row-equivalent to B, then $\mathrm{Col}(A) = \mathrm{Col}(B)$.

TRUE FALSE

(b) Distinct eigenvectors are linearly independent.

TRUE FALSE

(c) If 0 is an eigenvalue of an $n \times n$ matrix A, then $\mathrm{rank}(A) < n$.

TRUE FALSE

(d) If A is a 3×5 matrix such that $\mathrm{Null}(A)$ is 2 dimensional, then the equation $A\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ has infinitely many solutions.

TRUE FALSE

(e) Let A be a real 2×2 matrix, whose characteristic polynomial does not have real roots. Then A is digonalizable.

TRUE FALSE

(f) If A is an $n \times n$ matrix with fewer than n distinct eigenvalues, then A is not diagonalizable.

TRUE FALSE

(g) There exist non-zero vectors in $\mathbf{R}^3$ that are orthogonal to $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$.

TRUE FALSE

(h) Let A be a 2×2 matrix. If there is some basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ such that $[A]_{\mathcal{B}}$ is not diagonal, then A is not diagonalizable.

TRUE FALSE

(i) Every subspace of $\mathbf{R}^n$ has at least one orthogonal basis.

TRUE FALSE