**Name:**

**Instructions:** For each question, to receive full credit you must show all work. Explain your answers fully and clearly. You may refer to theorems in the book or from class unless the question specifically states otherwise. No calculators, books or notes of any form are allowed.

Note that the questions have different point values. Pace yourself accordingly.

Good luck!

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1. (10 points)

(a) Find the general solution of the following system of equations.

\[ \begin{align*}
  x_1 + x_2 + 3x_3 &= 0 \\
  2x_1 + x_2 + 4x_3 &= 1 \\
  3x_1 + x_2 + 5x_3 &= 2
\end{align*} \]

(b) Describe the geometric shape of the collection of all solutions to
the above equations considered as a subset of \( \mathbb{R}^3 \).
2. (8 points) Let $A$ be a $3 \times 3$ matrix such that the equation

$$Ax = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

has both

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

as solutions. Find another solution to this equation. Explain.
3. (10 points)

Let

\[ B = \{1 + 2x, x - x^2, x + x^2\} \]

(a) Show that \( B \) is a basis for \( \mathbb{P}_2 \)
(b) Let \( p(x) = 1 + 3x + x^2 \). Compute \([p(x)]_B\).
(c) Let $T : \mathbb{P}_2 \mapsto \mathbb{P}_2$ be the transformation $T(p(t)) = p'(t) - p(t)$.
Write down the matrix of $T$ with respect to the basis $B$. 
4. (20 points)

Let $W \subset \mathbb{R}^4$ be the subspace of vectors $(x_1, x_2, x_3, x_4)$ satisfying

$$2x_1 - x_3 + x_4 = 0$$

Find an orthonormal basis for $W$. 
5. (8 points)

Let $A$ be a $2 \times 2$ matrix such that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for $A$ with eigenvalue 2 and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector for $A$ with eigenvalue 1.

If $v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, compute $A^3v$. 
6. **(12 points)**

Let $W \subset \mathbb{R}^3$ be the plane with equation $x_1 - x_2 + x_3 = 0$. Let $T : \mathbb{R}^3 \rightarrow W$ be the orthogonal projection to $W$.

(a) Find the standard matrix associated to the linear map $T$. 

7. (10 points) Let $A$ be an $n \times n$-matrix and let

$$W = \{ v \in \mathbb{R}^n \text{ such that } Av = \lambda v \},$$

that is, the $\lambda$-eigenspace for $A$. Prove that $W$ is a subspace of $\mathbb{R}^n$. 
8. **(10 points)** Explain (algebraically) why one or the two facts is true. (Your choice.)

- Let $A$ be an $n \times n$ matrix.
  - If $\lambda$ is an eigenvalue of $A$, then $\det(A - \lambda I_n) = 0$.
- If $\{u_1, u_2, u_3\}$ is an orthogonal set, then these three vectors are linearly independent.
9. (10 points) Let $A$ be the matrix

$$
\begin{pmatrix}
4 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 \\
0 & 0 & 2 & 0 \\
1 & 0 & 0 & 2
\end{pmatrix}
$$

Determine if $A$ is diagonalizable, and if so, diagonalize it.
10. (28 points) In each question circle either True or False. No justification is needed. Answer 7 out of 9.

(a) Let $A$ and $B$ be $m \times n$ matrices. If $A$ is row-equivalent to $B$, then $\text{Col}(A) = \text{Col}(B)$.

TRUE  FALSE

(b) Distinct eigenvectors are linearly independent.

TRUE  FALSE

(c) If 0 is an eigenvalue of an $n \times n$ matrix $A$, then $\text{rank}(A) < n$.

TRUE  FALSE

(d) If $A$ is a $3 \times 5$ matrix such that $\text{Null}(A)$ is 2 dimensional, then the equation $Ax = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ has infinitely many solutions.

TRUE  FALSE

(e) Let $A$ be a real $2 \times 2$ matrix, whose characteristic polynomial does not have real roots. Then $A$ is diagonalizable.

TRUE  FALSE

(f) If $A$ is an $n \times n$ matrix with fewer than $n$ distinct eigenvalues, then $A$ is not diagonalizable.

TRUE  FALSE

(g) There exist non-zero vectors in $\mathbb{R}^3$ that are orthogonal to $e_1$, $e_2$ and $e_3$.

TRUE  FALSE

(h) Let $A$ be a $2 \times 2$ matrix. If there is some basis $B = \{b_1, b_2\}$ such that $[A]_B$ is not diagonal, then $A$ is not diagonalizable.

TRUE  FALSE

(i) Every subspace of $\mathbb{R}^n$ has at least one orthogonal basis.

TRUE  FALSE