MA 242 – Linear Algebra
Final Exam

## Name:

**Instructions:** For each question, to receive full credit you must show all work. Explain your answers fully and clearly. You may refer to theorems in the book or from class unless the question specifically states otherwise. No calculators, books or notes of any form are allowed.

Note that the questions have different point values. Pace yourself accordingly.

Good luck!

| Question | Score | Out of |
|----------|-------|--------|
| 1        |       | 10     |
| 2        |       | 8      |
| 3        |       | 10     |
| 4        |       | 15     |
| 5        |       | 8      |
| 6        |       | 12     |
| 7        |       | 10     |
| 8        |       | 10     |
| 9        |       | 10     |
| 10       |       | 28     |
| Total    |       | 121    |

1. **(10 points)**

   (a) Find the general solution of the following system of equations.

$$x_1 + x_2 + 3x_3 = 0$$
$$2x_1 + x_2 + 4x_3 = 1$$
$$3x_1 + x_2 + 5x_3 = 2$$

   **Solution:**

   The reduced row echelon form of the augmented matrix is:

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   We therefore get $x_1 = -x_3 - 1$, $x_2 = -2x_3 - 1$, with $x_3$ the free variable.

   (b) Describe the geometric shape of the collection of all solutions to the above equations considered as a subset of $\mathbf{R}^3$.

   **Solution:** A line through the point $(-1, -1, 0)$ in the direction of $(-1, -2, 1)$.

2. (**8 points**) Let $A$ be a $3 \times 3$ matrix such that the equation

$$A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{1}$$

has both $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ as solutions. Find another solution to this equation. Explain.

**Solution:**

The difference $y = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$ is a solution to $A\mathbf{x} = 0$, i.e.

lies in the nullspace of $A$. Adding any multiple of $y$ to $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ will yield

another solution of equation **??**.

3. **(10 points)**

Let

$$\mathcal{B} = \{1 + 2x, x - x^2, x + x^2\}$$

(a) Show that $\mathcal{B}$ is a basis for $\mathbb{P}_2$

**Solution:** Using the "standard" basis $1, x, x^2$, the vectors of $\mathcal{B}$ are the columns of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

It thus suffices to check that the three columns of $A$ are lin. independent. This is equivalent to the matrix having 3 pivots, non-zero determinant, etc. Pick your favorite condition, and check it.

(b) Let $p(x) = 1 + 3x + x^2$. Compute $[p(x)]_\mathcal{B}$.

**Solution:** We have to solve

$$Ax = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

The solution is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(c) Let $T : \mathbb{P}_2 \mapsto \mathbb{P}_2$ be the transformation $T(p(x)) = p'(x) - p(x)$. Write down the matrix of $T$ with respect to the basis $\mathcal{B}$.

**Solution:** Let us find the matrix of $T$ with respect to the standard basis $E = \{1, x, x^2\}$. $T(1) = -1, T(x) = 1 - x, T(x^2) = 2x - x^2$. Thus,

$$T_E^E = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

The change of basis matrix $\underset{E \leftarrow B}{P}$ is

$$\underset{E \leftarrow B}{P} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

We have $T_B^B = \underset{E \leftarrow B}{P}^{-1} T_E^E \underset{E \leftarrow B}{P}$

4. **(20 points)**

   Let $W \subset \mathbb{R}^4$ be the subspace of vectors $(x_1, x_2, x_3, x_4)$ satisfying

   $$2x_1 - x_3 + x_4 = 0$$

   Find an orthonormal basis for $W$.

   **Solution:** This is a three-dimensional subspace of $\mathbb{R}^4$, presented as the nullspace of the matrix

   $$\begin{pmatrix} 2 & 0 & -1 & 1 \end{pmatrix}$$

   The parametric solution is

   $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

   and the three vectors appearing on the RHS form a basis for $W$. Performing the Gram-Schmidt algorithm yields and orthogonal basis proportional to:

   $$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 5 \end{pmatrix}$$

5. **(8 points)**

   Let $A$ be a $2 \times 2$ matrix such that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for $A$ with

   eigenvalue 2 and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector for $A$ with eigenvalue 1.

   If $\mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, compute $A^3 \mathbf{v}$.

   **Solution:** Let $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Then $v = v_1 + v_2$. We have

   $$A^3 v = A^3(v_1 + v_2) = A^3 v_1 + A^3 v_2 = 2^3 v_1 + 1^3 v_2 = \begin{pmatrix} 8 \\ 8 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \end{pmatrix}$$

6. **(12 points)**

   Let $W \subset \mathbb{R}^3$ be the plane with equation $x_1 - x_2 + x_3 = 0$. Let $T : \mathbb{R}^3 \mapsto W$ be the orthogonal projection to $W$. Find the standard matrix associated to the linear map $T$.

   **Solution:**

   $$\begin{pmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{pmatrix}$$

7. **(10 points)** Let $A$ be an $n \times n$-matrix and let

$$W = \{\mathbf{v} \in \mathbf{R}^n \text{ such that } A\mathbf{v} = \lambda\mathbf{v}\},$$

that is, the $\lambda$-eigenspace for $A$. Prove that $W$ is a subspace of $\mathbf{R}^n$

**Solution:** Suppose that $u, v \in W$, i.e. $Au = \lambda u$, $Av = \lambda v$. We must show that

(a) $u + v \in W$. Proof: $A(u + v) = Au + Av = \lambda u + \lambda v = \lambda(u + v)$, so $u + v \in W$

(b) $cu \in W$ for an arbitrary scalar $c$. Proof: $A(cu) = c(Au) = c(\lambda u) = \lambda(cu)$, so $cu \in W$.

8. **(10 points)** Explain (algebraically) why **one or the two** facts is true. (Your choice.)

- Let $A$ be an $n \times n$ matrix.
  If $\lambda$ is an eigenvalue of $A$, then $\det(A - \lambda I_n) = 0$.

- If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set, then these three vectors are linearly independent.

**Solution:** (First statement) If $\lambda$ is an eigenvalue of $A$, then there exists a non-zero vector $v$ such that $Av = \lambda v$, or equivalently $(A - \lambda I)v = 0$. This means that $A - \lambda I$ has a nullspace of dimension at least 1, so that $rank(A - \lambda I) < n$. This means that $det(A - \lambda I) = 0$

(Second statement) Suppose that $c_1 u_1 + c_2 u_2 + c_3 u_3 = 0$. Dotting this equation by $u_1, u_2, u_3$ respectively, we get $c_1 = c_2 = c_3 = 0$, thus any linear dependence relation is the trivial one (i.e. with all constants 0).

9. **(10 points)** Let $A$ be the matrix

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

Determine if $A$ is diagonalizable, and if so, diagonalize it.

**Solution:** The characteristic polynomial is $(4-\lambda)^2(2-\lambda)^2$, so the eigenvalues are $4, 2$, each with multiplicity 2. The matrix will therefore be diagonalizable if each eigenspace has dimension 2. $A - 2I$, $A - 4I$ each have rank 2, so the nullspaces have the required dimension. A basis of eigenvectors for the $\lambda = 4$ eigenspace is given by $\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. A basis of eigenvectors for $\lambda = 2$ is $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Thus $A = PDP^{-1}$, where

$$P = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

10. **(28 points)** In each question circle either True of False. No justification is needed. Answer 7 out of 9.

(a) Let $A$ and $B$ be $m \times n$ matrices. If $A$ is row-equivalent to $B$, then $\mathrm{Col}(A) = \mathrm{Col}(B)$.

TRUE                                                   **FALSE**

(b) Distinct eigenvectors are linearly independent.

TRUE                                                   **FALSE**

(c) If 0 is an eigenvalue of an $n \times n$ matrix $A$, then $\mathrm{rank}(A) < n$.

**TRUE**                                                 FALSE

(d) If $A$ is a $3 \times 5$ matrix such that $\mathrm{Null}(A)$ is 2 dimensional, then the equation $A\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ has infinitely many solutions.

**TRUE**                                                 FALSE

(e) Let $A$ be a real $2 \times 2$ matrix, whose characteristic polynomial does not have real roots. Then $A$ is digonalizable.

**TRUE**                                                 FALSE

(f) If $A$ is an $n \times n$ matrix with fewer than $n$ distinct eigenvalues, then $A$ is not diagonalizable.

TRUE                                                   **FALSE**

(g) There exist non-zero vectors in $\mathbf{R}^3$ that are orthogonal to $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$.

TRUE                                                   **FALSE**

(h) Let $A$ be a $2 \times 2$ matrix. If there is some basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ such that $[A]_{\mathcal{B}}$ is not diagonal, then $A$ is not diagonalizable.

TRUE                                                   **FALSE**

(i) Every subspace of $\mathbf{R}^n$ has at least one orthogonal basis.

**TRUE**                                                 FALSE