MATH 242, SAMPLE MIDTERM SOLUTIONS

1.) The REF of the system is given by the augmented matrix

$$\begin{bmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is free, while x_1, x_2 are basic. The general solution is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$$

This is the parametric equation of a line through $(-5, 3, 0)$ parallel to $(-4, 3, 0)$.

2. a) Recall that a set of vectors is linearly independent iff the system $Ax = 0$ has only the trivial solution, where $A = [v_1 \ \ v_2 \ \ v_3]$. Passing to the augmented matrix $[v_1 \ \ v_2 \ \ v_3 \ \ 0]$, and row-reducing, we get the REF

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus, the system has only the trivial solution, and so the three vectors are linearly independent.

b) Such an example does not exist. Proceeding as in a), we would get a 3×4 matrix A, and the system $Ax = 0$ will now have a non-trivial solution, since there will be at least one free variable present.

3. a) $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a linear transformation, if for any two vectors $u, v \in \mathbb{R}^n$ and any scalar c,

$$T(u + v) = T(u) + T(v)$$

and

$$T(cu) = cT(u)$$

b) Looking for instance at the images of $e_1 = (1, 0)$ and $e_2 = (0, 1)$, we see that the matrix is

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

c) Looking at the images of e_1, e_2 as in b), we see that $T(e_1) = -e_2$ and $T(e_2) = -e_1$, so the matrix is

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

4.a) Recall that b is in the span of the columns of a matrix A iff the system $Ax = b$ is consistent. Looking at the augmented matrix of the system

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is already in EF we see that it is consistent, so b is indeed in the span of the columns of A.

b) Again, setting up the augmented matrix, we get

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

This is already in REF. We see that the system is consistent, and has one free variable, x_3. Thus the system has infinitely many solutions.

5.a) Recall that if $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ is one-to-one, then $m \leq n$, and if T is onto, then $m \geq n$. This last condition means that no map from $\mathbb{R}^4$ to $\mathbb{R}^7$ can be onto.

b) Yes, for instance $T(x_1, x_2, x_3) = (x_1, x_2)$.

c) No. In fact, the following argument shows that if a map $\mathbb{R}^n \mapsto \mathbb{R}^n$ is one-to-one, it is automatically onto. Let A be the standard matrix of the map. Since T is one-to-one, $Ax = 0$ has only the trivial solution. This means the system has no free variables, and so every column of A is a pivot column. (Thus A is row equivalent to the identity matrix). But this also implies that the system $Ax = b$ has a solution for every b, thus T is onto.

6.) Only BA makes sense yielding a 3×2 matrix. (AB does not make sense since A has two columns and B three rows). The product is

$$\begin{bmatrix} 1 & 2 \\ 9 & 0 \\ 1 & -4 \end{bmatrix}$$

7. a) T
b) T - for instance, take A to be a 2×2 zero matrix, and b any non-zero vector
c) F - take for instance the matrix $[e_1\ e_2\ e_3\ e_4]$ where the e_i are the standard vectors in $\mathbb{R}^5$.
d) F - e.g. T is 0
e) T
f) T - think of this in terms of linear transformations. Having a pivot in each row means the corresponding linear transformation is onto. The composition of two maps that are onto is onto.
g) F
h) T - there will be four free variables.