MA 242 – Linear Algebra
Exam #2

Name:

Instructions: For each question, to receive full credit you must show all work. Explain your answers fully and clearly. You may refer to theorems in the book or from class unless the question specifically states otherwise. No calculators, books or notes of any form are allowed.

Note that the questions have different point values. Pace yourself accordingly.

Good luck!

Question	Score	Out of
1		20
2		10
3		15
4		20
5		10
6		10
7		25
Total		110

1. **(20 points)** Let

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 & 1 \\ 1 & 2 & 3 & 5 & 1 \\ 2 & 4 & 1 & 5 & 1 \\ 1 & 2 & 1 & 3 & 1 \end{pmatrix}.$$

The reduced echelon form of this matrix is

$$B = \begin{pmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find a basis for $\mathrm{Null}(A)$.

(b) Find a basis for $\mathrm{Row}(A)$.

(c) Find a basis for $\text{Col}(A)$.

(d) Is it possible to find a 4×4 invertible matrix D such that $\text{Col}(D) = \text{Col}(A)$? Explain.

2. **(10 points)** Let A and B be $n \times n$ invertible matrices such that

$$\det(AB^2A^{-1}) = 1.$$

Show that $\det(B) = \pm 1$.

3. **(15 points)**

(a) Let V be a vector space and let $H \subseteq V$. Define what it means for H to be a **subspace** of V.

(b) Let $W \subseteq \mathbf{R}^4$ be composed of vectors of the form $\begin{pmatrix} a+b+3c \\ -a+2b+3c \\ 2a-b \\ 3a-b+c \end{pmatrix}$. Explain why W is a subspace of $\mathbf{R}^4$.

(c) What is the dimension of W (defined in the previous part)? Explain.

4. **(20 points)**

(a) Let V be a vector space. Define what it means for $\mathbf{v}_1, \ldots, \mathbf{v}_n$ to be **linearly independent** in V.

(b) Verify that $\{1 + x^2, -1 + x + 3x^2, 1 + 4x + 7x^2\}$ are linearly independent in $\mathbf{P}_2$.

(c) It turns out that

$$\mathcal{B} = \{1 + x^2, -1 + x + 3x^2, 1 + 4x + 7x^2\}$$

is in fact a basis of $\mathbf{P}_2$. Find a polynomial $f(x) \in \mathbf{P}_2$ such that

$$[f(x)]_{\mathcal{B}} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}.$$

(d) Compute $[x + 4x^2]_{\mathcal{B}}$. ($\mathcal{B}$ is the basis defined in the previous part of this question.)

5. **(10 points)** Find the inverse of the matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 3 \end{pmatrix}$$

6. **(10 points)** In each part, give an example of what is listed or explain why no such example can exist. Justify your answers.

   (a) A non-invertible 3×3 matrix such that all nine entries are non-zero.

   (b) A 4×7 matrix A such that $A\mathbf{x} = \mathbf{b}$ is solvable for all $\mathbf{b}$ and $\dim(\text{Null}(A)) = 5$.

7. **(25 points) – Answer 6 of the following 8 questions**

Circle either TRUE or FALSE. No justification is needed.

(a) Let A be a 5×5 matrix such that $\text{rank}(A) = 3$. If T_A is the linear transformation attached to A then T_A is neither one-to-one nor onto.

TRUE FALSE

(b) If A is a square matrix such that A^5 is the zero matrix then A is also the zero matrix.

TRUE FALSE

(c) If A is an invertible $n \times n$ matrix and A is row equivalent to B then the columns of B form a basis of $\mathbf{R}^n$.

TRUE FALSE

(d) If A and B are 5×5 matrices such that $\text{rank}(A) = \text{rank}(B) = 2$ then $\text{rank}(AB) = 2$.

TRUE FALSE

(e) If A and B are 5×5 matrices such that $\text{rank}(A) = \text{rank}(B) = 5$ then $\text{rank}(AB) = 5$.

TRUE FALSE

(f) If three vectors in $\mathbf{R}^3$ are linearly dependent, then the volume of the parellelpiped that they span equals zero.

TRUE FALSE

(g) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ are a linearly independent collection of vectors in some vector space V, then there is a subset of these vectors that form a basis of V.

TRUE FALSE

(h) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis of $\mathbf{R}^3$. Then the matrix

$$A = \begin{pmatrix} [\mathbf{b}_1]_{\mathcal{B}} & [\mathbf{b}_2]_{\mathcal{B}} & [\mathbf{b}_3]_{\mathcal{B}} \end{pmatrix}$$

is the identity matrix.

TRUE FALSE