SOLUTIONS

MA 242 – Final Exam

Name:

Instructions: For each question, to receive full credit you must show all work. Explain your answers fully and clearly. You may refer to theorems in the book or from class unless the question specifically states otherwise. No calculators, books or notes of any form are allowed except for one two-sided formula sheet.

Note that the questions have different point values. Pace yourself accordingly.

Good luck!

Question	Score	Out of
1		12
2		12
3		16
4		32
5		12
6		24
7		16
8		40
Total		164

1. (12 points) Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 & 6 \\ 2 & -1 & 2 \\ 1 & 1 & h \end{pmatrix}$$

For what values of the parameter h does the system $Ax = b$ have a solution for every $b \in \mathbb{R}^3$? Explain your reasoning.

$$A \sim \begin{pmatrix} 1 & 2 & 6 \\ 0 & -5 & -10 \\ 0 & -1 & h-6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & -1 & (h-6) \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & (h-4) \end{pmatrix}$$

$Ax = b$ has a sol^{n} $\forall b \in \mathbb{R}^3$

$\Longleftrightarrow$

A has a pivot in every row

$\Longleftrightarrow$

$$\boxed{h \neq 4}$$

2. (12 points) Let A be the matrix

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}$$

Determine if A is invertible, and if so, find A^{-1}.

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right)$$

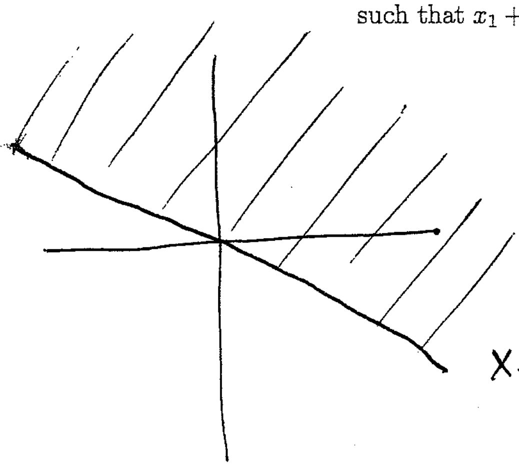
$$A^{-1} = \begin{pmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{pmatrix}$$

3. (16 points)

(a) Determine if the set of vectors S consisting of

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

such that $x_1 + 2x_2 \geq 0$ forms a subspace of $\mathbb{R}^2$. Explain.



$x_2 = -\frac{1}{2} x_1$

No: $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in S$

$-\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \notin S$

So S is not closed under scalar mult.

(b) Determine if the set of vectors S of the form

$$\begin{pmatrix} s+t \\ 2s-t \\ t \end{pmatrix}$$

for all values of s, t forms a subspace in $\mathbb{R}^3$. Explain.

$$= \left\{ s\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Spans are subspaces. $\boxed{\text{YES}}$

4. (32 points) Find the standard matrices of each of the following linear transformations.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^3$, given by reflection in the x_2, x_3–plane.

$$T(e_1) = -e_1$$
$$T(e_2) = e_2$$
$$T(e_3) = e_3$$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by the orthogonal projection to the line spanned by the vector $(2, 1, -1)$.

normalize n so that $\vec{u} = \begin{pmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}$

$$T(e_1) = \text{Proj}_{\vec{u}_1} e_1 = \frac{2}{\sqrt{6}} \vec{u} \quad ((e_1 \cdot \vec{u})\vec{u})$$
$$T(e_2) = \text{Proj}_{\vec{u}} e_2 = \frac{1}{\sqrt{6}} \vec{u} \quad ((e_2 \cdot \vec{u})\vec{u})$$
$$T(e_3) = \text{proj}_{\vec{u}} e_3 = -\frac{1}{\sqrt{6}} \vec{u} \quad ((e_3 \cdot \vec{u})\vec{u})$$

So : $$A = \begin{bmatrix} 4/6 & 2/6 & -2/6 \\ 2/6 & 1/6 & -1/6 \\ -2/6 & -1/6 & 1/6 \end{bmatrix}$$

(c) $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T(x_1, x_2) = (x_1 + 3x_2, 4x_1 - x_2, x_1)$$

$T(e_1) = T(1,0) = (1,4,1)$

$T(e_2) = T(0,1) = (3,-1,0)$

$$A = \begin{bmatrix} 1 & 3 \\ 4 & -1 \\ 1 & 0 \end{bmatrix}$$

(d) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

$$T(e_1) = T\left(\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix}1\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3/2\\3/2\end{pmatrix}$$

$$T(e_2) = T\left(\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix}1\\2\end{pmatrix} - \frac{1}{2}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-1/2\\1/2\end{pmatrix}$$

$$A = \begin{bmatrix} 3/2 & -1/2 \\ 3/2 & 1/2 \end{bmatrix}$$

5. (12 points) Let W be the subspace of $\mathbb{R}^4$ spanned by

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

Find an orthogonal basis for W using the Gram-Schmidt process.

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{\|u_1\|^2} u_1 = v_2 - \frac{3}{4} u_1 = \begin{pmatrix} -3/4 \\ +1/4 \\ +1/4 \\ +1/4 \end{pmatrix}$$

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{\|u_1\|^2} u_1 - \frac{(v_3 \cdot u_2)}{\|u_2\|^2} u_2.$$

$$= v_3 - 0 - 0 = v_3.$$

So $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ forms an orthogonal basis for W.

Let $\{u_1, u_2\}$ be the orthonormal basis from (b)

(c) Find the orthogonal projection $\hat{y}$ of $y = (1, 1, 2)$ onto the subspace W.

$$\hat{y} = (y \cdot u_1)u_1 + (y \cdot u_2)u_2$$

$$= \left(\frac{2}{\sqrt{2}}\right)u_1 + \left(\frac{2}{\sqrt{3}}\right)u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2/3 \\ -2/3 \\ 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 5/3 \\ 1/3 \\ 2/3 \end{pmatrix}$$

(d) Find the distance from y to the subspace W.

$$z = y - \hat{y} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 5/3 \\ 1/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 2/3 \\ 4/3 \end{pmatrix}$$

$$\text{dist}(y, W) = \|z\| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{16}{9}}$$

$$= \sqrt{\frac{8}{3}} = 2\sqrt{\frac{2}{3}}$$

7. (16 points) Explain why each of the following statements is true.

(a) If A and B are square matrices of the same size, and x is an eigenvector of AB such that $Bx \neq 0$, then Bx is an eigenvector of BA. **Give an algebraic explanation.**

x is an eigenvector of AB

$\Rightarrow \quad ABx = \lambda x$ for some λ, $x \neq 0$

$$\Rightarrow \quad (BA)Bx = B(ABx) = B(\lambda x) = \lambda Bx$$

so: if $Bx \neq 0 \Rightarrow Bx$ is an eig. vector of BA w./eigenvalue λ.

(b) If W is a subspace of $\mathbb{R}^n$, and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection to W, then T is not invertible unless $W = \mathbb{R}^n$.

$W = \text{Range}(T)$

if $W \neq \mathbb{R}^n$, then $\dim(\text{Range}(T)) < n$

$\Rightarrow \dim(\text{null}(T)) > 0 \Rightarrow T$ is not invertible.

If $W = \mathbb{R}^n$, Then $T = Id$.

8. **(40 points)**

Circle either TRUE or FALSE. No justification is needed. Correct answers receive 4 points, blank answers 2, and incorrect answers 0.

(a) If A and B are invertible matrices of the same size, then $A+B$ is invertible.

TRUE FALSE

(b) If A is a 3×5 matrix such that the solution of the system $Ax = 0$ has two free variables, then the linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ with standard matrix A is onto.

TRUE FALSE

(c) The system $Ax = b$ is only consistent if b lies in the column space of A.

TRUE FALSE

(d) A set of nonzero orthonormal vectors is linearly independent.

TRUE FALSE

(e) Given a set of linearly dependent vectors $\{v_1, \cdots, v_p\}$, v_1 can be written as a linear combination of $v_2, \cdots, v_p$.

TRUE FALSE

(f) If A is a 5×5 matrix, v_1 is an eigenvector of A with eigenvalue 2, and v_2, v_3 are linearly independent eigenvectors with eigenvalue 4, then $\{v_1, v_2, v_3\}$ is a linearly independent set.

TRUE FALSE

(g) If v is an eigenvector of A, the v is an eigenvector of $A+3I$, where I denotes the identity matrix.

TRUE FALSE

(h) If the product of the matrices A and B makes sense, then $null(B) \subset null(AB)$ (i.e. the nullspace of B is contained in the nullspace of AB).

TRUE FALSE

(i) If the product of the matrices A and B makes sense, then $rank(AB) \leq rank(A)$.

TRUE FALSE

(j) If A is diagonalizable, then so is A^T

TRUE FALSE