SOLUTIONS

MA 242 – Linear Algebra
Exam #1

Name:

Instructions: To receive full credit you must show all work. Explain your answers fully and clearly. You may refer to theorems/facts in the book or from class. No calculators, books or notes of any form are allowed. Good luck!

Question	Score	Out of
1		12
2		12
3		12
4		24
5		18
6		18
7		10
8		32
Total		138

1. **(12 points)**
Consider the system of equations

$$\begin{aligned} x_1 + 2x_2 - 3x_4 &= 1 \\ x_2 + x_4 &= 2 \\ x_1 - x_2 + x_3 &= -1. \end{aligned}$$

Row-reduce this system. Identify the basic and free variables. Find the general solution to these equations in parametric form. What geometric shape does the solution space form?

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -3 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 1 & -1 & 1 & 0 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -3 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & -3 & 1 & 3 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -3 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 6 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -5 & -3 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 6 & 4 \end{array}\right]$$

x_4 is free

x_1, x_2, x_3 are basic.

$x_1 = -3 + 5x_4$

$x_2 = 2 - x_4$

$x_3 = 4 - 6x_4$

x_4 is free.

OR

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 4 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 5 \\ -1 \\ -6 \\ 1 \end{pmatrix}$$

The solution set is a line through $\begin{pmatrix} -3 \\ 2 \\ 4 \\ 0 \end{pmatrix}$ parallel to $\begin{pmatrix} 5 \\ -1 \\ -6 \\ 1 \end{pmatrix}$.

2. **(12 points)**

Determine if the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 7 \\ -3 \\ 9 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix},$$

are linearly independent.

$$A = \begin{pmatrix} 1 & 7 & -1 \\ 1 & -3 & 1 \\ 2 & 9 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 7 & -1 \\ 0 & -10 & 2 \\ 0 & -5 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 7 & -1 \\ 0 & -10 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

only 2 pivots (not one in each col)

$\Rightarrow$ $\{v_1, v_2, v_3\}$ is linearly dependent.

(b) Determine if the vector

$$b = \begin{pmatrix} 2 \\ 12 \\ 9 \end{pmatrix}$$

is in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. If yes, write it as an explicit linear combination of the $\mathbf{v}$'s.

Must check consistency of $Ax = b$

$$\left[\begin{array}{ccc|c} 1 & 7 & -1 & 2 \\ 1 & -3 & 1 & 12 \\ 2 & 9 & -1 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -1 & 2 \\ 0 & -10 & 2 & 10 \\ 0 & -5 & 1 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -1 & 2 \\ 0 & -10 & 2 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

consistent

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -1 & 2 \\ 0 & 1 & -1/5 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2/5 & 9 \\ 0 & 1 & -1/5 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_1 = 9,\ x_2 = -1$ is a sol

$$\boxed{b = 9v_1 - v_2}$$

Yes, $b \in \operatorname{Span}\{v_1, v_2, v_3\}$

Now solve to produce weights c_1, c_2, c_3 such that

$$b = c_1 v_1 + c_2 v_2 + c_3 v_3$$

3. **(12 points)** Find the inverses of the following matrices

- $$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{3\cdot 1 - 2\cdot 1}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

- $$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix}}$$

4. **(24 points)**

- Let

$$A = \begin{pmatrix} 2 & 4 \\ 1 & h \end{pmatrix}$$

Determine the values of h for which the system $Ax = b$ is consistent for all $b \in \mathbb{R}^2$

$$\begin{pmatrix} 2 & 4 \\ 1 & h \end{pmatrix} \sim \begin{pmatrix} 2 & 4 \\ 0 & h-2 \end{pmatrix}$$

For $Ax = b$ to be consistent $\forall b$, $h-2$ must be a pivot. $\Rightarrow$ $\underline{\underline{h \neq 2}}$

- Let C be a 2×2 matrix, and $b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. If $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ are both solutions of the system $Cx = b$, find a third solution different from v_1 and v_2.

$V_h = V_1 - V_2$ is a solution to the homogeneous system $Ax = 0$.

$= \begin{pmatrix} -2 \\ -3 \end{pmatrix}$.

Any vector of the form $V = V_1 + t V_h$. is a solution $\forall\, t \in \mathbb{R}$. For instance, taking $t = 1$, $V = 2V_1 - V_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ is a sol^{n}.

- Describe the possible reduced echelon forms of the matrix C in the previous part.

Since $Cx = b$ does not have a unique sol^{n} $\Rightarrow$ C has at most 1 pivot.

C cannot have 0 pivots, as then $C = 0$, and the system $Cx = b \neq 0$ would be inconsistent.

So $C \sim \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$ or $C \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

- Define what it means for a map $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ to be one-to-one.

Many equivalent definitions:

i) T is one-to-one $\Leftrightarrow$ the eq^{n} $T(x) = 0$ has only the trivial sol^{n}.

ii) T is 1-1 if every $b \in \mathbb{R}^m$ is the image of at most 1 $x \in \mathbb{R}^n$

NOT REALLY ~~DEF~~:

iii) T is 1-1 if its standard matrix has a pivot in every column.

-1

5. **(18 points)**

- Find the standard matrix of each of the following linear transformations.
- For each state whether it is one-to-one and whether it is onto.

(a) Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ be given by $T(x_1, x_2) = (2x_1 - 2x_2, 4x_1 + x_2, 0)$

$T(e_1) = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$ $T(e_2) = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow A = \begin{bmatrix} 2 & -2 \\ 4 & 1 \\ 0 & 0 \end{bmatrix}$

- T cannot be onto
- T is 1-1: $A \sim \begin{bmatrix} 2 & -2 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}$ ← pivot in each column.

(b) The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T(2e_1) = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \text{ and } T(e_2 + e_1) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

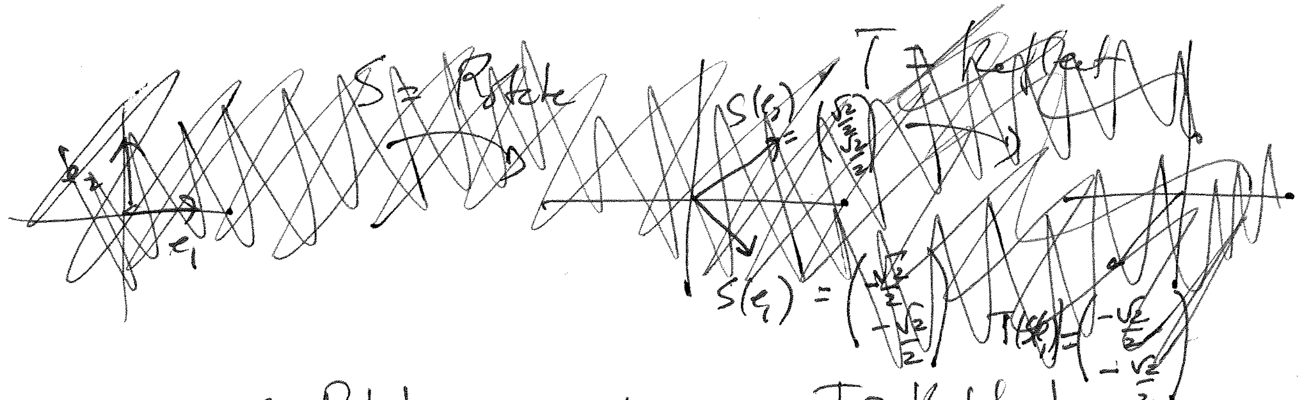
$\Rightarrow T(e_1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $T(e_2) = T(e_2 + e_1) - T(e_1)$

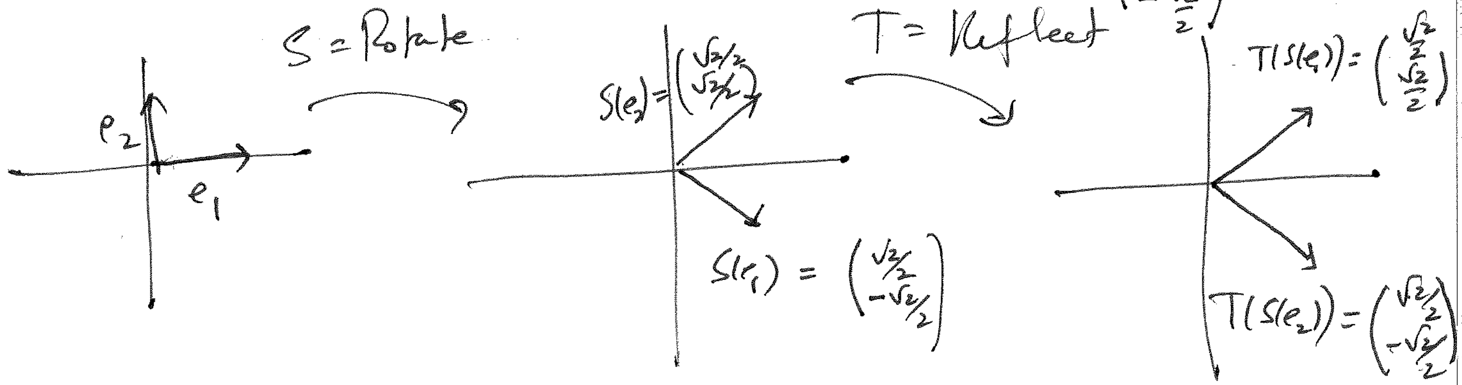
$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$\Rightarrow A = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$. $A \sim \begin{bmatrix} -1 & 3 \\ 0 & 4 \end{bmatrix}$ ← 2 pivots

T is onto & 1-1.

(c) Let $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ be the linear transformation defined by first rotating by $\pi/4$ radians clockwise and then reflecting over the x-axis. Determine the matrix associated to T.





$$A = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}.$$

A is invertible $\Rightarrow$ T is onto & 1-1.

6. **(18 points)**

(a) Do there exist 2×2 matrices A, B such that $AB = 0$ but $BA \neq 0$?
If yes, give an example, if no, explain why not.

$AB = 0$

Yes: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0.$

(b) Do there exist 2×2 matrices A, B such that $AB = I$ but $BA \neq I$?
If yes, given an example, if no, explain why not.

No. It follows from the invertible matrix thm. that if $AB = I$ then $B = A^{-1}$, and so $BA = I$ automatically.

(c) Does there exist a linear transformation from $\mathbf{R}^4$ to $\mathbf{R}^4$ that is onto and such that $T(1,1,1,1) = 0$? If yes, give an example. If no, explain why not.

No: $T\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = 0 \Rightarrow T(x) = 0$ has a non-trivial sol

$\Rightarrow T$ is not 1-1

$\Rightarrow T$ is not onto

(Since $T: \mathbb{R}^4 \to \mathbb{R}^4$

T is 1-1 $\Leftrightarrow$ T is onto).

7. **(10 points)**
Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & -1 \end{pmatrix}.$$

Only one of the products AB and BA makes sense. Determine which one and compute that product.

only AB makes sense:

$$AB = \begin{pmatrix} 5 & 7 & 0 \\ 7 & 11 & -6 \end{pmatrix}$$

8. **(32 points)** Circle either TRUE or FALSE. No justification is needed. **Correct answers score 4 points each. Incorrect answers score 0 points. Leaving a question blank scores 2 points.**

(a) If A is a 3×2 matrix, then the system $Ax = b$ cannot be consistent for all b.

TRUE FALSE

(b) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is onto, then the equation $T(x) = 0$ has a unique solution.

TRUE FALSE

(c) If the set of vectors $\{v_1, v_2, v_3\}$ is linearly independent, then so is $\{2v_1, -v_2, 3v_4\}$

TRUE FALSE

(d) If the system $Ax = b$ is consistent, then so is the system $Ax = 2b$

TRUE FALSE

(e) There exists a 3×4 matrix such that $Ax = b$ is consistent for every b, and such that $Ax = 0$ has two free variables.

TRUE FALSE

(f) A linear combination of solutions to a homogenous system is always a solution of the same system.

TRUE FALSE

(g) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a function such that $T(0) = 0$, then T is a linear transformation.

TRUE FALSE

(h) There exists a 5×5 matrix A with two identical columns such that $Ax = 0$ has a unique solution .

TRUE FALSE