SOLUTIONS,

MA 242 – Midterm Exam II

Name:

Instructions: For each question, to receive full credit you must show all work. Explain your answers fully and clearly. You may refer to theorems in the book or from class unless the question specifically states otherwise. No calculators, books or notes of any form are allowed.

Note that the questions have different point values. Pace yourself accordingly.

Good luck!

Question	Score	Out of
1		12
2		10
3		10
4		16
5		10
6		20
7		20
8		40
Total		138

1. (12 points) Let

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \end{pmatrix}$$

- Find a basis for $Null(A)$.

$$A \sim \begin{pmatrix} 1 & 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -x_2 + 2x_4 - x_5 \\ x_2 \\ -x_4 - x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{V_1} + x_4 \underbrace{\begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}}_{V_2} + x_5 \underbrace{\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}}_{V_3}$$

$\{V_1, V_2, V_3\}$ is a basis for $NULL(A)$

- Find a basis for $Col(A)$

Cols 1 & 3 are pivot cols.

so $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}$ is a basis for $COL(A)$.

2. (10 points) Find the determinant of the following matrix. (Hint: use row operations).

$$A = \begin{pmatrix} -2 & 2 & 4 & 0 \\ 2 & 0 & -1 & 4 \\ 4 & -2 & 6 & 2 \\ -2 & 3 & 2 & 3 \end{pmatrix}$$

$$\det(A) = \begin{vmatrix} -2 & 2 & 4 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 14 & 2 \\ 0 & 1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} -2 & 2 & 4 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 11 & -2 \\ 0 & 1 & -2 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 2 & 4 & 0 \\ 0 & 0 & 7 & -2 \\ 0 & 0 & 11 & -2 \\ 0 & 1 & -2 & 3 \end{vmatrix}$$

$$= - \begin{vmatrix} -2 & 2 & 4 & 0 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 11 & -2 \\ 0 & 0 & 7 & -2 \end{vmatrix}$$

$$= -(-2)(1) \begin{vmatrix} 11 & -2 \\ 7 & -2 \end{vmatrix} = 2(-22+14)$$

$$= 2(-8) = -16.$$

3. (10 points) Let A be the matrix

$$A = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Determine if A is diagonalizable, and if so, diagonalize it (i.e. find matrices P, D such that $A = PDP^{-1}$).

Eigenvalues $\lambda = 5$, $\lambda = 2$, each of alg. multiplicity 2.

$A - 5I$ has rank 2, so $\dim(\mathrm{Null}(A-5I)) = 2$

$\Rightarrow$ $\lambda = 5$ eigenspace is 2-dimensional.

$$A - 2I = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \leftarrow \text{has rank 3}$$

So $\dim(\mathrm{NULL}(A-2I)) = 1 <$ alg mult. of $\lambda = 2$.

$\therefore$ A is NOT diagonalizable.

4. (16 points) Let A be the matrix

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

- Diagonalize A, i.e. find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$

$$\det(A - \lambda I) = \begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (1-\lambda)^2 - 3^2 = 0$$

$$(1-\lambda+3)(1-\lambda-3) = 0 \Rightarrow (4-\lambda)(-2-\lambda) = 0$$

Eigenvalues are $\lambda = 4, \lambda = -2$.

$\underline{\underline{\lambda = 4}}$: $(A - 4I) = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \sim \begin{pmatrix} -3 & 3 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector.

$\lambda = -2$ $(A + 2I) = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector

$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$

$A = PDP^{-1}$

- Find an explicit formula for A^m.

$$A^m = P D^m P^{-1}$$

$$= \left(-\frac{1}{2}\right)\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4^m & 0 \\ 0 & (-2)^m \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4^m & 0 \\ 0 & (-2)^m \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4^m & 4^m \\ (-2)^m & -(-2)^m \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 4^m + (-2)^m & 4^m - (-2)^m \\ 4^m - (-2)^m & 4^m + (-2)^m \end{bmatrix}$$

5. (10 points) Suppose that A, B are 3×3 invertible matrices such that $det(B) = -2$. Evaluate

$$det(AB^3A^2BA^{-1}B^{-1}A^{-1}B^{-1}A^{-1})$$

Recall $\det(A)\det(A^{-1}) = 1$

So the above is equal to

$$= \cancel{\det(A)}^3 \cancel{\det(A^{-1})}^3 \det(B)^3 \det(B) \det(B^{-1})^2$$

$$= \det(B)^2 = 4.$$

6. (20 points) Given an example of each or explain why it cannot happen

(a) A 2×2 matrix with characteristic polynomial $(\lambda-1)^2$ which is diagonalizable.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ is such a matrix.

A is diagonal hence diagonalizable.

(b) A 2×2 matrix which is diagonalizable but not invertible.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonal, hence diagonalizable, but not invertible.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is another example.

7. (20 points) Explain why each of the following statements is true.

(a) If A is a square matrix with eigenvalue λ, then λ^2 is an eigenvalue of A^2.

λ is an eigenvalue of $A \Rightarrow \exists\ x \neq 0$ s.t. $Ax = \lambda x$. We have

$$A^2 x = A(Ax) = A(\lambda x) = \lambda Ax = \lambda(\lambda x) = \lambda^2 x$$

$\Rightarrow$ x is an eigenvector of A^2 with eigenvalue $\underline{\underline{\lambda^2}}$.

(b) Eigenvalues are solutions to the characteristic equation $det(A - \lambda I) = 0$.

λ is an eigenvalue of A

$\Leftrightarrow \exists\ x \neq 0$ s.t. $Ax = \lambda x$

$\Rightarrow (A - \lambda I)x = 0$.

$\Rightarrow \dim \mathrm{Null}(A - \lambda I) > 0$ i.e. $A - \lambda I$ has a <u>nonzero</u> <u>nullspace</u>

$\Rightarrow A - \lambda I$ is <u>not</u> <u>invertible</u>

$\Rightarrow \det(A - \lambda I) = 0$

8. (40 points)

Circle either TRUE or FALSE. No justification is needed. Correct answers receive 4 points, blank answers 2, and incorrect answers 0.

(a) If B is a square matrix obtained from A by first interchanging two rows and then interchanging two columns, then $det(A) = det(B)$

TRUE FALSE

(b) If A and B are square matrices such that $A = PBP^{-1}$ for some invertible matrix P, then A and B have the same eigenvalues.

TRUE FALSE

(c) If a 3×3 matrix A has eigenvalues $0, 1, 4$ then A is diagonalizable.

TRUE FALSE

(d) If a 3×3 matrix A has eigenvalues $0, 1, 4$ then A is invertible.

TRUE FALSE

(e) A diagonal square matrix is diagonalizable.

TRUE FALSE

(f) There is a square matrix A such that A is invertible, but A^2 is not.

TRUE FALSE

(g) If u, v are eigenvectors of A, then so is $u + v$.

TRUE FALSE

(h) If A is an $n \times n$ matrix with fewer than n distinct eigenvalues, then A is not diagonalizable.

TRUE FALSE

(i) There is a 3×3 matrix A with real entries such that $AA^T = -2I$ (here I denotes the identity matrix.)

TRUE FALSE

(j) If u, v are vectors and A a square matrix such that $Au = 2u$ and $Av = 3v$, then the set $\{u, v\}$ is linearly independent.

TRUE FALSE