SOLUTIONS 412

MA 412 – Complex Variables
Exam #2

Name:

Instructions: To receive full credit you must show all work. Explain your answers fully and clearly. You may refer to theorems/facts in the book or from class. No calculators, books or notes of any form are allowed. Good luck!

Question	Score	Out of
1		18
2		12
3		16
4		10
5		10
6		10
7		24
Total		100

1. (18 points)

- Define what it means for a function $f(z)$ to be entire.

Several possible ~~Def~~

(1) f is entire if $f(z)$ is analytic at every point $z_0 \in \mathbb{C}$

OR

(2) f is entire if $f'(z_0)$ exists at every $z_0 \in \mathbb{C}$.

- Is the function $f(z) = e^z \sin(2z-1)$ entire ? Explain your reasoning.

Yes. • e^z is entire

• $\sin(z)$ is entire & so is $\sin(2z-1)$ since it is the composition of $\sin(z)$ with a polynomial

• The product of entire functions is entire $\Rightarrow f(z)$ is entire.

- Is the function $f(z) = \sqrt{z} = \exp(\frac{1}{2}Log(z))$ entire ? Explain your reasoning.

No: $\text{Log}(z)$ is not analytic for $z = x$, $x \leq 0$, so $\sqrt{z} = \exp(\frac{1}{2}\text{Log}(z))$ is not analytic there either.

It has a branch cut along the negative real axis.

$\therefore \sqrt{z}$ is not entire.

2. **(12 points)**
Evaluate the following multivalued expressions

- $\log(-2+2i)$

$$-2+2i = 2\sqrt{2}\exp\left(\frac{3\pi}{4}i\right)$$

$$\Rightarrow \log(-2+2i) = \ln\left(2\sqrt{2}\right) + i\left(\frac{3\pi}{4}+2n\pi\right)$$

$n \in \mathbb{Z}$

- $(-i)^i$

$$\log(-i) = -\frac{\pi}{2}i + i2n\pi \quad n \in \mathbb{Z}$$

$$(-i)^i = e^{i\log(-i)}$$

$$= e^{i\left(-\frac{\pi}{2}i + 2n\pi i\right)} \quad n \in \mathbb{Z}$$

$$= e^{-\left(-\frac{\pi}{2}+2n\pi\right)}$$

$$= e^{\left(\pi/2 - 2n\pi\right)} \quad n \in \mathbb{Z}$$

$$\left(\text{Same as } e^{\left(\frac{\pi}{2}+2n\pi\right)} \quad n \in \mathbb{Z}\right)$$

3. (16 points)

Determine the region in which the following functions are analytic, carefully drawing the branch cuts and singularities. Explain your reasoning.

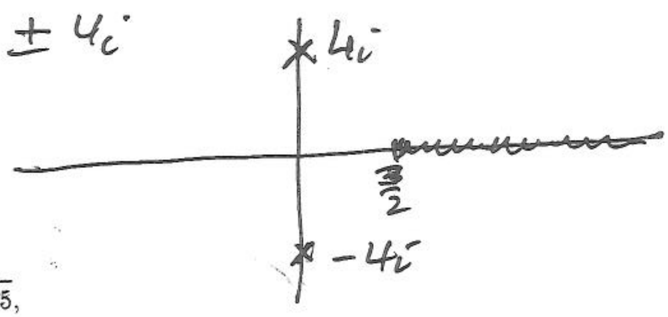
- $$\frac{Log(3-2z)}{z^2+16}$$

Problems occur when $3-2z \leq 0$
or $z^2+16=0$

branch cut

$3-2z \leq 0$
$\frac{3}{2} \leq z$ (z is real & $\geq \frac{3}{2}$)

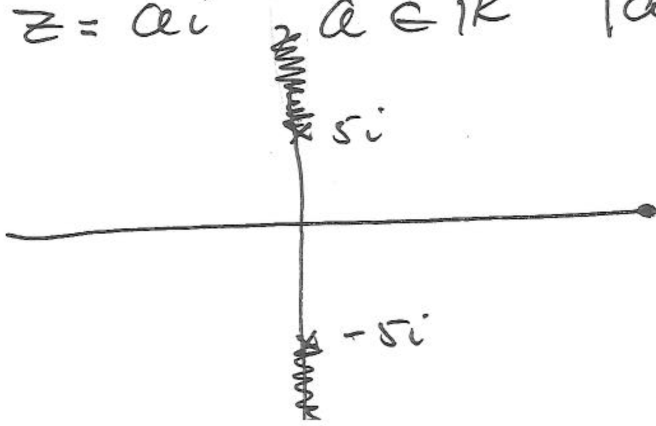
$z^2+16=0 \Rightarrow z = \pm 4i$



- $$\sqrt{z^2+25},$$

where the principal branch of the square root is taken.

$\sqrt{z^2+25} = \exp\left(\frac{1}{2} Log(z^2+25)\right)$
problems occur when z^2+25 is real & ≤ 0. This happens when $z = ai$, $a \in \mathbb{R}$ $|a| \geq 5$.

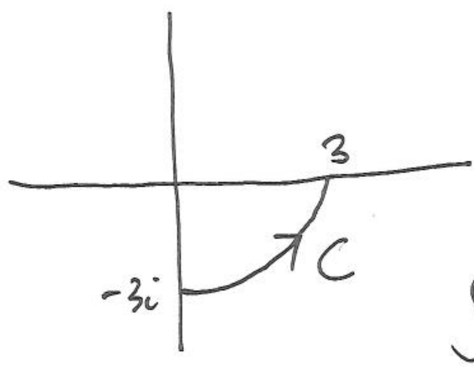


4. (10 points)

Compute the contour integral

$$\int_C \bar{z}\,dz$$

where C is the contour from $-3i$ to 3 along the circle $|z| = 3$ by parametrizing C and direct evaluation.



$$C: \ z(t) = 3e^{it} \quad -\frac{\pi}{2} \le t \le 0$$

$$z'(t) = 3ie^{it}$$

$$\int_C \bar{z}\,dz = \int_{-\pi/2}^{0} \overline{3e^{it}}\,(3ie^{it})\,dt$$

$$= \int_{-\pi/2}^{0} 3e^{-it}\,3ie^{it}\,dt$$

$$= \int_{-\pi/2}^{0} 9i\,dt = 9i\left(\frac{\pi}{2}\right)$$

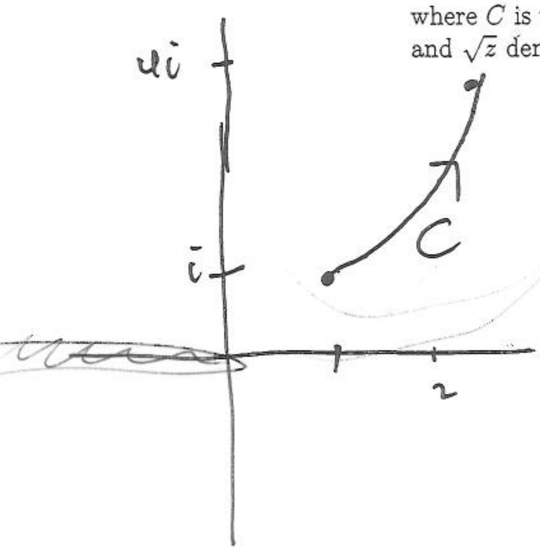
$$= \underline{\frac{9\pi i}{2}}$$

5. **(10 points)**

Evaluate the contour integral

$$\int_C \frac{dz}{\sqrt{z}}$$

where C is the contour from $z = 1+i$ to $2+4i$ along the parabola $y = x^2$ and $\sqrt{z}$ denotes the principal branch. (Hint: find an antiderivative).



$$\frac{1}{\sqrt{z}} = \exp\left(-\tfrac{1}{2}\operatorname{Log}(z)\right)$$

which is analytic in a neighborhood of C.

$$\frac{1}{\sqrt{z}} = \frac{d}{dz}\left(2\sqrt{z}\right)$$

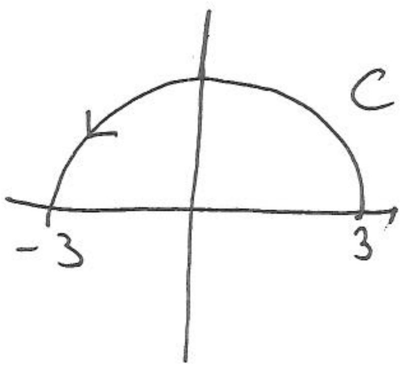
So $$\int_C \frac{dz}{\sqrt{z}} = 2\sqrt{z}\,\Big|_{1+i}^{2+4i} = 2\left(\sqrt{2+4i} - \sqrt{1+i}\right)$$

this cannot be simplified much more

6. **(10 points)**
Show that

$$\left|\int_C \frac{z-1}{z^3+2}\,dz\right| \le \frac{12}{25}\pi$$

where C is the part of the circle $|z| = 3$ from 3 to -3. Clearly show each step in your estimate and which inequalities are being used.



L = length of contour $= 3\pi$

To find M such that $\left|\frac{z-1}{z^3+2}\right| \le M$ on C,

use the triangle inequality:

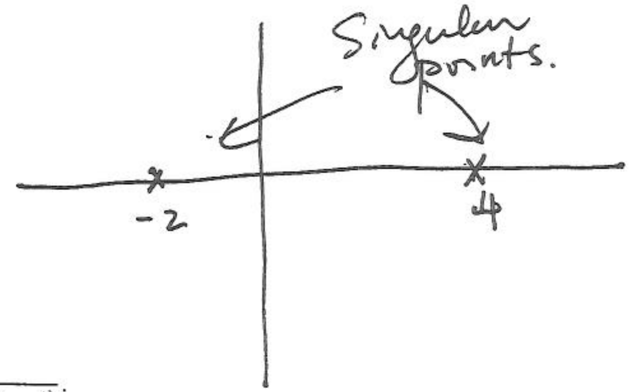
$|z-1| \le |z| + |1| = 4$ on C

$|z^3+2| \ge \big||z^3| - 2\big| = \big||z|^3 - 2\big|$
$= |27-2| = 25$

$\Rightarrow \frac{1}{|z^3+2|} \le \frac{1}{25}$

$\Rightarrow \left|\frac{z-1}{z^3+2}\right| \le \frac{4}{25} = M.$

$$\left|\int_C \frac{z-1}{z^3+2}\,dz\right| \le ML = \frac{12\pi}{25}.$$



7. **(24 points)**
Let

$$f(z) = \frac{z^3}{(z+2)^2(z-4)}.$$

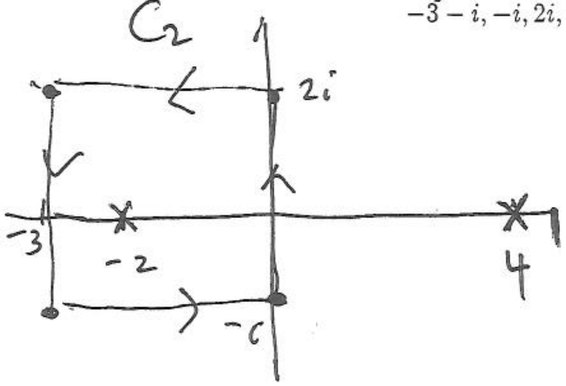
Evaluate the following contour integrals, in each case explaining your reasoning and referring to the relevant theorems.

(a) $\int_{C_1} f(z)dz$ where C_1 is the positively oriented circle $|z-i| = 1$

$f(z)$ is analytic on and inside C_1, so by the Cauchy-Goursat thm.

$$\int_{C_1} f(z)dz = 0$$

(b) $\int_{C_2} f(z)dz$ where C_2 is the positively oriented square with corners at $-3-i, -i, 2i, -3+2i$.



by the Cauchy integral formula

$$\int_{C_2} f(z)dz = \int_{C_2} \frac{g(z)}{(z+2)^2}dz \qquad g(z) = \frac{z^3}{z-4}$$

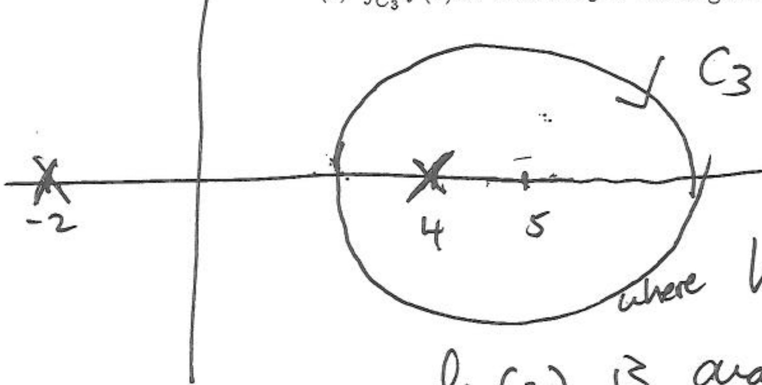
$$= 2\pi i\, g'(-2)$$

since $g(z)$ is analytic in C_2

$$g'(z) = \frac{(z-4)3z^2 - z^3}{(z-4)^2} = \frac{2z^3 - 12z^2}{\quad}$$

$$= 2\pi i\left(\frac{2(-8)-12(4)}{(-6)^2}\right) = 2\pi i\left(-\frac{64}{36}\right) = \frac{-2\pi i \cdot 16}{9}$$

(c) $\int_{C_3} f(z)dz$ where C_3 is the negatively oriented circle $|z-5| = 2$.



$$\int_{C_3} f(z)dz = \int_{C_3} \frac{h(z)}{z-4}\,dz$$

where $h(z) = \dfrac{z^3}{(z+2)^2}$, and

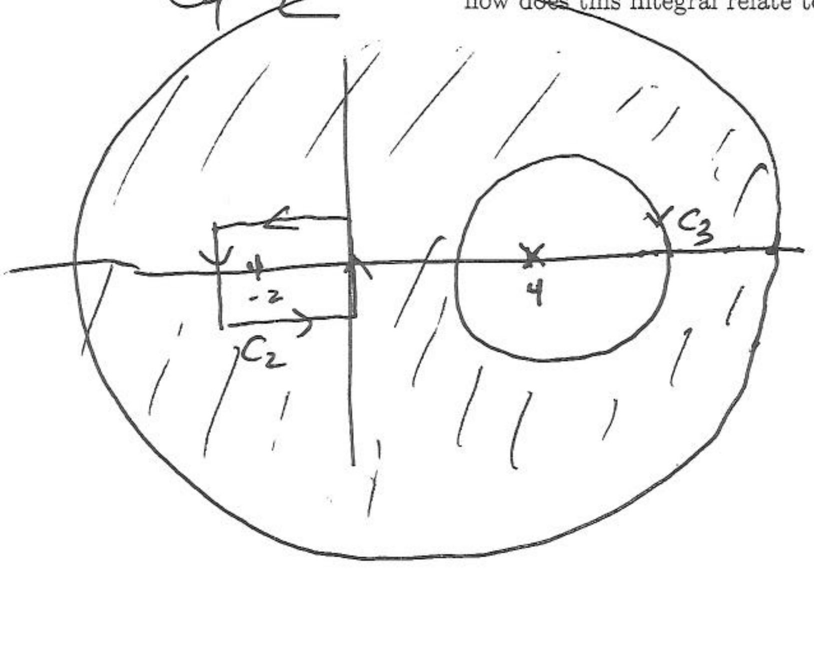
$h(z)$ is analytic in C_3.

By the CIF, $\displaystyle\int_{C_3} \frac{h(z)}{z-4} = (-)2\pi i\, h(4)$ because C_3 neg. oriente

$$= -2\pi i \left(\frac{4^3}{6^2}\right)$$

$$= -2\pi i \cdot \frac{16}{9}$$

(d) $\int_{C_4} f(z)dz$ where C_4 is the positively oriented circle $|z| = 8$. (Hint: how does this integral relate to those over C_2 and C_3 ?).



$f(z)$ is analytic in the shaded region.

By the C-G thm,

$$\int_{C_4} f(z)dz = \int_{C_2} f(z)dz + \int_{-C_3} f(z)dz$$

$$= \int_{C_2} f(z)dz - \int_{C_3} f(z)dz$$

$$= 0$$