412 Sample Final Sol.

Q1

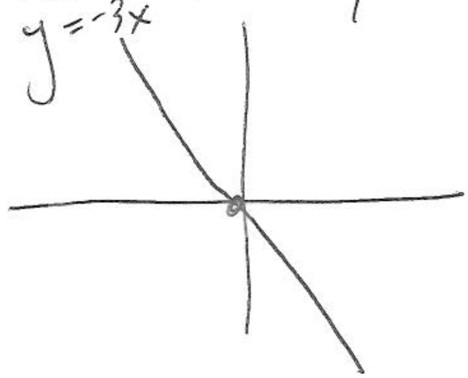
(1) The CR equations are a necessary condition. $u = y^2$ $\quad v = 3x^2$

$$u_x = v_y \Rightarrow 0 = 0$$

$$u_y = -v_x \Rightarrow 2y = -6x$$

$$\underline{y = -3x}$$

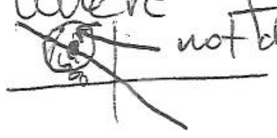
Since u_x, u_y, v_x, v_y are continuous at all points of $\mathbb{C}$, $f'(z)$ will exist along the line $y = -3x$



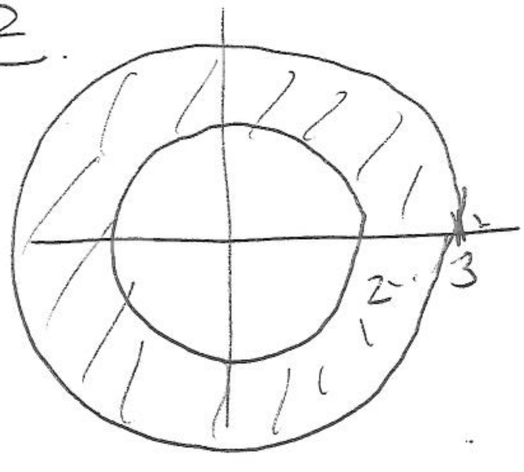
2). f is analytic at z_0 if $\exists\, \varepsilon > 0$ such that $f'(z)$ exists for all z such that $|z - z_0| < \varepsilon$

3) $f(z)$ is nowhere analytic.

$\forall\, \varepsilon > 0$, a disk centered at a point of the line $y = -3x$ will contain points where f is not differentiable.



Q2.

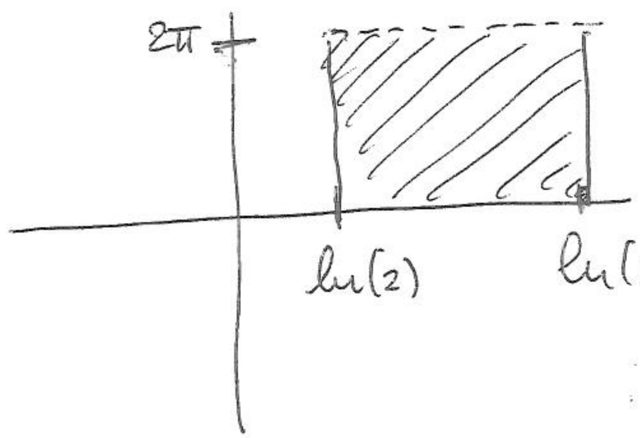


The annulus can be described as

$$z = re^{i\theta} \quad 0 \leq \theta < 2\pi$$
$$2 \leq r \leq 3$$

Therefore we can take R to be $z = \{ x + iy \mid \ln(2) \leq x \leq \ln(3), \ 0 \leq y < 2\pi \}$

R :



Q3 (1) $f(z) = \dfrac{1}{(i+z)^2} = \dfrac{d}{dz}\left(-\dfrac{1}{(i+z)}\right)$

$$\frac{1}{i+z} = \frac{1}{i} \cdot \frac{1}{1+\left(\frac{z}{i}\right)} = \frac{1}{i}\left(1 - \left(\frac{z}{i}\right) + \left(\frac{z}{i}\right)^2 \cdots\right)$$
$$= \frac{1}{i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{i}\right)^n$$
$$= \sum_{n=0}^{\infty}(-1)^n i^{(-n-1)} z^n$$

So $\dfrac{d}{dz}\left(\dfrac{1}{i+z}\right) = \displaystyle\sum_{n=0}^{\infty}(-1)^n i^{-n-1} \cdot n z^{n-1}$

$\therefore \dfrac{1}{(i+z)^2} = \displaystyle\sum (-1)^{n+1} i^{-n-1} \cdot n z^{n-1}$

This expansion is valid when $\left|\frac{z}{i}\right| < 1$

i.e. $|z| < 1$

Q3 (2)

$$\mathrm{Log}(1+z) = \int \frac{dz}{1+z}.$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

$$\int \frac{dz}{1+z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1}$$

This is valid when $|z| < 1$.

Q4

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\sin\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{-2n-1}}{(2n+1)!}$$

$$z^3 \sin\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{-2n+2}}{(2n+1)!}$$

$z = 0$ is an essential singularity, since arbitrarily high negative powers appear in the series.

Q5

(1) $$\frac{\exp(z)}{z^4} = \frac{1}{z^4}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\cdots\right)$$

$$= \frac{1}{z^4}+\frac{1}{z^3}+\frac{1}{2!\,z^2}+\frac{1}{3!\,z}+\frac{1}{4!}+\cdots$$

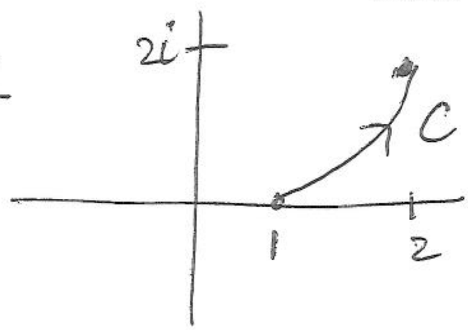
$$\operatorname{Res}_{z=0} f(z) = \frac{1}{3!}$$

(2) $$z^3\cos\left(\frac{1}{z}\right) = z^3\left(1-\frac{1}{2!\,z^2}+\frac{1}{4!\,z^4}-\frac{1}{6!\,z^6}+\cdots\right)$$

$$= z^3-\frac{z}{2!}+\frac{1}{4!\,z}-\frac{1}{6!\,z^3}+\cdots$$

$$\operatorname{Res}_{z=0} f(z) = \frac{1}{4!}$$

Q6



In a neighborhood of C, $\frac{1}{z}$ has an antiderivative

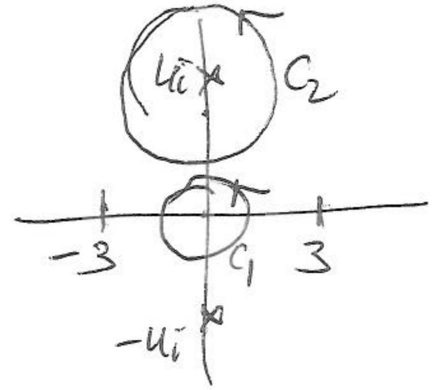
$$\frac{1}{z} = \frac{d}{dz}\operatorname{Log}(z)$$

So $$\int_C \frac{dz}{z} = \operatorname{Log}(2+2i)-\operatorname{Log}(1)$$

$$= \operatorname{Log}(2+2i)$$

$$= \ln(2\sqrt{2}) + i\left(\frac{\pi}{4}\right)$$

Q7 $\displaystyle\int_C \frac{dz}{(z^2-9)(z^2+16)}$



(1) $|z|=1$: C_1

Since $f(z) = \dfrac{1}{(z^2-9)(z^2+16)}$ is analytic in and on C_1

$\displaystyle\int_{C_1} f(z)\,dz = 0$ by C-G.

(2) $|z-3i|=2$: C_2.

C_2 contains the singularity $4i$

$f(z) = \dfrac{g(z)}{(z-4i)}$ $\quad g(z) = \dfrac{1}{(z^2-9)(z+4i)}$

by CIF, $\displaystyle\int_{C_2} \frac{g(z)}{z-4i} = 2\pi i\, g(4i)$

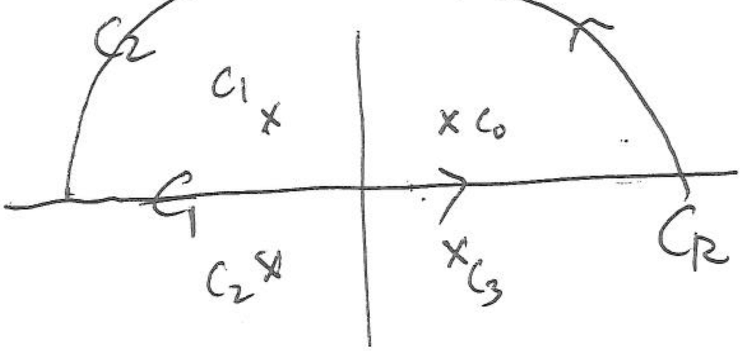
$$= \frac{2\pi i}{(-16-9)(4i+4i)}$$

$$= \frac{2\pi i}{-200i} = -\frac{\pi}{100}.$$

Q8

$$\frac{1}{z^4+1} = \frac{1}{(z-c_0)(z-c_1)(z-c_2)(z-c_3)}$$

~~where~~ c_0, c_1, c_2, c_3 are the 4th roots of -1



$$c_0 = e^{\frac{\pi}{4}i}$$
$$c_1 = e^{\frac{3\pi}{4}i}$$
$$c_2 = e^{\frac{5\pi}{4}i}$$
$$c_3 = e^{\frac{7\pi}{4}i}$$

Let C_R be the semi-circular contour above. $C_R = C_1 + C_2 \leftarrow$ circular part.

$\uparrow$ straight part

$$\int_{C_R} \frac{dz}{z^4+1} = \int_{C_1} \frac{dz}{z^4+1} + \int_{C_2} \frac{dz}{z^4+1} = 2\pi i \left(\operatorname{Res}_{z=c_0} f + \operatorname{Res}_{c_1} f \right.$$

$$= \int_{-R}^{R} \frac{dx}{x^4+1} + \int_{C_2} \frac{dz}{z^4+1}$$

Show $\int_{C_2} f\,dz \to 0$

① $$\left| \int_{C_2} \frac{dz}{z^4+1} \right| \le \underbrace{\pi R}_{L} \cdot \underbrace{\frac{1}{R^4-1}}_{M} = \frac{\pi R}{R^4-1}$$

$$\lim_{R\to\infty} \int_{C_2} \frac{dz}{z^4+1} = 0 \Rightarrow$$

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^4+1} = 2\pi i \left(\operatorname{Res}_{z=c_0} f + \operatorname{Res}_{c_1} f \right)$$

$$\operatorname{Res}_{z=c_0} f = \frac{1}{4c_0^3} = \frac{1}{4e^{3\pi i/4}}$$

$$\operatorname{Res}_{z=c_1} f = \frac{1}{4c_1^3} = \frac{1}{4e^{9\pi i/4}} = \frac{1}{4e^{\pi i/4}}$$

<u>So</u>:

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{2\pi i}{4}\left(e^{-3\pi i/4} + e^{-\pi i/4}\right)$$

$$= \frac{\pi i}{2}\left(\left(\frac{-1-i}{\sqrt{2}}\right) + \left(\frac{1-i}{\sqrt{2}}\right)\right)$$

$$= \frac{\pi i}{2}\,\frac{(-2i)}{\sqrt{2}} = \frac{2\pi}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}.$$